

ON THE AVERAGE PRIME DIVISORS

I. Kátai (Budapest, Hungary)

Dedicated to the memory of Professor M.V. Subbarao

Abstract. Let $\kappa(n) = \sum_{p|n} p$, $\omega(n)$ = number of prime divisors of n ,

$\rho(n) = \frac{\kappa(n)}{\omega(n)}$. Refining the result of W.D. Banks and his coauthors we prove that

$$\#\{n \leq x \mid \rho(n) = \text{integer}\} = (1 + o_x(1))c \frac{x}{\log \log x}$$

with some constant $c > 0$.

1. Introduction

Let \mathcal{P} = set of primes. p with and without suffixes always denote primes,
 $\omega(n) := \sum_{p|n} 1$,

$$\kappa(n) := \sum_{p|n} p; \quad \rho(n) = \frac{\kappa(n)}{\omega(n)},$$

$$R(x) := \#\{n \leq x \mid \rho(n) = \text{integer}\}.$$

The research was supported by the Hungarian National Foundation for Scientific Research under grants OTKA T043657 and T046993.

In [1] W.D. Banks and his coauthors proved that

$$(1.1) \quad c_1 < \frac{R(x) \log \log x}{x} < c_2$$

holds for $x > x_0$, where x_0, c_1, c_2 are positive constants.

By using our method evaluated in [2], [3] we can prove the following

Theorem. *We have*

$$(1.2) \quad R(x) = (1 + o_x(1))c \frac{x}{\log \log x} \quad (x \rightarrow \infty),$$

where c is a suitable positive constant.

Let φ be the Euler's totient function, $\pi(x)$ = number of primes up to x ,

$$\pi(x, k, l) = \sum_{\substack{p \leq x \\ p \equiv l \pmod{k}}} 1, \quad \pi_r(x) := \#\{n \leq x \mid \omega(n) = r\}.$$

Let $z \geq 1$ and

$$A(n|z) := \prod_{\substack{p^\alpha \parallel n \\ p \leq z}} p^\alpha, \quad B(n|z) = \frac{n}{A(n|z)}.$$

Let $P(n)$ be the largest prime factor of n . We shall write furthermore $x_1 = \log x$, $x_2 = \log x_1$, $x_3 = \log x_2, \dots$. Let furthermore $e(\alpha) := e^{2\pi i \alpha}$.

2. Lemmata

Lemma 1. *Let*

$$c_k(n) := \sum_{\substack{h=1 \\ (h,k)=1}}^k e\left(\frac{hn}{k}\right)$$

be the Ramanujan sum. Then

$$(2.1) \quad c_k(n) = \frac{\mu(t)\varphi(k)}{\varphi(t)}, \quad t = \frac{k}{(k, n)}$$

(see in Tenenbaum [4], p.35).

Lemma 2. *We have*

$$(2.2) \quad \#\left\{n \leq x \mid |\omega(n) - x_2| \geq \frac{1}{2}x_2\right\} \ll x/x_2^B,$$

where B is an arbitrary large constant.

This is an easy consequence of the Hardy–Ramanujan inequality, namely that

$$\pi_r(x) \leq c_1 \frac{x}{x_2} \frac{(x_2 + c)^{r-1}}{(r-1)!}.$$

Lemma 3. *Let $U(x, w)$ be the number of those integers $n \leq x$ for which there exists a square divisor d^2 such that $d > w$. Then*

$$(2.3) \quad U(x, w) \ll \frac{x}{w}.$$

The assertion is clear.

Lemma 4. *Let $G_L(x)$ be the number of integers $n \leq x$ having two prime divisors p_1 and p_2 satisfying $L < p_1 < p_2 < 4p_1$. Then*

$$(2.4) \quad G_L(x) \ll \frac{x}{\log L}.$$

Proof. Since

$$G_L(x) \leq \sum_{\substack{L < p_1 < p_2 \leq 4p_1 \\ p_1 p_2 \leq x}} \frac{x}{p_1 p_2} \leq cx \sum_{L < p_1 \leq \sqrt{x}} \frac{1}{p_1 \log p_1} \ll \frac{x}{\log L},$$

(2.4) holds.

