Annales Univ. Sci. Budapest., Sect. Comp. 27 (2007) 65-74

DISTRIBUTION OF q–ADDITIVE FUNCTIONS ON THE SET OF INTEGERS HAVING k PRIME FACTORS

L. Germán and I. Kátai

(Budapest, Hungary)

Dedicated to the memory of Professor M.V. Subbarao

1. Notations. $q \geq 2$ is an integer, $A_q = \{0, 1, \dots, q-1\}$; $n = \sum_{j=0}^{\infty} \epsilon_j(n) q^j$, $\epsilon_j(n) \in A_q$ is the q-ary expansion of n.

We say, that $f : \mathbb{N}_0 \to \mathbb{R}$ is q-additive, if $f(0) = 0$, and $f(n) = \sum_{j=0}^{\infty} f(\epsilon_j(n) q^j)$ holds for every $n \in \mathbb{N}$. The set of q-additive functions is denoted by $\mathcal{A}_q$.

$\mathcal{P}$ = set of primes, p with or without suffixes denote primes, $\omega(n)$ = number of distinct prime factors of n, $\Omega(n)$ = number of prime factors of n counted them with multiplicity. Let $P(n)$ be the largest prime factor of n. Let $\mathcal{P}_k = \{n \mid \omega(n) = k\}$, $\mathcal{N}_k = \{n \mid \Omega(n) = k\}$. Let $\pi(x) = \sum_{p \leq x} 1$, $\pi_k(x) = \#\{n \leq x, \ n \in \mathcal{P}_k\}$, $N_k(x) = \#\{n \leq x, \ n \in \mathcal{N}_k\}$. Let $x_1 = \log x$, $x_2 = \log x_1, \dots$. $\Phi(x)$ = Gaussian normal law.

The letters $c, c_1, c_2, \dots$ denote suitable positive constants not the same at every occurence. Let furthermore $\{y\}$ = fractional part of y, $\|y\| = \min(\{y\}, 1 - \{y\})$, $e(y) := e^{2\pi i y}$.

In this paper we shall formulate some generalizations of earlier results of the second named author.

The research was supported by the Hungarian National Foundation for Scientific Research under grants OTKA T043657 and T46993.

For $f \in \mathcal{A}_q$ define

$$m_j = \frac{1}{q}\sum_{b=0}^{q-1} f(bq^j), \qquad \sigma_j^2 = \frac{1}{q}\sum_{b=0}^{q-1}(f(bq^j) - m_j)^2, \tag{1.1}$$

$$M(N) = \sum_{l=0}^{N-1} m_j, \qquad D^2(N) = \sum_{l=0}^{N-1} \sigma_l^2. \tag{1.2}$$

Let ξ_j be independent random variables, $P(\xi_j = f(bq^j)) = 1/q$ $(b \in \in A_q)$, $\eta_N = \xi_0 + \ldots + \xi_{N-1}$, and so $E(\xi_j)$ (=mean value of ξ_j)=m_j, $E\eta_N = = M(n)$, $E(\eta_N - M(N))^2 = D^2(N)$, and so

$$q^{-N}\sum_{n=0}^{q^N-1}(f(n) - M(n))^2 = D^2(N). \tag{1.3}$$

In [1] it was proved, that

$$\frac{1}{\pi(q^N)}\sum_{p<q^N}(f(p) - (N))^2 < cD^2(N). \tag{1.4}$$

This inequality had been deduced from a nontrivial estimate for

$$\#\{p < q^N \mid \epsilon_{j_1}(p) = b_1,\ \epsilon_{j_2}(p) = b_2\},$$

where $0 \le j_1, j_2 < N$, $j_1 \ne j_2$, $b_1, b_2 \in A_q$. The proof depends on the Siegel-Walfisz theorem for primes in arithmetical progressions, on sieve results for the upper estimation of primes in short intervals, and on two theorems for trigonometric sums with prime variables which will be refered now to as Lemma 1 and 2.

