Annales Univ. Sci. Budapest., Sect. Comp. 26 (2006) 105-125

ON THE CHOICE-REVEALED EXTENSION OPERATORS

B. Bodó (Budapest, Hungary)

Abstract. In this paper we will discuss two set-extension operators. Both of them can be connected to the decision making, namely, they will be revealed by a choice function. However, we do not assume that this choice function is given on all subsets of the possible alternatives. The main aim to discover that hidden information which replaces the missed ones, supporting the decision maker.

1. Introduction

In a decision process the final aim is to select a subset from the set of possible alternatives Ω, consisting of the alternatives which are the most preferred by the decision maker. This selection can be done by different mechanisms. These mechanisms are based on the requirements of the decision maker and a technique for the evaluation, which of the possible alternatives fulfils better the prescribed requirements. This evaluation can be realized by numerical evaluation, pairwise comparison, etc. Here we will discuss some properties of the set-valued evaluation, described by choice functions.

One of the main problem of the general choice theory is under what kind of conditions can the choice function be represented by a binary relation. Without claiming the completeness we refer to [1], [2], [5], [7], [8], [9] which deal with this problem. In the literature a lot of papers (e.g. [1], [2], [6], etc.) deal with other rational properties of the choice function, too.

One of the reasons for the large interest in general choice theory is that it can model the individual choice. For example, it is suited to investigate the outcomes of such economic activities as tender-evaluation and market-analysis, and such social and political processes that consider voting procedures as gallup poll.

However, the individual choice is usually limited to the practically relevant subsets of the alternatives. Our interest turns to the problems, how this limitations effect to the irrelevant subsets, and how we can refine the relevancy of the subsets of alternatives.

2. Basic definitions and theorems

Let Ω denote the *set of alternatives* and let $\mathcal{B} \subseteq 2^{\Omega} \setminus \emptyset$. We will refer to this set system as *option set.*

We have to mention, that in some papers the term "option set" is used for the set of all possible alternatives. However, we will distinguish the set of all alternatives and its subsets.

Definition 2.1. $C : \mathcal{B} \to 2^{\Omega}$ is a *choice function* given on the subset $\mathcal{B} \subseteq 2^{\Omega}$ if $C(X) \subseteq X$ for all $X \in \mathcal{B}$. If $C : \mathcal{B} \to \mathcal{B}$, then we say that C is *injective.*

Definition 2.2. The triplet $\mathcal{D} = (\Omega, \mathcal{B}, C)$ will be called *real decision mechanism* if the following properties are satisfied:

1. $\emptyset \notin \mathcal{B}$;
2. The option set $\mathcal{B}$ covers Ω, i.e. $\Omega = \bigcup\limits_{X \in \mathcal{B}} X$;
3. $C(X) \neq \emptyset$ for all $X \in \mathcal{B}$;
4. The option set $\mathcal{B}$ may contain a set from 2^{Ω} at most once.

If we extend the option set to the whole $2^{\Omega} \setminus \emptyset$ then from the conditions of the real decision mechanism the second one is obviously unnecessary, so we have the following definition.

Definition 2.3. The triplet $\mathcal{D}^* = (\Omega, 2^{\Omega} \setminus \emptyset, C)$ will be called *perfect decision mechanism* if $C(X) \neq \emptyset$ for all $X \in 2^{\Omega} \setminus \emptyset$.

The greatest part of the papers dealing with decision mechanism uses the perfect decision mechanism.

The exclusion of $\emptyset$ from the option set has only practical meaning in the decision making. The condition $C(\emptyset) = \emptyset$, used in a lot of papers, is meaningless for the decision. From practical point of view it would be meaningful, if we excluded from $\mathcal{B}$ the whole set of alternatives. Indeed, the task of the decision making is to choose the best alternative(s) from the set of all possible alternatives using the given decision. If the mechanism directly defines this choice for the whole set of alternatives, then the task of decision making looses its meaning again. However, for the following analysis, as a technical tool, it will be allowed, but will not be demanded that $\Omega \in \mathcal{B}$.

In practice, to collect the elements of the option set is based on the manner of satisfaction of the requirements by the different alternatives, i.e. those alternatives belong to the same set of the option set, which apply the same technic (but maybe with different quality) to satisfy a group of requirements of decision procedure. Grouping the requirements satisfied by the same alternatives guarantees the possibility to fulfil the fourth property of the real decision mechanism.