Lemma 5. *Let $l_0 := \exp(x_2^A)$, where A is a large constant. Then*

$$(2.5) \quad \#\{n \leq x \mid A(n|l_0) > \exp(x_2^{A+1})\} \ll \frac{x}{x_2^{2A}}.$$

This can be deduced easily from the wellknown estimate

$$\psi(x, y) := \#\{n \leq x \mid P(n) \leq y\} \ll x \exp\left(-\frac{x_1}{2 \log y}\right)$$

(see for instance Tenenbaum [4]).

Lemma 6. *Let A and l_0 be as in Lemma 5. Then there exists a constant $b > 0$ such that*

$$(2.6) \quad \#\{n \leq x \mid \omega(A(n|l_0)) > bx_3\} \ll \frac{x}{x_2^{2A}}.$$

Proof. The left hand side is less than

$$\begin{aligned} & \frac{1}{2^{bx_3}} \sum_{n \leq x} \tau(A(n|l_0)) \leq \\ & \leq \frac{x}{2^{bx_3}} \sum_{P(d) \leq l_0} 1/d \ll \frac{x}{2^{bx_3}} \prod_{p < l_0} \frac{1}{1 - 1/p} \ll \frac{x}{2^{bx_3}} \exp(\log \log l_0), \end{aligned}$$

whence the assertion follows.

Lemma 7. *Let c, B be arbitrary constants,*

$$\frac{x}{x_1^c} \leq y \leq x, \quad k \leq x_1^B, \quad (l, k) = 1.$$

Then

$$(2.7) \quad \pi(x + y, k, l) - \pi(x, k, l) = \frac{\text{li}(x + y) - \text{li } x}{\varphi(k)} (1 + O(\exp(-c_2 \sqrt{x_1})))$$

uniformly in k, l .

Lemma 8. *Let \mathbb{Z}_q^* be the set of reduced residues classes mod q , $\lambda_{q,h}(s)$ be the number of solutions of $l_1 + l_2 + \dots + l_h \equiv s \pmod{q}$, where l_ν run over \mathbb{Z}_q^* independently. Then*

$$(2.8) \quad \lambda_{q,h}(s) = \frac{1}{q} \sum_{a=0}^{q-1} e\left(-\frac{sa}{q}\right) c_q(a)^h.$$

Consequently, if $q = \text{odd}$, p^ is its smallest prime divisor, then*

$$(2.9) \quad \left| \frac{\lambda_{q,h}(s)}{\varphi(q)^h} - \frac{1}{q} \right| \leq \frac{(q-1)}{q} \cdot \frac{1}{\varphi(p^*)^h},$$

while if $q = \text{even}$, then

$$(2.10) \quad \left| \frac{\lambda_{q,h}(s)}{\varphi(q)^h} - \frac{1}{q} \{1 + (-1)^{h+s}\} \right| \leq \frac{q-2}{q} \cdot \frac{1}{\varphi(p^*)^h},$$

where p^* is the smallest odd prime factor of q .

Proof. Clear.

3. Proof of the Theorem

3.1. Let \underline{v} be a fixed large number, and assume that $1 \leq h \leq Bx_2$, $\frac{x_2}{3} \leq q \leq 2x_2$. Let us choose a large constant A , and a positive constant c_0 . We shall consider the set

$$(3.1) \quad \mathcal{L} = \{l_j : j = 0, 1, 2, \dots\},$$

where

$$(3.2) \quad l_0 = \exp(x_2^A), \quad l_{j+1} = l_j + \frac{l_j}{(\log l_j)^{c_0}} \quad (j = 0, 1, 2, \dots).$$

Let $I(l_j) = [l_j, l_{j+1})$, $\beta(l_j) = \text{li } l_{j+1} - \text{li } l_j$, where $\text{li } y = \int_2^y \frac{du}{\log u}$. If $u \in \mathcal{L}$, $u = l_\nu$, then $\Delta u := l_{\nu+1} - l_\nu$, and so $I(u) = [u, u + \Delta u]$.