Lemma 1. (I.M. Vinogradov) *Let* $H = e^{0,5\sqrt{x_1}}$, $\alpha = \dfrac{a}{Q} + \dfrac{\theta}{Q^2}$, $(a, Q) = = 1$, $|\theta| \le 1$, $1 < Q \le x$,

$$S(x \mid \alpha) := \sum_{\substack{p<x \\ p\in\mathcal{P}}} e(\alpha p). \tag{1.5}$$

Let $R = \frac{1}{H} + \sqrt{\frac{1}{Q} + \frac{Q}{x}}$. *Then*

$$|S(x \mid \alpha)| \le cx \cdot x_1^3 R \tag{1.6}$$

(see in [2], Chapter 10, §2).

Lemma 2. *Let* $\epsilon_0 > 0$ *be a small positive constant,* $x \ge Y \ge \ge xe^{-\epsilon_0\sqrt{x_1}}$, $0 < Q \le e^{\epsilon_0\sqrt{x_1}}$, $(a, Q) = 1$. *Then*

$$\left| \sum_{x-Y<p\le x} e\left(\frac{a}{Q}p\right) \right| \le \frac{cY \log\log Q}{\sqrt{Q} \cdot x_1} \tag{1.7}$$

(see [2], Chapter 10, §3).

Repeating the argument without any important changes which was used to prove (1.4) we obtain

Theorem 1. *Let* $1 \le B < x^{1/3}$, $(B, q) = 1$, $f \in \mathcal{A}_q$, $x > q$. *Then*

$$\frac{1}{\pi(\frac{x}{B})} \sum_{p\le x/B} (f(Bp) - M(N_x))^2 < cD^2(N_x),$$

where $c > 0$ *is a numerical constant,* N_x *is the integer, for which* $x \le q^{N_x} < qx$.

In a joint paper of N.L. Bassily and I. Kátai [3] it was proved

Theorem A. *Let* $f \in \mathcal{A}_q$, *and* $\sup_{j\in\mathbb{N},\ b\in A_q} |f(bq^j)| < c$. *Let* $P(x)$ *be a polynomial with integer coefficients,* $r = \deg P(x) \ge 1$, *the leading coefficient of* P *be positive. Let* $N_x := \left[\frac{\log x}{\log q}\right]$. *Assume that* $D(N)/N^{1/3} \to \infty$ $(N \to \infty)$. *Then*

$$\lim_{x\to\infty} \frac{1}{x} \# \left\{ n < x \;\middle|\; \frac{f(P(n)) - M(Nr)}{D(Nr)} < y \right\} = \Phi(y)$$

and

$$\lim_{x\to\infty} \frac{1}{\pi(x)} \# \left\{ p < x \;\middle|\; \frac{f(P(p)) - M(Nr)}{D(Nr)} < y \right\} = \Phi(y).$$

The proof is based upon some theorems of I.M. Vinogradov and L.K. Hua for trigonometric sums, by which the following assertion was proved.

Theorem B. *Let $\lambda > 0$ be an arbitrary constant. $N = N_x = \left[\frac{\log x}{\log q}\right]$. Let h be fixed, $b_1, \dots, b_h \in A_q$ and $l_1, \dots, l_h$ be integers for which*

$$N^{-1/3} \le l_1 < l_2 < \dots < l_h \le rN - N^{1/3}. \tag{1.8}$$

Let

$$\Sigma_1 := \#\{n \le x \mid a_{l_j}(P(n)) = b_j,\ j = 1, \dots, h\}$$

and

$$\Sigma_2 := \#\{p \le x \mid a_{l_j}(P(p)) = b_j,\ j = 1, \dots, h\}.$$

Then

$$q^h \Sigma_1 = x + \mathcal{O}(x \cdot x_1^{-\lambda}), \quad q^h \Sigma_2 = \pi(x) + \mathcal{O}(\pi(x) \cdot x_1^{-\lambda}). \tag{1.9}$$

The implicit constants in the error terms on the right hand sides of the formulas (1.9) may depend on P, h, λ.

