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MACMAHON'S ANALYSIS AND A NEW PARTITION IDENTITY

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Dedicated to the memory of Professor M.V. Subbarao

Abstract. We prove a new 3-way partition identity analytically as well as combinatorially. In the analytical proof we use MacMahon's partition analysis. The main result is further extended by using n-reflected lattice paths.

1. Introduction, definitions and the main result

Partition identities play an important role in many areas like number theory, combinatorics, Lie theory, particle physics and statistical mechanics. The first partition identity which states *"the number of partitions of a positive integer into odd parts equals the number of its partitions into distinct parts"* is due to Euler [6, Theorem 344]. Other famous partition identities are Rogers-Ramanujan-MacMahon identities [6, Theorems 364-365], Schur's identity [8], Göllnitz-Gordon identities [4,5] and Agarwal-Andrews identities [2]. In Section 2 we shall prove analytically as well as combinatorially a 3-way partition identity and extend our main result in Section 3 by using n-reflected lattice paths defined by Agarwal and Andrews in [1]. First we recall the following definitions which will be used in the sequel.

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Definition 1. A partition of a positive integer n is a representation of n as a sum of positive integers, called parts (or summands) of the partition. The order of parts is irrelevant.

Definition 2. In a graphical representation (called the Ferrers graph) a partition is represented by horizontal rows of dots (aligned to the left). The number of dots in each row indicates the part size. For example the graph $6+4+2$ is

$$\begin{matrix} o & o & o & o & o & o \\ o & o & o & o & & \\ o & o & & & & \end{matrix}$$

The largest square of nodes contained in the graph is called the Durfee Square. The partition obtained by reading a Ferrers graph by columns is called the conjugate (of the given partition); the conjugate of $6+4+2$ is $3+3+2+$ $+2+1+1$. A partition is called self-conjugate if it is identical with its conjugate. Thus, the lone self-conjugate partition of 6 is $3+2+1$.

Definition 3. Ordered partitions are called *compositions*.

For example there are five partitions, viz: 4, 3+1, 2+2, 2+1+1, 1+1+1+1 and eight compositions, viz: 4, 3+1, 1+3, 2+2, 2+1+1, 1+2+1, 1+1+2, 1+1+1+1 of 4.

In our next section we shall prove the following

Theorem 1. *For $2 \le k \le n$ let $A_k(n)$ denote the number of partitions of the form $a_1 + a_2 + \ldots + a_k + \ldots + a_t$ such that $a_1 > a_2 > \ldots > a_k > \ldots > a_t$, $a_1 = n$, $a_1, a_2, \ldots, a_k$ are consecutive integers and $a_k - a_{k+1} \ge 2$ if $k \ne n$. Let B_k denote the number of compositions of n of the form $b_s + b_{s-1} + \ldots + b_1$, where $b_1 = b_2 = \ldots = b_{k-1} = 1$, $b_k \ne 1$ if $k \ne n$. Let $C_k(n)$ denote the number of self-conjugate partitions with largest part n and least part k. Then*

$$A_k(n) = B_k(n) = C_k(n) \qquad for\ all\ \ 2 \le k \le n. \tag{1.1}$$

Example. For $n = 4$ and $k = 2$ we have

$A_2(4) = 2$, the relevant partitions are 4+3+1, 4+3.

$B_2(4) = 2$, the relevant compositions are 1+2+1, 3+1.

$C_2(4) = 2$, the relevant self-conjugate partitions are 4+4+3+2, 4+4+2+2.

2. First proof of Theorem 1 (analytical)

Obviously,

$$A_n(n) = B_n(n) = C_n(n) = 1, \tag{2.1}$$

since the lone partition enumerated by $A_n(n)$ is $n + (n-1) + \ldots + 2 + 1$, the lone composition enumerated by $B_n(n)$ is $1 + 1 + 1 + \ldots + 1$ (n times) and the only self-conjugate partition enumerated by $C_n(n)$ is the partition whose Ferrers graph is Durfee Square of size n.