3.2. Let $Y \in [x^{1/2}, x]$. We shall consider such h -tuples (u_1, \dots, u_h) for which

$$(3.3) \quad l_0 \leq u_1 < \dots < u_h, \quad u_\nu \in \mathcal{L} \quad (\nu = 1, 2, \dots, h).$$

We say that it is feasible if $u_1, \dots, u_h \leq Y$, it is well spaced if $u_{j+1} \geq u_j$ ($j = 1, \dots, h-1$), and that it is completely suitable if

$$(u_1 + \Delta u_1) \dots (u_h + \Delta u_h) \leq Y.$$

3.3. Let

$$(3.4) \quad E_h(u_1, \dots, u_h) := \#\{p_1 \dots p_h \mid p_\nu \in I(u_\nu), \nu = 1, \dots, h\}.$$

In [3] we proved

Lemma 9. *If (u_1, \dots, u_h) is a well-spaced feasible h -tuple, then*

$$(3.5) \quad E_h(u_1, \dots, u_h) = \prod_{\nu=1}^h \beta(u_\nu) \cdot \left(1 + O\left(e^{-c_3 x_2^{A/2}}\right)\right).$$

3.4. For a given q and prime $p > l_0$ let $H(p) = H_q(p) \in [1, q-1]$ such that $H(p) \equiv p \pmod{q}$, and if $(l_0 <) p_1 < p_2 < \dots < p_h$ are primes then let $H(p_1 \dots p_h) = H(p_1) \dots H(p_h)$ be the word standing from the letters $H(p_1), \dots, H(p_h)$ concatenating them.

Let $\alpha = l_1 l_2 \dots l_h$, $l_\nu \in \mathbb{Z}_q^* = \{u \in [1, q-1], (u, q) = 1\}$. Furthermore let

$$(3.6) \quad E_h^{(q)}(u_1, \dots, u_h \mid \alpha) = \#\{p_1 \dots p_h \mid p_j \in I(u_j), H_q(p_j) = l_j, j = 1, \dots, h\}.$$

In [3] we proved also

Lemma 10. *We have*

$$(3.7) \quad E_h^{(q)}(u_1, \dots, u_h \mid \alpha) = \frac{1}{\varphi(q)^h} \prod_{\nu=1}^h \beta(u_\nu) \cdot (1 + O(\exp(-c_2 \sqrt{x_1}))),$$

if u_1, \dots, u_h is a well-spaced feasible h -tuple.

3.5. Let $\kappa_q(n) \equiv \sum_{p|n} p \pmod{q}$, especially $\kappa_q(p_1 \dots p_h) \equiv l_1 + l_2 + \dots + l_h \pmod{q}$, if $H_q(p_j) = l_j$ ($j = 1, \dots, h$). Thus the value of $\kappa_q(p_1 \dots p_h)$ does depend only on the value of $\alpha (= H_q(p_1 \dots p_h))$. We shall write $\kappa_q(\alpha) := l_1 + \dots + l_h \pmod{q}$.

From Lemma 8, 9, 10 we obtain

Lemma 11. *Let u_1, \dots, u_h be a well-spaced feasible h -tuple. Then, for every $s \pmod{q}$*

a) *in the case $q = \text{odd}$*

$$\begin{aligned} \sum_{\kappa_q(\alpha) \equiv s \pmod{q}} E_h^{(q)}(u_1, \dots, u_h \mid \alpha) &= \frac{1}{q} E_h(u_1, \dots, u_h) (1 + O(-c_2(\sqrt{x_1}))) = \\ &= O\left(\frac{1}{\varphi(p^*)^h} E_h(u_1, \dots, u_h)\right), \end{aligned}$$

where p^* is the smallest prime divisor of q (then $p^* \geq 3!$);

b) in the case $q = \text{even}$,

$$\begin{aligned} & \sum_{\kappa_q(\alpha) \equiv s \pmod{q}} E_h^{(q)}(u_1, \dots, u_h \mid \alpha) = \\ &= \frac{1}{q} \{1 + (-1)^{h+s}\} E_h(u_1, \dots, u_h) (1 + O(e^{-c_2 \sqrt{x_1}})) + \\ &+ O\left(E_h(u_1, \dots, u_h) \cdot \frac{1}{\varphi(p^*)^h}\right), \end{aligned}$$

where p^* is the smallest odd prime factor of q .