Definition 2.4. We say that the real decision mechanism is normal, if there exists a binary relation P on $\Omega \times \Omega$ such that either $C(X) = C_P^{ND}(X)$ or $C(X) = C_P^{MAX}(X)$ holds for every $X \in \mathcal{B}$, where

$$C_P^{ND}(X) = \{x \in X : y\overline{P}'x \;\; \forall y \in X\}$$

is the set of *non-dominated alternatives* of X, and

$$C_P^{MAX}(X) = \{x \in X : xP'y \;\; \forall y \in X\}$$

is the set of *maximal alternatives* of X.

In this case we will refer to C and P as *normal choice function* and *choice-representing relation*, respectively.

In the Definition 2.4 P' means the restriction of P to X. In the sequel, when the restriction is obvious, we will omit the mark $'$ from the nomination of the relation. Here and in the following the overlined relation denotes the complement of the relation.

It is easy to show examples for choice functions which are not normal on a given $\mathcal{B}$, what is more on 2^{Ω}.

Every choice function reveals two preferences relations (see [7] and [8]).

Definition 2.5. R is the *C-revealed Richter-relation* on $\mathcal{B}$ if

$$xRy \Leftrightarrow \exists X \in \mathcal{B} : x \in C(X), \;\; y \in X.$$

Definition 2.6. S is the *C-revealed Samuelson-relation* if

$$xSy \Leftrightarrow \exists X \in \mathcal{B} : x \in C(X), \;\; y \in X \setminus C(X).$$

The maximal or non-dominated alternatives with these preferences play very important role to recognize whether a decision mechanism is normal or not. Namely, in our investigations we will use the following propositions proved in [5].

Proposition 2.1. *Let $\mathcal{D} = (\Omega, \mathcal{B}, C)$ be a real decision mechanism. For all $X \in \mathcal{B}$*

$$C_S^{ND}(X) \subseteq C(X) \subseteq C_R^{MAX}(X). \tag{1}$$

(Here the inclusions are strict for some $X \in \mathcal{B}$ if C is not normal.)

Proposition 2.2. *Let $\mathcal{D} = (\Omega, \mathcal{B}, C)$ be a real decision mechanism, R and S be the C-revealed Richter and Samuelson relations, respectively, and let S^d denote the dual of the relation S, i.e. $S^d = \overline{S^{-1}}$. Then*

$$R \subseteq S^d$$

is equivalent with the equalities

$$C_S^{ND}(X) = C(X) = C_R^{MAX}(X) \qquad \forall X \in \mathcal{B}.$$

Let us observe, that C_R^{MAX} and C_S^{ND} can be applied for $X \notin \mathcal{B}$, too, but in this case it can not be guaranteed that $C_R^{MAX}(X) \neq \emptyset$ and $C_S^{ND}(X) \neq \emptyset$. Furthermore, R is, in general, reflexive relation, however, if there exists $x \in \Omega$ such that $x \notin C(X)$ for all $X \in \mathcal{B}$, then the reflexivity property will be injured.

The following proposition is almost trivial, but it will be very useful in the further discussion.

Proposition 2.3. *Suppose that $\mathcal{D} = (\Omega, \mathcal{B}, C)$ is a normal real decision mechanism with the choice-representing relation P. Then the following equalities are valid*

$$\begin{aligned} C_P^{MAX}(C(X)) &= C(X) \qquad && \forall X \in \mathcal{B}, \\ C_P^{MAX}(C_P^{MAX}(X)) &= C_P^{MAX}(X) \qquad && \forall X \in (2^\Omega \setminus \emptyset) \setminus \mathcal{B}. \end{aligned} \tag{2}$$

Proof. Under the condition of the proposition it is enough to prove that (2) fulfils for all $X \in 2^\Omega \setminus \emptyset$. From the definition of the choice function follows that

$$C_P^{MAX}(C_P^{MAX}(X)) \subseteq C_P^{MAX}(X) \qquad \forall X \in 2^\Omega.$$

Otherwise, if $x \in C_P^{MAX}(X)$ then xPy for all $y \in X$, consequently, xPy for all $y \in C_P^{MAX}(X)$, i.e. $x \in C_P^{MAX} C_P^{MAX}(X)$, so

$$C_P^{MAX}(X) \subseteq C_P^{MAX}(C_P^{MAX}(X)) \qquad \forall X \in 2^\Omega. \qquad \diamond$$