We shall prove the theorem for $2 \leq k < n$. Let $A_k^m(n)$ denote the number of partitions enumerated by $A_k(n)$ into m parts and let $A_k^m(n,\nu)$ denote the number of partitions of ν enumerated by $A_k^m(n)$. Then by using MacMahon's partition analysis we have

$$\sum_{n,\nu} A_k^m(n,\nu) z^n q^\nu = \tag{2.2}$$

$$= \Omega_{\geq} \sum_{n_1,n_2,\ldots,n_{m-k+1}\geq 0} z^{n_1} q^{kn_1 - \frac{k(k-1)}{2} + n_2 + \ldots + n_{m-k+1}} \lambda_1^{n_1 - n_2 - (k+1)} \cdot$$

$$\cdot \lambda_2^{n_2 - n_3 - 1} \ldots \lambda_{m-k}^{n_{m-k} - n_{n-k+1} - 1} \lambda_{m-k+1}^{n_{m-k+1} - 1},$$

where the variables $\lambda_1, \lambda_2, \ldots, \lambda_{m-k+1}$ handle the inequalities satisfied by n_j while the n_j themselves become free. The linear operator $\Omega_{\geq}$ when applied to the Laurent series in $\lambda_1, \lambda_2, \ldots, \lambda_{m-k+1}$ annihilates terms with any negative exponents and in the remaining terms sets $\lambda_i = 1$.

(2.2) can be written as

$$\sum_{n,\nu} A_k^m(n,\nu) z^n q^\nu =$$

$$= q^{-\binom{k}{2}} \Omega_{\geq} \sum_{n_1,n_2,\ldots,n_{m-k+1}\geq 0} (zq^k\lambda_1)^{n_1} \left(q\frac{\lambda_2}{\lambda_1}\right)^{n_2} \ldots \left(q\frac{\lambda_{m-k}}{\lambda_{m-k-1}}\right)^{n_{m-k}} \cdot$$

$$\cdot \left(q\frac{\lambda_{m-k+1}}{\lambda_{m-k}}\right)^{n_{m-k+1}} \lambda_1^{-(k+1)} \lambda_2^{-1} \ldots \lambda_{m-k}^{-1} \lambda_{m-k+1}^{-1} =$$

$$= q^{-\binom{k}{2}} \Omega_{\geq} \frac{\lambda_1^{-(k+1)} \lambda_2^{-1} \lambda_3^{-1} \ldots \lambda_{m-k}^{-1} \lambda_{m-k+1}^{-1}}{(1 - zq^k\lambda_1)\left(1 - q\frac{\lambda_2}{\lambda_1}\right) \ldots \left(1 - q\frac{\lambda_{m-k}}{\lambda_{m-k-1}}\right)\left(1 - q\frac{\lambda_{m-k+1}}{\lambda_{m-k}}\right)}.$$

Now applying Lemma 11.2.3 [3, p.559], which states (for nonnegative α)

$$\Omega_{\geq} \frac{\lambda^{-\alpha}}{(1 - \lambda x)\left(1 - \frac{y}{\lambda}\right)} = \frac{x^\alpha}{(1-x)(1-xy)},$$

to each $\lambda_1, \lambda_2, \ldots, \lambda_{m-k+1}$, we obtain

$$\sum_{n,\nu} A_k^m(n,\nu) z^n q^\nu = \frac{z^{m+1} q^{\binom{m+1}{2}+k}}{(1-zq^k)(1-zq^{k+1})\ldots(1-zq^m)}. \tag{2.3}$$

Setting $q = 1$ in (2.3) we get

$$\sum_{n} A_k^m(n) z^n = \frac{z^{m+1}}{(1-z)^{m-k+1}}. \tag{2.4}$$

Next, let $B_k^m(n)$ denote the number of compositions enumerated by $B_k(n)$ into m parts. Following Riordan [7, p.125] we see that

$$\sum_{n} B_k^m(n) z^n = z^{k-1}(z + z^2 + \ldots)^{m-k+1} - z^k(z + z^2 + \ldots)^{m-k} =$$

$$= \frac{z^{m+1}}{(1-z)^{m-k+1}}. \tag{2.5}$$

Finally, let $C_k^m(n)$ denote the number of self-conjugate partitions enumerated by $C_k(n)$ with Durfee Square of size m and let $C_k^m(n,N)$ denote the number of self-conjugate partitions of N enumerated by $C_k^m(n)$. Then, following the standard technique of partition theory, we see that

$$\sum_{n,N} C_k^m(n,N) z^n q^N = \frac{z^{m+1} q^{m^2+2k}}{(1-zq^{2k})(1-zq^{2k+2})\ldots(1-zq^{2m})}. \tag{2.6}$$

