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A LARGE SIEVE ESTIMATE FOR DIRICHLET POLYNOMIALS AND ITS APPLICATIONS

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Dedicated to the memory of Professor M.V. Subbarao

1. Introduction

In this article, we will introduce a large sieve mean-value estimate for Dirichlet polynomials, from which we deduce the Bombieri-Vinogradov theorem and a Bombieri-type mean-value theorem for exponential sums over primes.

Estimation of sums of the form

$$S(x) = \sum_{n \le x} \Lambda(n) f(n), \tag{1.1}$$

where $\Lambda(n)$ is the von Mangoldt function, and $f(n)$ a certain arithmetical function, plays an important role in number theory. Let $z \ge 1$ and $k \ge 1$. By Heath-Brown's identity [1], for any $n < 2z^k$

$$\Lambda(n) = \sum_{j=1}^{k} (-1)^{j-1} \binom{k}{j} \sum_{\substack{n_1 n_2 \cdots n_{2j} = n \\ n_{j+1}, \dots, n_{2j} \le z}} \cdots \sum (\log n_1) \mu(n_{j+1}) \cdots \mu(n_{2j}). \tag{1.2}$$

This can be applied to estimate sums like (1.1). Suppose

$$z^k > x, \tag{1.3}$$

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so that Heath-Brown's identity applies for $n \le x$. We then split up each range of summation in (1.2) into intervals $n \sim N$, i.e. $n \in (N, 2N]$, and find that $S(x)$ is a linear combination of $O(\log^{2k} x)$ sums of the form

$$\sum_{\substack{n_1 n_2 \cdots n_{2k} \le x \\ n_j \sim N_j}} \cdots \sum (\log n_1) \mu(n_{k+1}) \cdots \mu(n_{2k}) f(n_1 n_2 \cdots n_{2k}), \tag{1.4}$$

where $\prod N_j < x$, and $2N_j \le z$ if $j > k$. Note that some of the intervals $(N, 2N]$ may contain only the integer 1.

2. A large sieve mean-value estimate

Usually, one takes $k = 5$ in Heath-Brown's identity (1.2). The following applications are examples in such a setting.

We estimate (1.1) via (1.4) with $k = 5$. To this end, let

$$X^{2/5} < Y \le X$$

and $M_1, ..., M_{10}$ be positive real numbers such that

$$Y \le M_1 \cdots M_{10} < X, \quad \text{and} \quad 2M_6, ..., 2M_{10} \le X^{1/5}. \tag{2.1}$$

For $j = 1, ..., 10$ define

$$a_j(m) = \begin{cases} \log m & \text{if } j = 1, \\ 1 & \text{if } j = 2, ..., 5, \\ \mu(m) & \text{if } j = 6, ..., 10. \end{cases} \tag{2.2}$$

For a complex variable s and a Dirichlet character χ, put

$$f_j(s, \chi) = \sum_{m \sim M_j} \frac{a_j(m) \chi(m)}{m^s},$$

and

$$F(s, \chi) = f_1(s, \chi) \cdots f_{10}(s, \chi). \tag{2.3}$$

The following hybrid estimate for $|F|$ will be very important in our later argument.

Theorem 1. *Let $F(s,\chi)$ be as in (2.3), and $A \geq 1$ arbitrary. Then for any $1 \leq R \leq X^{2A}$ and $0 < T \ll X^A$,*

$$\sum_{\substack{r\sim R\\ d|r}} \sum_{\chi \bmod r}^{*} \int_{-T}^{T} \left| F\left(\frac{1}{2}+it,\chi\right)\right| dt \ll \tag{2.4}$$

$$\ll \left(\frac{R^2}{d}T + \frac{R}{d^{1/2}}T^{1/2}X^{3/10} + X^{1/2}\right)\log^c X.$$

Here $c > 0$ is an absolute constant independent of A, but the constant implied in $\ll$ depends on A.

Theorem 1 with $d = 1$ was established in [4], and in this general form in [2] and [7]. In the following applications, we will only need Theorem 1 with $d = 1$.

Note that in Theorem 1 we have $r > R \geq 1$; therefore $\chi \bmod r$ never takes the principal primitive character $\chi^0 \bmod 1$. To access the strength of Theorem 1, we note that, by taking absolute value directly, a trivial bound for $|F|$ is

$$F\left(\frac{1}{2}+it,\chi\right) \ll \sum_{m\sim M_1} \frac{|a_1(m)|}{m^{1/2}} \cdots \sum_{m\sim M_{10}} \frac{|a_{10}(m)|}{m^{1/2}} \ll$$

$$\ll (M_1\cdots M_{10})^{1/2}\log X \ll X^{1/2}\log X.$$

Consequently, a trivial bound for the left-hand side of (2.4) is

$$\ll \sum_{\substack{r\sim R\\ d|r}} \sum_{\chi \bmod r} \int_{T}^{2T} X^{1/2}\log^c X dt \ll \frac{R^2}{d}TX^{1/2}\log^c X. \tag{2.5}$$

Compared with (2.5), the first term on the right-hand side of (2.4) saves in the X aspect, the second term in all the R, T and X aspects, and the third term in the R and T aspects.

3. The Bombieri-Vinogradov theorem

As an application of Theorem 1, we will establish the following Bombieri-Vinogradov theorem.