The deduction of Theorem A from Theorem B is quite straightforward. One consider the normalized sums

$$a_k(x) := \frac{1}{x} \sum_{n \le x} \left(\frac{f(P(n)) - M(rN)}{D(rN)} \right)^k,$$

$$b_k(x) := \frac{1}{\pi(x)} \sum_{p \le x} \left(\frac{f(P(p)) - M(rN)}{D(rN)} \right)^k,$$

$$c_k(x) := \frac{1}{x} \sum_{n \le x^r} \left(\frac{f(n) - M(rN)}{D(rN)} \right)^k.$$

From Lemma 1 it follows almost immediately that $a_k(x) - c_k(x) \to$ $\to 0,\ b_k(x) - c_k(x) \to 0$ $(x \to \infty)$, for every fixed $k = 0, 1, 2 \dots$.

By standard method of probability theory for sums of independent random variables one can obtain that

$$\lim_{x \to \infty} c_k = \int x^k d\Phi,$$

whence by using the Frechet-Shohat theorem (see in Galambos [4]) Theorem A immediately follows.

By using the method of proof of Theorem B one can obtain the following assertion which we quote as

Lemma 3. *Let* $N=\left[\frac{\log x}{\log q}\right]$, $\lambda>0$ *be an arbitrary constant,* $h\in\mathbb{N}$. *Then*

$$\sup_{N^{1/3}\le l_1<\ldots<l_h\le N-N^{1/3}}\ \sup_{b_1,\ldots,b_h\in A_q}\left|\frac{q^h}{\pi(x/B)}\#\{Bp\le x\mid \epsilon_{l_j}(Bp)=b_j,\right.$$

$$\left. j=1,\ldots h\}-1\right|\le c\pi\left(\frac{x}{B}\right)x_1^{-\lambda}$$

uniformly as $1\le B\le x^{1/3}$, $(B,q)=1$.

Hence by using the argument which was applied getting by Theorem A from Theorem B, we obtain

Theorem 2. *Let* $f\in\mathcal{A}_q$, $\sup_{j\in\mathbb{N},\ b\in A_q}|f(bq^j)|<c$. $N=\left[\frac{\log x}{\log q}\right]$. *Assume that* $D(N)/N^{1/3}\to\infty$. *Let* $1\le B<x^{1/3}$, $(B,q)=1$. *Then*

$$\sup_{\substack{B<x^{1/3}\\(B,q)=1}}\ \sup_{y\in[-\infty,\infty]}\left|\frac{1}{\pi\left(\frac{x}{B}\right)}\#\left\{p<x/B\ \middle|\ \frac{f(Bp)-M(N)}{D(N)}<y\right\}-\Phi(y)\right|\le\tau(x),$$

where $\tau(x)\to 0$ *as* $x\to\infty$.

According to a theorem of H. Delange [5] $f\in\mathcal{A}_q$ has a limit distribution, if and only if

$$\sum_{j=0}^{\infty}\sum_{a\in A_q} f(aq^j)\text{ is convergent and}\tag{1.10}$$

$$\sum_{j=0}^{\infty}\sum_{a\in A_q} f^2(aq^j)<\infty.\tag{1.11}$$

Kátai proved that $f\in\mathcal{A}_q$ has a limit distribution on the set of primes, if and only if (1.10) and (1.11) hold true. The sufficiency of the conditions has been proved by J. Coquet and I. Kátai, independently, earlier. Here we formulate now without proof

Theorem 3. *Let* $f\in\mathcal{A}_q$, *assume that (1.10), (1.11) hold. Let* $\xi_0,\xi_1,\xi_2,\ldots$ *be independent random variables,* $P(\xi_j=f(bq^j))=1/q$ *if* $j\ge 1$, $b\in$