Setting $q = 1$ in (2.6), we obtain

$$\sum_{n} C_k^m(n) z^n = \frac{z^{m+1}}{(1-z)^{m-k+1}}. \tag{2.7}$$

A comparison of (2.4), (2.5) and (2.7) leads to the identity

$$A_k^m(n) = B_k^m(n) = C_k^m(n). \tag{2.8}$$

Summing over m, (2.8) leads to (1.1). This completes the analytical proof of Theorem 1.

Remark. For $k \neq n$ (2.4), (2.5) and (2.7) give the explicit formula

$$A_k^m(n) = B_k^m(n) = C_k^m(n) = \binom{n-k-1}{n-m-1}, \tag{2.9}$$

and the generating function

(2.10)

$$\sum_n A_k(n)q^n = \sum_n B_k(n)q^n = \sum_n C_k(n)q^n = \sum_{m \geq k} \frac{z^{m+1}}{(1-z)^{m-k+1}} = \frac{z^{k+1}}{1-2z}.$$

(2.10) gives the explicit formula

$$A_k(n) = B_k(n) = C_k(n) = 2^{n-k-1}. \tag{2.11}$$

Second proof of (1.1) (combinatorial). Let $\mathcal{A}_m$ denote the collection of partitions enumerated by $A_k^m(n)$ and $\mathcal{B}_m$ denote the collection of compositions enumerated by $B_k^m(n)$. We first establish a bijection between $\mathcal{A}_m$ and $\mathcal{B}_m$. The mapping $\phi : \mathcal{A}_m \to \mathcal{B}_m$ is

$$\phi(\pi) = a_m + (a_{m-1} - a_m) + \ldots + (a_{k-1} - a_k) + \ldots + (a_1 - a_2), \tag{2.12}$$

where $\pi = a_1 + a_2 + \ldots + a_m \in \mathcal{A}_m$.

The inverse mapping $\phi^{-1} : \mathcal{B}_m \to \mathcal{A}_m$ is easily seen to be

$$\phi^{-1}(\mu) = (b_1+b_2+\ldots+b_m)+(b_1+b_2+\ldots+b_{m-1})+\ldots+(b_1+b_2)+b_1, \tag{2.13}$$

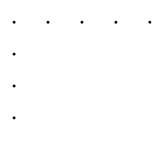
where $\mu = b_1 + b_2 + \ldots + b_m \in \mathcal{B}_m$.

Remark. The mapping ϕ holds good between $\mathcal{A} = \bigcup_m \mathcal{A}_m$ and $\mathcal{B} = \bigcup_m \mathcal{B}_m$, too.

Example. For $n = 4, k = 2$ we see that
$\phi(4+3+1) = 1+2+1, \quad \phi^{-1}(1+2+1) = 4+3+1,$
$\phi(4+3) = 3+1, \quad \phi^{-1}(3+1) = 4+3.$

Next, we shall establish a bijection between $\mathcal{A}$ and $\mathcal{C}$, where $\mathcal{C}$ is the collection of all self-conjugate partitions enumerated by $C_k(n)$. We first define a k-bend.

Definition. We call a right bend



a k-bend if the number of dots in the first row and first column are both equal to k. Thus by 1-bend we mean a single dot . , by 2-bend

$$\begin{array}{ll} \cdot & \cdot \\ \cdot & \end{array}$$

by a 3-bend

$$\begin{array}{lll} \cdot & \cdot & \cdot \\ \cdot & & \\ \cdot & & \end{array}$$

etc. Now we define the mapping $\psi : \mathcal{A} \to \mathcal{C}$ by

$$\psi(\pi) = \nu, \tag{2.14}$$

where $\pi = a_1 + a_2 + \ldots + a_t \in \mathcal{A}$ and $\nu \in \mathcal{C}$ such that the graph of ν consists of a_1-bend, a_2-bend,..., a_t-bend.