Theorem 2. (Bombieri-Vinogradov) *Set*

$$\psi(y;q,a)=\sum_{\substack{n\le y\\ n\equiv a(\mathrm{mod}\, q)}}\Lambda(n).$$

Then for any $A>0$ there exists a constant $B=B(A)>0$, such that

$$\sum_{q\le Q}\max_{y\le x}\max_{(a,q)=1}\left|\psi(y;q,a)-\frac{y}{\varphi(q)}\right|\ll\frac{x}{L^A},$$

where $\varphi(q)$ is the Euler totient function, $Q=x^{1/2}L^{-B}$ and $L=\log x$.

Proof. Introducing the Dirichlet characters,

$$\psi(y;q,a)-\frac{y}{\varphi(q)}=\frac{1}{\varphi(q)}\sum_{\chi \bmod q}\bar{\chi}(a)\sum_{m\le y}(\Lambda(m)\chi(m)-\delta_\chi)-\frac{y-[y]}{\varphi(q)}, \tag{3.1}$$

where $\delta_\chi=1$ or 0 according as χ is principal or not. Let

$$W(\chi)=\sum_{y/2<m\le y}(\Lambda(m)\chi(m)-\delta_\chi);$$

we remark that $W(\chi)$ depends on y although this is not made explicit. Then

$$\begin{aligned}\sum_{q\le Q}\max_{y\le x}\max_{(a,q)=1}&\left|\psi(y;q,a)-\frac{y}{\varphi(q)}\right|\ll\\ &\ll L+L\sum_{q\le Q}\frac{1}{\varphi(q)}\max_{y\le x}\sum_{\chi \bmod q}|W(\chi)|\ll\\ &\ll L+L\sum_{r\le Q}\sum_{\substack{q\le Q\\ r|q}}\frac{1}{\varphi(q)}\max_{y\le x}\sum_{\chi \bmod r}{}^{*}|W(\chi\chi^0)|,\end{aligned} \tag{3.2}$$

where χ^0 mod q is the principal character. For χ mod r and χ^0 mod q in the last line, we have

$$W(\chi\chi^0)-W(\chi)\ll\sum_{\substack{y/2<m\le y\\ (m,q)>1}}\Lambda(m)\ll(\log q)(\log y)\ll L^2.$$

Therefore we can replace $W(\chi\chi^0)$ by $W(\chi)+O(L^2)$ in the last term of (3.2). Now $W(\chi)+O(L^2)$ is independent of q, and the last term in (3.2) can be bounded by

$$\begin{aligned}&\ll L\sum_{r\le Q}\Big\{\sum_{\substack{q\le Q\\ r|q}}\frac{1}{\varphi(q)}\Big\}\max_{y\le x}\sideset{}{^*}\sum_{\chi \bmod r}\{|W(\chi)|+L^2\}\ll\\ &\ll QL^5+L^3\sum_{r\le Q}\frac{1}{r}\max_{y\le x}\sideset{}{^*}\sum_{\chi \bmod r}|W(\chi)|,\end{aligned}$$

where we have used the elementary estimates $\varphi(rt)\ge\varphi(r)\varphi(t)$ and

$$\sum_{\substack{q\le Q\\ r|q}}\frac{1}{\varphi(q)}\ll\frac{1}{\varphi(r)}\sum_{t\le Q/r}\frac{1}{\varphi(t)}\ll\frac{L^2}{r}.$$

Therefore Theorem 2 is a consequence of the estimate

$$\sum_{r\sim R}\frac{1}{r}\sideset{}{^*}\sum_{\chi \bmod r}\max_{y\le x}|W(\chi)|\ll xL^{-A} \tag{3.3}$$

for $R\le Q$ and arbitrary $A>0$.

Suppose first that $R\le L^C$ with an arbitrary $C>0$. Then, by the Siegel-Walfisz theorem,

$$W(\chi)\ll x\exp(-c\sqrt{\log x})$$

for some constant $c>0$, and hence (3.3) is true in this case.

If $L^C<R\le Q$, then we always have $\delta_\chi=0$ for all primitive character modulo $r\sim R$, and consequently

$$W(\chi)=\sum_{y/2<m\le y}\Lambda(m)\chi(m). \tag{3.4}$$

To (3.4), we apply Heath-Brown's identity with $k=5$. In (2.1) we take

$$Y=x^{2/5},\quad X=x; \tag{3.5}$$

it is important that X and Y do not depend on y. Define $a_j(m)$, $f_j(s,\chi)$ and $F(s,\chi)$ as in §2. For

$$2x^{2/5}=2Y<y\le X=x, \tag{3.6}$$

$W(\chi)$ is a linear combination of $O(L^{10})$ terms, each of which is of the form

$$\sigma(\mathbf{M}) := \sum_{\substack{m_1\sim M_1 \\ y/2<m_1\cdots m_{10}\le y}} \cdots \sum_{m_{10}\sim M_{10}} a_1(m_1)\chi(m_1)\cdots a_{10}(m_{10})\chi(m_{10}),$$

where $\mathbf{M}$ denotes the vector $(M_1, M_2, ..., M_{10})$ with M_j as in (2.1). Note that some of the intervals $(M_j, 2M_j]$ may contain only the integer 1. By using Perron's summation formula with $T = y$, and then shifting the contour to the left, the above $\sigma(\mathbf{M})$ is

$$= \frac{1}{2\pi i}\int_{1+1/L-iy}^{1+1/L+iy} F(s,\chi)\frac{y^s-(y/2)^s}{s}ds + O(L^2) =$$

$$= \frac{1}{2\pi i}\left\{\int_{1+1/L-iy}^{1/2-iy} + \int_{1/2-iy}^{1/2+iy} + \int_{1/2+iy}^{1+1/L+iy}\right\} + O(L^2).$$

