A SIMPLY–OBTAINED UPPER BOUND FOR q(n)

N. Robbins (San Francisco, CA, USA)

Dedicated to the memory of Professor M.V. Subbarao

Abstract. Using simple analytic methods, we obtain an upper bound for q(n), the number of partitions of the natural number n into distinct parts (or into odd parts).

1. Introduction

If n is a natural number, let p(n), q(n) denote respectively the number of unrestricted partitions of n, the number of partitions of n into distinct parts (or into odd parts). Using the circle method, G.H. Hardy and S. Ramanujan [3] obtained the asymptotic formula

(1)
$$p(n) \sim \frac{1}{n\sqrt{48}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right)$$

By similar methods, P. Hagis [2] obtained the following asymptotic formula for q(n):

(2)
$$q(n) \sim 18^{\frac{1}{4}} (24n+1)^{-\frac{3}{4}} \exp\left(\frac{\pi}{12}\sqrt{48n+2}\right).$$

As is mentioned in [1] (Chapter 14), using more elementary methods, van Lint [4] obtained the following upper bound for p(n):

(3)
$$p(n) < \frac{\pi}{\sqrt{6(n-1)}} \exp\left(\pi \frac{\sqrt{2n}}{3}\right).$$

•

The purpose of this note is to obtain, in similar fashion, an upper bound for q(n), namely

(4)
$$q(n) < \frac{\pi}{\sqrt{12(n-1)}} \exp\left(\frac{\pi^2}{12} + \pi \frac{\sqrt{n-1}}{3}\right).$$

2. The main result

Theorem.

$$q(n) < \frac{\pi}{\sqrt{12(n-1)}} \exp\left(\frac{\pi^2}{12} + \pi \frac{\sqrt{n-1}}{3}\right)$$

Proof. We begin with the generating function

$$\sum_{n \ge 0} q(n)x^n = F(x) = \prod_{n \ge 1} (1 - x^{2n-1})^{-1},$$

where x is a real variable and 0 < x < 1. Taking logarithms, we have

$$\log F(x) = -\sum_{n \ge 1} \log(1 - x^{2n-1}) = \sum_{n \ge 1} \sum_{m \ge 1} \frac{(x^{2n-1})^m}{m} =$$
$$\sum_{m \ge 1} \frac{1}{m} \sum_{n \ge 1} (x^{2n-1})^m = \sum_{m \ge 1} \frac{1}{m} \left(\frac{x^m}{1 - x^{2m}}\right) = \sum_{m \ge 1} \frac{1}{m} \left(\frac{x^m}{1 - x^m}\right) \left(\frac{1}{1 + x^m}\right)$$

Recall from the proof of (3) that for 0 < x < 1, we have

$$\frac{1}{m}\left(\frac{x^m}{1-x^m}\right) < \frac{1}{m^2}\left(\frac{x}{1-x}\right).$$

Therefore we have

$$\log F(x) < \sum_{m \ge 1} \frac{1}{m^2} \left(\frac{x}{1-x} \right) \left(\frac{1}{1+x^m} \right) < \sum_{m \ge 1} \frac{1}{m^2} \left(\frac{x}{1-x} \right) \left(\frac{1}{1+x} \right),$$

that is

$$\log F(x) < \sum_{m \ge 1} \frac{1}{m^2} \left(\frac{x}{1 - x^2} \right) = \frac{x}{1 - x^2} \sum_{m \ge 1} \frac{1}{m^2} = \frac{\pi^2}{6} \left(\frac{x}{1 - x^2} \right).$$

Also, we have

$$q(n)\sum_{k=n}^{\infty} x^k \le \sum_{m=n}^{\infty} q(m)x^m < F(x),$$

so that

$$q(n)\left(\frac{x^n}{1-x}\right) < F(x).$$

This implies

$$\log q(n) < \log(1-x) + n \log \frac{1}{x} + \frac{\pi^2}{6} \left(\frac{x}{1-x^2}\right).$$

Let $x = (1+t)^{-1}$, so that 1 - x = xt and $\frac{1}{x} = 1 + t$, where $0 \le t < \infty$. Then we have

$$\log q(n) < \log t + \log x + n \log(1+t) + \frac{\pi^2}{6} \left(\frac{t+1}{(t+1)^2 - 1} \right),$$

that is

$$\log q(n) < \log t + (n-1)\log(1+t) + \frac{\pi^2}{6} \left(\frac{t+1}{(t+1)^2 - 1}\right),$$

which implies

$$\log q(n) < \log t + (n-1)t + \frac{\pi^2}{6} \left(\frac{t+1}{(t+1)^2 - 1} \right).$$

Let $v = (t+1)^2 - 1$, so that $t+1 = \sqrt{1+v}$, where $0 \le v < \infty$. Now

$$(1+v)^{\frac{1}{2}} < 1 + \frac{v}{2},$$

so we have

$$\log q(n) < \log \frac{v}{2} + (n-1)\frac{v}{2} + \frac{\pi^2}{6} \left(\frac{1+\frac{v}{2}}{v}\right).$$

Let

$$g(v) = \log \frac{v}{2} + (n-1)\frac{v}{2} + \frac{\pi^2}{6} \left(\frac{1+\frac{v}{2}}{v}\right).$$

Then

$$g'(v) = \frac{1}{v} + \frac{n-1}{2} - \frac{\pi^2}{6v^2} = \frac{3(n-1)v^2 + 6v - \pi^2}{6v^2}.$$

Note that g'(v) = 0 when

$$v = v_0 = \frac{-6 + \sqrt{36 + 12(n-1)\pi^2}}{6(n-1)}$$

Furthermore, g'(v) < 0 for $v < v_0$ and g'(v) > 0 for $v > v_0$. Therefore g(v) must have a minimum value when at $v = v_0$. Let

$$v_1 = \frac{\pi}{3\sqrt{n-1}}$$

Then

$$\log q(n) < g(v_1) = \log \frac{\pi}{\sqrt{12(n-1)}} + \frac{n-1}{2} \left(\frac{\pi}{3\sqrt{n-1}} + \frac{\pi^2}{6} \right).$$

The conclusion now follows.

References

- Apostol T., Introduction to analytic number theory, Springer Verlag, 1984.
- [2] Hagis P., On a class of partitions with distinct summands, Trans. Amer. Math. Soc., 112 (1964), 401-416.
- [3] Hardy G.H. and Ramanujan S., Asymptotic formulas in combinatory analysis, Proc. London Math. Soc., 17 (2) (1918), 75-115.
- [4] van Lint J.H., Combinatorial Theory Seminar, Lecture Notes in Mathematics 382, Springer Verlag, 1974.

(Received October 4, 2006)

N. Robbins

Mathematics Department San Francisco State University San Francisco, CA 94132, USA