$\in A_q,\ P(\xi_0 = f(b)) = 1/\varphi(q)$ *if* $b \in A_q,\ (b,q) = 1$. *Let* $\theta := \sum_{l=0}^{\infty} \xi_l$. *Then the right hand side is almost surely convergent. Let* $F(y) := P(\theta < y)$. *Let* $1 \le B < x^{1/3},\ (B,q) = 1,$

$$F_{x,B}(y) := \frac{1}{\pi\left(\frac{x}{B}\right)} \# \left\{p < \frac{x}{B} : f(pB) < y\right\}.$$

Then

$$\max_{\substack{1 \le B < x^{1/3}\\ (B,q)=1}} \sup_y |F_{x,B}(y) - F(y)| \le \delta_x,$$

$\delta_x \to 0\ (x \to \infty)$.

2. Let $k = k(x) \in \mathbb{N}$ be a function of x, such that $1 \le k \le \delta(x)x_2$, where $\delta(x) \to 0$ monotonically, as $x \to \infty$. It is known, that

$$\pi_k(x) = (1 + o_x(1))\frac{x}{x_1}\frac{x_2^{k-1}}{(k-1)!}, \tag{2.1}$$

$$N_k(x) = (1 + o_x(1))\frac{x}{x_1}\frac{x_2^{k-1}}{(k-1)!} \tag{2.2}$$

uniformly as $1 \le k \le \delta(x)x_2$. Let $j_x := [1,\ \delta(x)x_2]$. Furthermore, it is clear that uniformly in $k \in J_x$

$$\frac{1}{\pi_k(x)}\#\{n \le x \mid n \in \mathcal{P}_k,\ (n,q) > 1\} \to 0 \quad (x \to \infty), \tag{2.3}$$

$$\frac{1}{N_k(x)}\#\{n \le x \mid n \in \mathcal{N}_k,\ (n,q) > 1\} \to 0 \quad (x \to \infty). \tag{2.4}$$

Let us write every $n \in \mathcal{P}_k$ (and $n \in \mathcal{N}_k$) as $n = \pi p$, where $p = P(n)$.

Lemma 4. *Assume that* $\delta(x) \to 0\ (x \to \infty)$, *monotonically. Then there exists some suitable* $\epsilon(x) \to 0\ (x \to \infty)$ *monotonically such that*

$$\frac{1}{\pi_k(x)}\#\{n \le x \mid n \in \mathcal{P}_k,\ \pi > p^{\epsilon(x)}\} \to 0 \quad (x \to \infty), \tag{2.5}$$

and

$$\frac{1}{N_k(x)}\#\{n \le x \mid n \in \mathcal{N}_k,\ \pi > p^{\epsilon(x)}\} \to 0 \quad (x \to \infty), \tag{2.6}$$

uniformly as $k \in J_x$.

Proof of Lemma 4. The case $k = 1, 2$ is clear. We assume that $k \ge 3$. From (2.1), (2.2) one can deduce easily that

$$\frac{1}{\pi_k(x)}\#\{n \in \mathcal{P}_k,\ n \le x,\ n \ne \text{ square free}\} \to 0,$$

$$\frac{1}{N_k(x)}\#\{n \in \mathcal{N}_k,\ n \le x,\ n \ne \text{ square free}\} \to 0,$$

as $x \to \infty$ uniformly in $k \in J_x$. Consequently it is enough to prove (2.5).