The integral on the two horizontal segments above can be easily estimated as

$$\ll \max_{1/2\le\sigma\le 1+1/L}|F(\sigma\pm iy,\chi)|\frac{y^\sigma}{y} \ll \max_{1/2\le\sigma\le 1+1/L} x^{1-\sigma}L\frac{y^\sigma}{y} \ll$$
$$\ll \left(\frac{x}{y}\right)^{1/2} L \ll x^{3/10}L$$

on using the trivial estimate

$$F(\sigma\pm iy,\chi) \ll |f_1(\sigma\pm iy,\chi)|\cdots|f_{10}(\sigma\pm iy,\chi)| \ll$$
$$\ll (M_1^{1-\sigma}L)M_2^{1-\sigma}\cdots M_{10}^{1-\sigma} \ll x^{1-\sigma}L.$$

Thus, for y satisfying (3.6),

$$\sigma(\mathbf{M}) = \frac{1}{2\pi}\int_{-y}^{y} F\left(\frac{1}{2}+it,\chi\right)\frac{y^{\frac{1}{2}+it}-(y/2)^{\frac{1}{2}+it}}{\frac{1}{2}+it}dt + O(L^2) \ll$$

$$\ll y^{1/2}\int_{-y}^{y}\left|F\left(\frac{1}{2}+it,\chi\right)\right|\frac{dt}{|t|+1} + L^2.$$

Recalling that F does not depend on y, we have

$$\max_{2Y<y\le x} |W(\chi)| \ll L^{10}x^{1/2}\int_{-x}^{x}\left|F\left(\frac{1}{2}+it,\chi\right)\right|\frac{dt}{|t|+1}+L^{12}.$$

On the other hand, one has trivially

$$\max_{y\le 2Y}|W(\chi)|\ll Y.$$

Now the left-hand side of (3.3) is

$$\begin{aligned}&\ll \frac{1}{R}\sum_{r\sim R}\sideset{}{^*}\sum_{\chi \bmod r}\max_{2Y<y\le x}|W(\chi)|+\frac{1}{R}\sum_{r\sim R}\sideset{}{^*}\sum_{\chi \bmod r}\max_{y\le 2Y}|W(\chi)|\ll\\ &\ll \frac{x^{1/2}}{R}L^{10}\sum_{r\sim R}\sideset{}{^*}\sum_{\chi \bmod r}\int_{-x}^{x}\left|F\left(\frac{1}{2}+it,\chi\right)\right|\frac{dt}{|t|+1}+RY.\end{aligned}$$

The last term is acceptable; it therefore follows that (3.3) is a consequence of the estimate that, for $0<T\le x$,

$$\sum_{r\sim R}\sideset{}{^*}\sum_{\chi \bmod r}\int_{T}^{2T}\left|F\left(\frac{1}{2}+it,\chi\right)\right|dt\ll Rx^{1/2}(T+1)L^{-A}. \tag{3.7}$$

By Theorem 1, the left-hand side of (3.7) is now

$$\begin{aligned}&\ll (R^2T+RT^{1/2}x^{3/10}+x^{1/2})L^c\ll\\ &\ll Rx^{1/2}(T+1)L^c(Rx^{-1/2}+x^{-1/5}+R^{-1}).\end{aligned}$$

The above quantity is acceptable provided that $L^C<R\le x^{1/2}L^{-B}$ with B and C sufficiently large in terms of A. This establishes (3.7) and (3.3), and hence the Theorem.

4. Exponential sums over primes: small q

In this section, we are concerned with the asymptotic behavior of the sum

$$S(x,\alpha)=\sum_{m\le x}\Lambda(m)e(m\alpha),$$

where $\alpha \in [0,1]$ satisfies the rational approximation

$$\alpha = \frac{a}{q} + \lambda, \quad \text{with} \quad (a,q)=1, \quad 1 \le a \le q. \tag{4.8}$$

The results below (Lemmas 3 and 4) are just an easy application of the Siegel-Walfisz theorem, but here we present the argument explicitly since some of the materials will be used in §5.

By the orthogonality of Dirichlet characters $S(x,\alpha)$ can be written as

$$\begin{aligned} S(x,\alpha) &= \sum_{\substack{m\le x\\(m,q)=1}} \Lambda(m)e(m\alpha) + O\left\{\sum_{\substack{m\le x\\(m,q)>1}} \Lambda(m)\right\} = \\ &= \sum_{\substack{h=1\\(h,q)=1}}^{q} e\left(\frac{ah}{q}\right) \sum_{\substack{m\le x\\ m\equiv h(\bmod q)}} \Lambda(m)e(m\lambda) + O\{(\log q)(\log x)\} = \\ &= \frac{1}{\varphi(q)} \sum_{\chi \bmod q} C(\bar\chi, a) \sum_{m\le x} \Lambda(m)\chi(m)e(m\lambda) + O(L^2), \end{aligned}$$

where

$$C(\chi,a) = \sum_{h=1}^{q} \chi(h) e\left(\frac{ah}{q}\right).$$

Now recall $C(\chi^0,a)=\mu(q)$. Thus,

$$\begin{aligned} S(x,\alpha) &= \frac{\mu(q)}{\varphi(q)} \sum_{m\le x} e(m\lambda) + \\ &+ \frac{1}{\varphi(q)} \sum_{\chi \bmod q} C(\bar\chi,a) \sum_{m\le x} (\Lambda(m)\chi(m) - \delta_\chi) e(m\lambda) + O(L^2). \end{aligned} \tag{4.9}$$