Clearly, the inverse mapping $\psi^{-1} : \mathcal{C} \to \mathcal{A}$ is given by

$$\psi^{-1}(\nu) = b_1 + b_2 + \ldots + b_s, \tag{2.15}$$

where $\nu \in \mathcal{C}$ such that the graph of ν consists of b_1-bend, b_2-bend,..., b_s-bend.

To illustrate the bijection ψ we consider the case when $n = 4, k = 2$, we see that

$$4+3+1 \quad \underset{\psi^{-1}}{\overset{\psi}{\rightleftharpoons}} \quad \begin{array}{llll} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & & \\ (4 & +4 & +3 & +2) \end{array}$$

$$4+3 \quad \underset{\psi^{-1}}{\overset{\psi}{\rightleftharpoons}} \quad \begin{array}{llll} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & & \\ \cdot & \cdot & & \\ (4 & +4 & +2 & +2) \end{array}$$

The mapping $\psi.\phi^{-1}$ is clearly a bijection between $\mathcal{B}$ and $\mathcal{C}$.

3. n-reflected lattice paths and an extension of Theorem 1

Agarwal and Andrews [1] called a lattice path from $(0,0)$ to (n,n) n-reflected if for each (x,y) on the path $(n-y, n-x)$ is also on the path. For example there are four 2-reflected lattice path, viz.,

$(0,0),(1,0),(2,0),(2,1),(2,2);$
$(0,0),(1,0),(1,1),(2,1),(2,2);$
$(0,0),(0,1),(1,1),(1,2),(2,2);$
$(0,0),(0,1),(0,2),(1,2),(2,2).$

Here we denote by $D_k(n)$ the number of n-reflected lattice paths such that (i) the points $(n-i,n)$, $1 \leq i \leq n$ do not lie on the path and (ii) if (k,y) and $(k,y-r)$ (for some suitable y and r) lie on the path, but $(k,y-r-1)$ does not lie on the path, then the points $(k-j,y-r)$, $1 \leq j \leq k$ also lie on the path (but $(k+1,y-r)$ does not lie on the path).

Example. The two 4-reflected lattice paths enumerated by $D_2(4)$ are
(0,0), (1,0), (2,0), (2,1), (3,1), (3,2), (4,2), (4,3), (4,4) (here $y=1, r=1$),
(0,0), (1,0), (2,0), (2,1), (2,2), (3,2), (4,2), (4,3), (4,4) (here $y=2, r=2$).

It was observed in [1] that the n-reflected lattice paths are in one-to-one correspondence with the self-conjugate partitions with largest part $\leq n$. In this correspondence it is easy to see that the conditions satisfied by the n-reflected lattice paths enumerated by $D_k(n) \Leftrightarrow$ the conditions satisfied by the self-conjugate partitions enumerated by $C_k(n)$, that is the largest part is n and the smallest part is k. This leads to the following 4-way extension of Theorem 1:

Theorem 2. *For $2 \leq k \leq n$ we have*

$$A_k(n) = B_k(n) = C_k(n) = D_k(n). \tag{3.1}$$

We note that the lone n-reflected lattice path enumerated by $D_n(n)$ is $(0,0),(1,0),(2,0),...,(n,0),(n,1),(n,2),...,(n,n)$, which corresponds to a self-conjugate partition whose graph is a Durfee Square of size n.

4. Conclusion

We see that for $n > k$, if any subset of $\{1,2,\ldots,n-k-1\}$ is adjoined to the set $\{n-k+1,\ldots,n-1,n\}$, we get a partition enumerated by $A_k(n)$. This shows that $A_k(n)$ equals the number of subsets of $\{1,2,\ldots,n-k-1\}$ which is 2^{n-k-1}. We conclude by noting that our discussion of the preceding sections was a natural way to look at the following

Theorem 3. *For $n \geq 2$*

$$A_n(n) = B_n(n) = C_n(n) = D_n(n) = 1,$$

and for $2 \le k < n$

$$A_k(n) = B_k(n) = C_k(n) = D_k(n) = E_k(n) = 2^{n-k-1},$$

where $E_k(n)$ *denotes the number of subsets of* $\{1, 2, \ldots, n-k-1\}$.

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