Let $y \in [\sqrt{x}, x],\ p \in \mathcal{P},\ p < x^{1/4}$,

$$T = T(y,p) := \sum_{\substack{n \le y \\ n \in \mathcal{P}_{k-1} \\ P(n) < p}} \log n.$$

Then

$$T = \sum_{\substack{t \in \mathcal{P} \\ t < p}} (\log t^\alpha)\pi_{k-2}\left(\frac{y}{t^\alpha}\right) < \frac{cy}{x_1}\frac{x_2^{k-3}}{(k-3)!}\sum_{\substack{t < p \\ t^\alpha < x^{1/4}}} \frac{(\log t^\alpha)}{t^\alpha} + \mathcal{O}(yx^{-1/5}) \le$$

$$\le \frac{cy}{x_1}\frac{x_2^{k-3}}{(k-3)!}\log p + \mathcal{O}(yx^{-1/5}),$$

whence we obtain that

$$\#\{n \le y,\ n \in \mathcal{P}_{k-1},\ P(n) < p\} \le \frac{cy}{x_1^2}\frac{x_2^{k-3}}{(k-3)!}\log p + \mathcal{O}(yx^{-1/5}). \tag{2.7}$$

Let $\epsilon > 0$ be fixed. We shall estimate those $n = \pi p \in \mathcal{P}_k$, for which $n \le x$, $\pi > p^{\epsilon(x)}$. We can drop the integers ($\mathcal{P}_k \ni$) n up to $\frac{x}{x_1}$. If $n \in \left[\frac{x}{x_1}, x\right]$ then $x \ge p\pi > p^{1+\epsilon}$, and so $p \le x^{\frac{1}{1+\epsilon}}$. Let us count first those $n \in \mathcal{P}_k$ for which $p \le x^\varrho$, where ϱ is a small positive number. As we observed we may assume that $n =$ squarefree.

From (2.7) we obtain that

$$\begin{aligned}
&\#\{n \le x,\ n \in \mathcal{P}_k,\ P(n) \le x^{\varrho},\ n = \text{squarefree}\} \le \\
\le &\sum_{p \le x^{\varrho}} \#\left\{m \le \frac{x}{p},\ m \in \mathcal{P}_{k-1},\ P(m) < p\right\} \le \\
\le &\frac{cx}{x_1^2} \frac{x_2^{k-3}}{(k-3)!} \sum_{p < x^{\varrho}} \frac{\log p}{p} + \mathcal{O}(x^{4/5} x_2) \le \\
\le &c\varrho \frac{x}{x_1} \frac{x_2^{k-3}}{(k-3)!} + \mathcal{O}(x^{4/5} x_2).
\end{aligned}$$

Let $n = \pi p \in \mathcal{P}_k,\ \pi > p^{\epsilon},\ p > x^{\varrho}$. Then $x^{\epsilon\varrho} \le p^{\epsilon} < \pi < x^{1-\varrho}$, and

(2.8)
$$\#\{n = \pi p \in \mathcal{P}_k,\ n \le x,\ x^{\epsilon\varrho} < \pi < x^{1-\varrho}\} \le \frac{c x x_2^{k-2}}{(k-2)!} \sum_{\substack{\pi \in \mathcal{P}_{k-1} \\ \pi \in [x^{\epsilon\varrho}, x^{1-\varrho}]}} \frac{1}{\pi \log \frac{x}{\pi}}.$$

Let $U_0 := x^{1-\varrho},\ U_j := 2^{-j} U_0$ $(j = 0, 1, \ldots, j_0)$, where j_0 is the smallest integer for which $U_{j_0} < x^{\epsilon\varrho}$. To estimate the sum on the right hand side of (2.8) we subdivide it into subsums $\sum\limits_{\pi \in [U_{j+1}, U_j]}$. We have

$$\sum_{\pi \in [U_{j+1}, U_j]} \frac{1}{\pi \log \frac{x}{n}} \le \frac{1}{U_{j+1} \log \frac{x}{U_j}} \pi_{k-1}(U_j) \le \frac{c}{(\log U_j) \log x^{\varrho} 2^j} \frac{x_2^{k-2}}{(k-2)!}.$$