To investigate the inner sum over m, we let

$$W(\chi,\lambda) = \sum_{x/2<m\le x} (\Lambda(m)\chi(m) - \delta_\chi) e(m\lambda); \tag{4.10}$$

note that $W(\chi,\lambda)$ also depends on x. Now suppose $q \le L^A$ with arbitrary $A>0$. Then by the Siegel-Walfisz theorem,

$$W(\chi,\lambda) = \int_{x/2}^{x} e(u\lambda)\, d\left\{\sum_{m\le u} (\Lambda(m)\chi(m) - \delta_\chi)\right\} = \int_{x/2}^{x} e(u\lambda)\, dR(u),$$

where $R(u) \ll u \exp(-c\sqrt{\log u})$. Therefore, partial integration gives

$$W(\chi,\lambda) \ll |R(x)| + \left|\lambda \int_1^x e(u\lambda)R(u)du\right| \ll (1+|\lambda|x)x\exp(-c\sqrt{L}),$$

and hence the inner sum over m in (4.9) has the same upper bound with a smaller c. Applying the bound $|C(\bar{\chi},a)| \le \sqrt{q}$ to (4.9), we conclude that

$$S(x,\alpha) = \frac{\mu(q)}{\varphi(q)} \sum_{m\le x} e(m\lambda) + O\{\sqrt{q}(1+|\lambda|x)x\exp(-c\sqrt{L})\}. \tag{4.11}$$

This may be summarized in the following.

Lemma 3. *Let α be as in (4.8), with $q \le L^A$ and $A > 0$ arbitrary. Then there exist two positive constants c_1 and c_2, such that for $|\lambda| \le x^{-1}\exp(c_1\sqrt{L})$,*

$$S(x,\alpha) = \frac{\mu(q)}{\varphi(q)} \sum_{m\le x} e(m\lambda) + O\{x\exp(-c_2\sqrt{L})\}. \tag{4.12}$$

The following conditional result may be compared with Lemma 3.

Lemma 4. *Under GRH,*

$$S(x,\alpha) = \frac{\mu(q)}{\varphi(q)} \sum_{m\le x} e(m\lambda) + O\{\sqrt{qx}(1+|\lambda|x)L^c\}. \tag{4.13}$$

5. A Bombieri-type theorem for exponential sums over primes

To extend the range of q in Lemma 3 is as difficult as to do this in the Siegel-Walfisz theorem. However, on average, the range of q can be extended considerably, as is shown in the following theorem of Bombieri-Vinogradov's type.

Theorem 5. *Let α be as in (4.8) and $\varepsilon > 0$ arbitrary. For any $A > 0$, there exists a constant $B = B(A) > 0$, such that if Q and θ satisfy*

$$1 \le Q \le x^{1/3}L^{-B}, \qquad \theta = Q^{-3}L^{-B}, \tag{5.1}$$

then

$$\sum_{q\le Q}\max_{y\le x}\max_{(a,q)=1}\max_{|\lambda|\le\theta}\left|S(y,\alpha)-\frac{\mu(q)}{\varphi(q)}\sum_{m\le y}e(m\lambda)\right|\ll xL^{-A}.$$

Theorem 5 with

$$1\le Q\le x^{1/4},\quad \theta=\min(Q^{-4},L^{-B}),$$

where $B=B(A)>0$ was established by Wolke [8]. Theorem 5 in the present form was proved in [6] by a different argument. Here we will derive it from Theorem 1. A result similar to Theorem 5 for non-linear exponential sums over primes has been established in [5].

We remark that, in the special case $Q=1$, we must have $a=q=1$ in (4.8), and hence the theorem states that, for $|\lambda|\le L^{-B}$,

$$S(x,\lambda)=\sum_{m\le x}e(m\lambda)+O(xL^{-A}). \tag{5.2}$$

On the other hand, we may take $\lambda=0$ in (4.8), so that now the theorem reduces to

$$\sum_{q\le Q}\max_{y\le x}\max_{(a,q)=1}\left|S\left(y,\frac{a}{q}\right)-\frac{\mu(q)}{\varphi(q)}y\right|\ll xL^{-A}. \tag{5.3}$$

We note that (5.2) and (5.3) are not covered by Lemma 3.

To derive from Theorem 5 a result for almost all q, we denote by $\mathcal{Q}$ the set of $q\le Q$ such that

$$\left|S(x,\alpha)-\frac{\mu(q)}{\varphi(q)}\sum_{m\le x}e(m\lambda)\right|\gg x^{2/3}L^B.$$

Then Theorem 5 gives

$$\sum_{q\in\mathcal{Q}}x^{2/3}L^B\ll\frac{x}{L^A},$$

and hence

$$|\mathcal{Q}|\ll x^{1/3}\log^{-A-B}x.$$

Therefore, for all $q\le Q$ except on a set $\mathcal{Q}$ of cardinality $O(x^{1/3}\log^{-A-B}x)$,

$$S(x,\alpha)=\frac{\mu(q)}{\varphi(q)}\sum_{m\le x}e(m\lambda)+O(x^{2/3}\log^B x).$$

Proof of Theorem 5. We begin by modifying the definition of $W(\chi,\lambda)$ in (4.10) slightly, so that now it depends on y instead of x, i.e.

$$W(\chi,\lambda)=\sum_{y/2<m\le y}(\Lambda(m)\chi(m)-\delta_\chi)e(m\lambda). \tag{5.4}$$