Lemma 12. *Let us summarize $E_h(u_1, \dots, u_h)$ over those feasible well spaced h -tuples for which $\prod_{\nu=1}^h u_\nu < Y < \prod_{\nu=1}^h (u_\nu + \Delta u_\nu)$, and then over all $h \leq Bx_2$. The amount is less than $\ll Y/x_2 A c$.*

This assertion is proved in [3].

4. Completion of the proof

Let us consider the integers $n \leq x$. Let l_0 be as in (3.2). Drop those integers for which (1) $\left| \omega(n) - x_2 \right| \geq \frac{x_2}{x}$, or (2) $B(n|l_0)$ is not square free, or (3) $B(n|l_0)$ has two prime divisors p_1, p_2 such that $p_1 < p_2 < 4p_1$, or (4) $A(n|l_0) > \exp(x_2^{A+1})$, or (5) $\omega(A(n|l_0)) > bx_3$, or (6) if $n = A(n|l_0)B(n|l_0)$, $B(n|l_0) = p_1 \dots p_h$, and by $p_\nu \in I(n_\nu)$, $\nu = 1, \dots, h$, then by $Y = \frac{x}{A(n|l_0)} \prod u_\nu < Y < \prod (u_\nu + \Delta u_\nu)$.

By using our lemmas we obtain that the size of the dropped integers is $O\left(\frac{x}{x_2^2}\right)$. Now we classify the others.

Let D be a fixed integer, for which $P(D) \leq l_0$, $D \leq \exp(x_2^{A+1})$, $\omega(D) \leq bx_3$. Let $Y = \frac{x}{D}$. Let \mathcal{E}_D be the set of those integers $n \leq x$, for which $A(n|l_0) = D$, and which are not dropped. Let $\mathcal{E}_D^{(h)}$ be that subset for which $B(n|l_0)$ contains exactly h prime divisors, $B(n|l_0) = p_1 \dots p_h$. If n is not dropped, then $p_\nu \in I(u_\nu)$, $\nu = 1, \dots, h$ such that (u_1, \dots, u_h) is a feasible, well spaced, completely suitable h tuple.

Let now u_1, \dots, u_h be fixed, $\mathcal{E}_D^{(h)}(u_1, \dots, u_h)$ be the numbers $p_1 \dots p_h$ such that $p_\nu \in I(u_\nu)$, $\nu = 1, \dots, h$.

Let $t = \omega(D)$, $q = t + h$. It is clear that $n = Dp_1 \dots p_h$ satisfies $\kappa(n) \equiv \equiv O(\text{mod } \omega(n))$, if $\kappa(D) + p_1 + \dots + p_h \equiv O(\text{mod } q)$. Let $\kappa(D) \equiv -s \pmod{q}$. Then $\kappa(n) \equiv O(\text{mod } q)$ holds if and only if $p_j \equiv l_j \pmod{q}$ ($j = 1, \dots, h$) and $l_1 + \dots + l_h \equiv s \pmod{q}$.

Our assertion easily follows from Lemma 11.

References

- [1] **Banks W.D., Garaev M.Z., Luca F. and Shparlinski I.E.**, Uniform distribution of the fractional part of the average prime divisor, *Forum Math.*, **17** (2005), 885-901.
- [2] **DeKoninck J.-M. and Kátai I.**, On the distribution of subsets of primes in the prime factorization of integers, *Acta Arith.*, **72** (1995), 169-200.
- [3] **DeKoninck J.-M. and Kátai I.**, On the local distribution of certain arithmetic functions, *Liet. Math. Rink.*, **46**(3) (2006), 315-331.

(Received January 15, 2007)

I. Kátai

Department of Computer Algebra

Eötvös Loránd University

Pázmány Péter sét. 1/C

H-1117 Budapest, Hungary

katai@compalg.inf.elte.hu