Since

$$\begin{aligned}
(\log U_j) \log x^{\varrho} 2^j &= \{(1-\varrho)x_1 - j \log 2\}\{\varrho x_1 + j \log 2\} = \\
&= [x_1 - (\varrho x_1 + j \log 2)](\varrho x_1 + j \log 2)
\end{aligned}$$

and

$$\varrho x_1 \le \varrho x_1 + j \log 2 \le \varrho x_1 + j_0 \log 2 \le (1 - \epsilon\varrho)x_1 - \log 2,$$

furthermore $j_0 \log 2 < (1 - \varrho - \epsilon\varrho)x_1$, after some computation we obtain that

$$\sum_{j=0}^{j_0} \frac{1}{[x_1 - (\varrho x_1 + j \log 2)][\varrho x_1 + j \log 2]} \le \frac{c \log 1/\varrho}{x_1},$$

whence

$$\#\{n \le x,\ n = \pi p \in \mathcal{P}_k,\ x^{\epsilon\varrho} < \pi < x^{1-\varrho},\ n = \text{squarefree}\} \le \frac{cx \log 1/\varrho}{x_1} \frac{x_2^{k-2}}{(k-2)!}.$$

Collecting our inequalities Lemma 4 follows. By using Lemma 4, furthermore Theorem 2 we shall obtain

Theorem 4. *Let $f \in \mathcal{A}_q$, $k \in J_x$, and $\sup_{\substack{j=1,2,\dots \\ b \in A_q}} |f(bq^j)| < c$. Let $N = N_x = \left[\frac{\log x}{\log q}\right]$. Assume that $D(n)/N^{1/3} \to \infty$. Then*

$$\sup_{k \in J_x} \sup_{y \in \mathbb{R}} \left| \frac{1}{\pi_k(x)} \#\{n \le x,\ n \in \mathcal{P}_k \ \Big| \ \frac{f(n) - M(N)}{D(N)} < y\} - \Phi(y) \right| \to 0$$

as $x \to \infty$, and

$$\sup_{k \in J_x} \sup_{y \in \mathbb{R}} \left| \frac{1}{N_k(x)} \#\{n \le x,\ n \in \mathcal{N}_k \ \Big| \ \frac{f(n) - M(N)}{D(N)} < y\} - \Phi(y) \right| \to 0$$

as $x \to \infty$.

Theorem 5. *Let $f \in \mathcal{A}_k$, and assume that (1.10), (1.11) hold. Let $k \in J_x$,*

$$H_x^{(k)}(y) := \frac{1}{\pi_k(x)} \#\{n < x,\ n \in \mathcal{P}_k,\ f(n) < y\},$$

$$G_x^{(k)}(y) := \frac{1}{N_k(x)} \#\{n < x,\ n \in \mathcal{N}_k,\ f(n) < y\}.$$

Let $F(y)$ be defined as in Theorem 3. Then

$$\lim_{x \to \infty} \sup_{k \in J_x} |H_x^{(k)} - F(y)| = 0,$$

$$\lim_{x \to \infty} \sup_{k \in J_x} |G_x^{(k)} - F(y)| = 0$$

if y is a continuity point of F.

References

[1] **Kátai I.,** Distribution of digits of primes in q-ary canonical forms, *Acta Math. Hung.,* **47** (1986), 341-359.

[2] **Карацуба А.А.,** *Основы аналитической теории чисел,* Наука, Москва, 1975. (Karatsuba A.A., The basics of analytic number theory, Nauka, Moscow, 1975. (in Russian))

[3] **Bassily N.L. and Kátai I.,** Distribution of the values of q-additive functions on polynomial sequences, *Acta Math. Hung.,* **68** (1995), 353-361.

[4] **Galambos J.,** *Advanced probability theory,* Marcel Dekker Inc., New York, 1988.

[5] **Delange H.,** Sur les fonctions q-additives ou q-multiplicatives, *Acta Arith.,* **21** (1972), 285-298.

(Received December 11, 2006)

L. Germán and I. Kátai
Department of Computer Algebra
Eötvös Loránd University
Pázmány Péter s. 1/C
H-1117 Budapest, Hungary
katai@compalg.inf.elte.hu