Thus (4.9), with x replaced by y, gives

$$\sum_{q\le Q}\max_{y\le x}\max_{(a,q)=1}\max_{|\lambda|\le\theta}\left|S(y,\alpha)-\frac{\mu(q)}{\varphi(q)}\sum_{m\le y}e(m\lambda)\right|\ll \tag{5.5}$$

$$\ll QL^2+L\sum_{q\le Q}\frac{1}{\varphi(q)}\max_{y\le x}\max_{(a,q)=1}\max_{|\lambda|\le\theta}\sum_{\chi \bmod q}|C(\bar\chi,a)W(\chi,\lambda)|\ll$$

$$\ll QL^2+L\sum_{r\le Q}\sum_{\substack{q\le Q\\ r|q}}\frac{1}{\varphi(q)}\max_{y\le x}\max_{(a,q)=1}\max_{|\lambda|\le\theta}\sideset{}{^*}\sum_{\chi \bmod r}|C(\bar\chi\chi^0,a)W(\chi\chi^0,\lambda)|,$$

where $\chi^0 \bmod q$ is the principal character. For $\chi \bmod r$ and $\chi^0 \bmod q$ in the last line, we have

$$W(\chi\chi^0,\lambda)-W(\chi,\lambda)\ll\sum_{\substack{m\le y\\ (m,q)>1}}\Lambda(m)\ll L^2.$$

Therefore we can replace $W(\chi\chi^0,\lambda)$ by $W(\chi,\lambda)+O(L^2)$ in the last term of (5.5). Since

$$|C(\bar\chi\chi^0,a)|\le r^{1/2},$$

the last term in (5.5) is bounded by

$$\ll L\sum_{r\le Q}\left\{\sum_{\substack{q\le Q\\ r|q}}\frac{r^{1/2}}{\varphi(q)}\right\}\max_{y\le x}\max_{|\lambda|\le\theta}\sideset{}{^*}\sum_{\chi \bmod r}\{|W(\chi,\lambda)|+L^2\}\ll$$

$$\ll L^3\sum_{r\le Q}\frac{1}{r^{1/2}}\max_{y\le x}\max_{|\lambda|\le\theta}\sideset{}{^*}\sum_{\chi \bmod r}\{|W(\chi,\lambda)|+L^2\}\ll$$

$$\ll Q^{3/2}L^5+L^3\sum_{r\le Q}\frac{1}{r^{1/2}}\max_{y\le x}\max_{|\lambda|\le\theta}\sideset{}{^*}\sum_{\chi \bmod r}|W(\chi,\lambda)|.$$

The term $Q^{3/2}L^5$ is acceptable if Q satisfies (5.1) with B sufficiently large in terms of A. Therefore, the theorem is a consequence of the estimate

$$J:=\sum_{r\sim R}\frac{1}{r^{1/2}}\sideset{}{^*}\sum_{\chi \bmod r}\max_{y\le x}\max_{|\lambda|\le\theta}|W(\chi,\lambda)|\ll xL^{-A}, \tag{5.6}$$

where $R \le Q$, and $A > 0$ is arbitrary.

We consider two cases according as R small or big. The case when R is big is handled in Lemma 6, where it is proved that there exists some constant $C = C(A) > 0$, such that (5.6) is true if $L^C < R \le Q$. The proof applies, among other things, Theorem 1 and Heath-Brown's identity. The case when R is small is treated in Lemma 7. It is proved, by the zero-density estimate, that (5.6) is true for $R \le L^C$ and arbitrary $C > 0$. The desired assertion now follows from Lemmas 6 and 7.

Lemma 6. *Let J be as in (5.6). Then for arbitrary $A > 0$, there exists a constant $C = C(A) > 0$, such that for*

$$L^C < R \le Q,$$

we have

$$J \ll xL^{-A}.$$

Proof. Let

$$Y = x^{2/5}, \quad X = x,$$

and define $a_j(m)$, $f_j(s,\chi)$, and $F(s,\chi)$ as in §2. Suppose

$$2Y < y \le u \le X,$$

and to the sum

$$\sum_{y/2 < m \le u} \Lambda(m)\chi(m) \tag{5.7}$$

we apply Heath-Brown's identity as in the last section. Thus, (5.7) is a linear combination of $O(L^{10})$ terms, each of which is of the form

$$\sigma(u;\mathbf{M}) := \underset{y/2 < m_1 \cdots m_{10} \le u}{\sum_{M_1 < m_1 \le 2M_1} \cdots \sum_{M_{10} < m_{10} \le 2M_{10}}} a_1(m_1)\chi(m_1) \cdots a_{10}(m_{10})\chi(m_{10}),$$

where $\mathbf{M}$ denotes the vector $(M_1, M_2, ..., M_{10})$ with M_j as in (2.1). We may estimate $\sigma(\mathbf{M})$ by an argument similar to that after (3.6) in the proof the Bombieri-Vinogradov theorem; actually, by using Perron's summation formula with $T = x$, and then shifting the contour to the left, the above $\sigma(u;\mathbf{M})$ is

$$\sigma(u;\mathbf{M}) = \frac{1}{2\pi} \int_{-x}^{x} F\left(\frac{1}{2} + it, \chi\right) \frac{u^{\frac{1}{2}+it} - (y/2)^{\frac{1}{2}+it}}{\frac{1}{2} + it} dt + O(L^2).$$

Since $R > L^C$ (so $\chi \neq \chi^0$), our $W(\chi, \lambda)$ in (5.4) can be written as

$$W(\chi,\lambda) = \sum_{m\sim y} \Lambda(m)\chi(m)e(m\lambda) = \int_{y/2}^{y} e(u\lambda)d\left\{\sum_{y/2<m\leq u} \Lambda(m)\chi(m)\right\}, \tag{5.8}$$

and consequently $W(\chi, \lambda)$ is a linear combination $O(L^{10})$ terms, each of which is of the form

$$\int_{y/2}^{y} e(u\lambda)d\sigma(u;\mathbf{M}) = \frac{1}{2\pi}\int_{-x}^{x} F\left(\frac{1}{2}+it,\chi\right)\int_{y/2}^{y} u^{-1/2+it}e(u\lambda)dudt+$$

$$+O\{(1+|\lambda|x)L^2\}.$$

Changing variables in the inner integral, we deduce from the above formulae that

$$\begin{aligned}\max_{2Y<y\leq X}|W(\chi,\lambda)| \ll L^{10}\max_{\mathbf{M}}\Bigg|\int_{-x}^{x} F\left(\frac{1}{2}+it,\chi\right)\times \\ \times\int_{y/2}^{y} u^{-1/2}e\left(\frac{t}{2\pi}\log u+\lambda u\right)dudt\Bigg| + \theta x L^{12},\end{aligned} \tag{5.9}$$

where the maximum is taken over all $\mathbf{M} = (M_1, M_2, ..., M_{10})$. This will be used later in combination with the trivial bound

$$\max_{y\leq 2Y}|W(\chi,\lambda)| \ll Y. \tag{5.10}$$

Now we estimate the contribution of $|W(\chi, \lambda)|$ to the J in (5.6). The contribution of (5.10) is

$$\ll R^{2/3}Y \ll Q^{2/3}x^{2/5} \ll x^{28/45},$$

which is acceptable; and the contribution of the term $\theta x L^{12}$ in (5.9) is

$$\ll R^{3/2}\theta x L^{12} \ll xL^{-B+12},$$

by the definition of θ in (5.1), which is also acceptable if B is sufficiently large. To estimate the contribution of the first term on the right-hand side of (5.9) we note that

$$\frac{d}{du}\left(\frac{t}{2\pi}\log u+\lambda u\right) = \frac{t}{2\pi u}+\lambda, \quad \frac{d^2}{du^2}\left(\frac{t}{2\pi}\log u+\lambda u\right) = -\frac{t}{2\pi u^2}.$$

Thus, by the first and second derivative tests, the inner integral in (5.9) can be bounded by

$$\ll y^{-1/2}\min\left\{\frac{y}{(|t|+1)^{1/2}}, \frac{y}{\min\limits_{y/2<u\le y}|t+2\pi\lambda u|}\right\}\ll$$

$$\ll\begin{cases} x^{1/2}/(|t|+1)^{1/2} & \text{if } |t|\le T_0,\\ x^{1/2}/|t| & \text{if } T_0<|t|\le T,\end{cases} \tag{5.11}$$

where

$$T_0=4\pi x\theta. \tag{5.12}$$

Here the choice of T_0 is to ensure that $|t+2\pi\lambda u|>|t|/2$ whenever $|t|>T_0$; in fact,

$$|t+2\pi\lambda u|\ge|t|-2\pi|u|\theta>\frac{|t|}{2}+\frac{T_0}{2}-2\pi x\theta\ge\frac{|t|}{2}.$$

It therefore follows that

$$J\ll\frac{x^{1/2}L^{10}}{R^{1/2}}\sum_{r\sim R}\sideset{}{^*}\sum_{\chi \bmod r}\max_{y\le x}\max_{|\lambda|\le\theta}\max_{\mathbf{M}}\left\{\int\limits_{|t|\le T_0}\left|F\left(\frac{1}{2}+it,\chi\right)\right|\frac{dt}{\sqrt{|t|+1}}+\right.$$
$$\left.+\int\limits_{T_0<|t|\le T}\left|F\left(\frac{1}{2}+it,\chi\right)\right|\frac{dt}{|t|}\right\}+xL^{-B+12}.$$

The two maxima over y and over λ above can be deleted because the quantity within the braces is now independent of these two variables. Also, we may assume F is the function for which the maximum over $\mathbf{M}$ is obtained. Therefore, Lemma 6 is a consequence of the following two estimates: if $0<T_1\le T_0$, then

$$\sum_{r\sim R}\sideset{}{^*}\sum_{\chi \bmod r}\int_{T_1}^{2T_1}\left|F\left(\frac{1}{2}+it,\chi\right)\right|dt\ll R^{1/2}x^{1/2}(T_1+1)^{1/2}L^{-A}; \tag{5.13}$$

and if $T_0<T_2\le x$, then

$$\sum_{r\sim R}\sideset{}{^*}\sum_{\chi \bmod r}\int_{T_2}^{2T_2}\left|F\left(\frac{1}{2}+it,\chi\right)\right|dt\ll R^{1/2}x^{1/2}T_2L^{-A}. \tag{5.14}$$

By Theorem 1 the left-hand side of (5.13) is now

$$\begin{aligned} &\ll (R^2T_1 + RT_1^{1/2}x^{3/10} + x^{1/2})L^c \ll \\ &\ll R^{1/2}x^{1/2}(T_1+1)^{1/2}L^c\{R^{3/2}T_0^{1/2}x^{-1/2} + R^{1/2}x^{-1/5} + R^{-1/2}\}. \end{aligned}$$

Since $T_0 \asymp \theta x$ (see (5.12)), the above quantity is acceptable provided that θ satisfies (5.1) and $R > L^C$ with sufficiently large B and C. This establishes (5.13).

By Theorem 1 again, the left-hand side of (5.14) is

$$\begin{aligned} &\ll (R^2T_2 + RT_2^{1/2}x^{3/10} + x^{1/2})L^c \ll \\ &\ll R^{1/2}x^{1/2}T_2L^c\{R^{3/2}x^{-1/2} + R^{1/2}x^{-1/5} + R^{-1/2}\}, \end{aligned}$$

which is acceptable provided that $L^C < R \le x^{1/3}\log^{-B} x$ with a sufficiently large C. This establishes (5.14), and Lemma 6 now follows.

Now we treat the case $R \le L^C$.

Lemma 7. *Let $A > 0$ be arbitrary and $C = C(A)$ be determined as in Lemma 6. Let θ be as in Theorem 5, and $R \le L^C$. Then there exists $B = B(A) > 0$ such that*

$$J \ll xL^{-A}.$$

Proof. We begin with $W(\chi, \lambda)$ defined in (5.4). Now we have

$$W(\chi,\lambda) = \int_{y/2}^{y} e(u\lambda)d\left\{\sum_{n\le u}(\Lambda(m)\chi(m) - \delta_\chi)\right\}. \tag{5.15}$$

To the quantity within the braces, we apply the explicit formula

$$\sum_{m\le u}(\Lambda(m)\chi(m) - \delta_\chi) = -\sum_{|\gamma|\le T}\frac{u^\rho}{\rho} + O\left\{\left(\frac{u}{T}+1\right)\log^2(qT)\right\},$$

where $\rho = \pm i\gamma$ runs over non-trivial zeros of the function $L(s,\chi)$, and $T \ge 2$ is a parameter. Take $T = x$; then the above O-term is $O(L^2)$. Hence by partial summation,

$$W(\chi,\lambda) = -\sum_{|\gamma|\le x}\int_{y/2}^{y} u^{\rho-1}e(u\lambda)du + O\{(1+|\lambda|x)L^2\}. \tag{5.16}$$

The integral in (5.16) can be estimated similarly by the first and second derivative tests. Thus,

$$\begin{aligned}\int_{y/2}^{y} u^{\rho-1} e(u\lambda) du &= \int_{y/2}^{y} u^{\beta-1} e\left(\frac{\gamma}{2\pi}\log u + \lambda u\right) du \ll \\ &\ll y^{\beta-1}\min\left\{\frac{y}{(|\gamma|+1)^{1/2}}, \frac{y}{\min\limits_{y/2<u\le y}|\gamma+2\pi\lambda u|}\right\} \ll \\ &\ll \begin{cases} x^{\beta}/(|\gamma|+1)^{1/2} & \text{if } |\gamma|\le T_0, \\ x^{\beta}/|\gamma| & \text{if } T_0<|\gamma|\le x, \end{cases}\end{aligned}$$

where $T_0 = 4\pi x\theta$, the same as in (5.12). Inserting this into (5.16) and then taking summation over χ mod r and $r \sim R \le L^C$, we have
(5.17)

$$\begin{aligned} J &\ll \sum_{r\sim R}\sum_{\chi \bmod r} \max_{y\le x}\max_{|\lambda|\le\theta} |W(\chi,\lambda)| \ll \\ &\ll \sum_{r\sim R}\sum_{\chi \bmod r}\sum_{|\gamma|\le T_0} \frac{x^{\beta}}{\sqrt{|\gamma|+1}} + \sum_{r\sim R}\sum_{\chi \bmod r}\sum_{T_0<|\gamma|\le x}\frac{x^{\beta}}{|\gamma|} + \theta x L^{2C+2} =: \\ &=: J_1 + J_2 + \theta x L^{2C+2}, \end{aligned}$$

say.

By (5.1) we have $\theta \ll L^{-B}$, and hence the last term is

$$\ll xL^{-B+2C+2},$$

which is acceptable if B is sufficiently large.

The term J_2 will be bounded by Vinogradov's zero-free region, which states that for any χ mod r, there exists a constant $c_3 > 0$ such that $L(\sigma + it, \chi) \ne 0$ in the region

$$\sigma \ge 1 - \frac{c_3}{\log r + \log^{4/5}(|t|+2)}$$

except for the possible Siegel zero. However, since $r \le L^C$, the Siegel zero does not exist in the present situation. It follows that $L(s,\chi)$ is zero-free for $\sigma \ge 1 - \eta(T)$ and $|t| \le T$, where

$$\eta(\tau) = \frac{c_3}{2\log^{4/5}(\tau+2)} \quad \text{for} \quad \tau \ge 0.$$

Consequently, the inner sum in J_2 is

$$\ll x \sum_{T_0<|\gamma|\le x} \frac{x^{\beta-1}}{|\gamma|} \ll x\exp\{-\eta(x)\log x\} \sum_{T_0<|\gamma|\le x} \frac{1}{|\gamma|} \ll x\exp\left\{-\frac{c_3}{3}L^{1/5}\right\}.$$

Therefore,

$$J_2 \ll x\exp\left\{-\frac{c_3}{4}L^{1/5}\right\},$$

which is also acceptable.

To bound J_1, we write

$$\sum_{\chi \bmod r}\sum_{|\gamma|\le T_0} \frac{x^\beta}{\sqrt{|\gamma|+1}} \ll xL \max_{T_1\le T_0}(T_1+1)^{-1/2} \sum_{\chi \bmod r}\sum_{\gamma\sim T_1} x^{\beta-1}.$$

The last double sums can be estimated by Ingham's zero-density theorem that

$$\sum_{\chi \bmod r} N(\sigma,\tau,\chi) \ll (r\tau)^{\frac{3-3\sigma}{2-\sigma}}(\log r\tau)^9,$$

where $N(\sigma,\tau,\chi)$ denotes the number of zeros $\rho=\beta+i\gamma$ of $L(s,\chi)$ with $\sigma\le$ $\le\beta\le 1$ and $|\gamma|\le\tau$. Thus,

$$\sum_{\chi \bmod r}\sum_{\gamma\sim T_1} x^{\beta-1} \ll -\int_{1/2}^{1-\eta(T_0)} x^{\sigma-1} d\left\{\sum_{\chi \bmod r} N(\sigma,T_1,\chi)\right\} \ll$$
$$\ll \log^9(rT_1) \max_{1/2\le\sigma\le 1-\eta(T_0)} (rT_1)^{\frac{3-3\sigma}{2-\sigma}} x^{\sigma-1},$$

and therefore,

$$\sum_{\chi \bmod r}\sum_{|\gamma|\le T_0} \frac{x^\beta}{\sqrt{|\gamma|+1}} \ll xL^{11+C} \max_{T_1\le T_0}\max_{1/2\le\sigma\le 1-\eta(T_0)} \times$$
$$\times\exp\left\{-(1-\sigma)L+\left(\frac{3-3\sigma}{2-\sigma}-\frac{1}{2}\right)\log T_1\right\}. \tag{5.18}$$

Denote by $f(T_1,\sigma)$ the exponential function above; we will analyze $f(T_1,\sigma)$ in detail.

Suppose first $4/5\le\sigma\le 1-\eta(T_0)$, so that

$$\frac{3-3\sigma}{2-\sigma}-\frac{1}{2}\le 0.$$

From this and the zero-free region it follows that

$$\max_{T_1\le T_0}\max_{4/5\le\sigma\le 1-\eta(T_0)} f(T_1,\sigma) \ll \max_{4/5\le\sigma\le 1-\eta(T_0)} \exp\{(\sigma-1)L\} \ll$$
$$\ll \exp\left\{-\frac{c_3}{2}L^{1/5}\right\}.$$

Secondly we consider $3/5 \le \sigma \le 4/5$, which implies that

$$\frac{3-3\sigma}{2-\sigma}-\frac{1}{2}\ge 0.$$

Since $T_1 \le T_0 \ll \theta x \ll xL^{-B}$, we have $\log T_1 \le L$, and consequently

$$\max_{T_1\le T_0}\max_{3/5\le\sigma\le 4/5} f(T_1,\sigma) \ll \max_{3/5\le\sigma\le 4/5} \exp\left\{-(1-\sigma)L+\left(\frac{3-3\sigma}{2-\sigma}-\frac{1}{2}\right)L\right\} =$$
$$= \max_{3/5\le\sigma\le 4/5} \exp\left\{-\frac{\sigma(\sigma-1/2)}{2-\sigma}L\right\} =$$
$$= x^{-3/70}.$$

Finally we deal with the case $1/2 \le \sigma \le 3/5$. Now we have

$$\frac{3-3\sigma}{2-\sigma}-\frac{1}{2}\ge\frac{6}{7},$$

and consequently,

$$\max_{T_1\le T_0}\max_{1/2\le\sigma\le 3/5} f(T_1,\sigma) \ll$$
$$\ll \max_{1/2\le\sigma\le 3/5} \exp\left\{-(1-\sigma)L+\left(\frac{3-3\sigma}{2-\sigma}-\frac{1}{2}\right)\log x\right\}\times$$
$$\times \exp\left\{-\left(\frac{3-3\sigma}{2-\sigma}-\frac{1}{2}\right)\log\frac{x}{T_0}\right\}.$$

By $T_0 \ll xL^{-B}$ again, the above quantity is

$$\ll \max_{1/2\le\sigma\le 3/5} \exp\left\{-\frac{\sigma(\sigma-1/2)}{2-\sigma}L\right\}\exp\left\{-\frac{6}{7}B\log\log x\right\} \ll$$
$$\ll L^{-6B/7}.$$

Inserting these estimates into (5.18), we get

$$J_1 \ll xL^{C-6B/7+11}\sum_{r\sim R}1 \ll xL^{2C-6B/7+11},$$

which is acceptable if B is sufficiently large. Lemma 7 now follows from (5.17) and the above estimates for J_1 and J_2.

6. Application of Theorem 1 in the Waring-Goldbach problem

Theorem 1 also enables one to deal with enlarged major arcs in the Waring-Goldbach problem. It is useful to a wide circle of problems of Waring-Goldbach type, and has been successfully applied to a number of additive problems concerning primes. The reader is referred to [3] for a survey.

References

[1] **Heath-Brown D.R.,** Prime numbers in short intervals and a generalized Vaughan's identity, *Can. J. Math.,* **34** (1982), 1365-1377.

[2] **Liu J.Y.,** On Lagrange's theorem with prime variables, *Quart. J. Math. (Oxford),* **54** (2003), 453-462.

[3] **Liu J.Y.,** An iterative method in the Waring-Goldbach problem, *Chebyshevskiĭ Sb.,* **5** (2005), 164-179.

[4] **Liu J.Y. and Liu M.C.,** The exceptional set in the four prime squares problem, *Illinois J. Math.,* **44** (2000), 272-293.

[5] **Liu J.Y. and Ye J.M.,** Mean-value estimates for nonlinear Weyl sums over primes, *Japan. J. Math. (N.S.),* **31** (2005), 379-390.

[6] **Liu J.Y. and Zhan T.,** The ternary Goldbach problem in arithmetic progrssions, *Acta Arith.,* **82** (1997), 197-227.

[7] **Liu J.Y. and Zhan T.,** The exceptional set in Hua's theorem for three squares of primes, *Acta Math. Sin. (Engl. Ser.),* **21** (2005), 335-350.

[8] **Wolke D.,** Some applications to zero-density theorems for L-functions, *Acta Math. Hung.,* **61** (1993), 241-258.

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