Corollary 2.1. *Let $\mathcal{D} = (\Omega, \mathcal{B}, C)$ be a real decision mechanism, R and S be the C-revealed Richter and Samuelson relations, respectively, and assume that $R \subseteq S^d$ holds. Then*

$$C_S^{ND}(C(X)) = C_R^{MAX}(C(X)) = C(X) \qquad \forall X \in \mathcal{B},$$
$$C_R^{MAX}(C_R^{MAX}(X)) = C_R^{MAX}(X) \qquad \forall X \in 2^\Omega \setminus \mathcal{B},$$
$$C_S^{ND}(C_S^{ND}(X)) = C_S^{ND}(X) \qquad \forall X \in 2^\Omega \setminus \mathcal{B}.$$

Proof. Taking into consideration the Proposition 2.2 we have that $C(X) = C_R^{MAX}(X) = C_S^{ND}(X) = C_{S^d}^{MAX}(X)$, i.e. the decision mechanism is normal with the relations R and S^d. The statement of the Proposition 2.3 gives the demanded equalities with the choice $P = R$ or $P = S^d$. $\diamondsuit$

3. The extension operator $\mathcal{K}_C$ revealed by the choice function C

In his paper [4] G.A.Koshevoy has introduced the operator

$$\tilde{C}(X) = \bigcup \{Y \in 2^\Omega \setminus \emptyset : C(Y) = C(X)\}, \qquad (3)$$

revealed by a choice function $C : 2^\Omega \to 2^\Omega$. Under the assumption that C is an *ordinally rationalizable* choice function, i.e. if there exists an order (reflexive, antisymmetric and transitive relation) P on Ω such that $C(X) = C_P^{MAX}(X)$ for all $X \in 2^\Omega$, it has been proved that $\tilde{C}$ is a closure operator on 2^Ω, i.e. it satisfies properties required in the following definition with $\mathcal{A} = 2^\Omega$.

Definition 3.1. $\mathcal{K}$ is a *closure operator* on the set system $\mathcal{A}$ if the following requirements are fulfilled for all set belonging to $\mathcal{A}$:

1. $X \subseteq \mathcal{K}(X)$;
2. $\mathcal{K}(\mathcal{K}(X)) = \mathcal{K}(X)$;
3. $X \subseteq Y$ implies $\mathcal{K}(X) \subseteq \mathcal{K}(Y)$.

The antisymmetry of the relation P is very strict in a real decision making, since it means that $C(X)$ has only a single element for all $X \in 2^\Omega$.

In the following we will restrict the Koshevoy-operator to a subset $\mathcal{B} \subseteq 2^\Omega$ and instead of assuming about C to be ordinally rationalizable, we will give assumptions for the C-revealed Richter- and Samuelson-relations.

Definition 3.2. Let $\mathcal{D} = (\Omega, \mathcal{B}, C)$ be a real decision mechanism. The operator $\mathcal{K}_C : \mathcal{B} \to 2^\Omega$ defined by

$$\mathcal{K}_C(X) = \bigcup \{Y \in \mathcal{B} : C(Y) = C(X)\} \tag{4}$$

will be called a *C-revealed extension of the sets belonging to the option set* $\mathcal{B}$.

The operator $\mathcal{K}_C$ is *injective* if $\mathcal{K}_C : \mathcal{B} \to \mathcal{B}$.

The set system

$$\mathcal{B}_{\mathcal{K}_C} = \mathcal{B} \cup \{\mathcal{K}_C(X) : X \in \mathcal{B}\} \tag{5}$$

will be called $\mathcal{K}_C$*-extension of the option set* $\mathcal{B}$.

Proposition 3.1. *Let* $\mathcal{D} = (\Omega, \mathcal{B}, C)$ *be a real decision mechanism and let* $\mathcal{K}_C$ *be defined by (4). Then*

$$X \subseteq \mathcal{K}_C(X) \qquad \forall X \in \mathcal{B}.$$

Proof. It follows from that fact that X appears among the sets which compose the set $\mathcal{K}_C(X)$. $\diamondsuit$

Proposition 3.2. *Let* $\mathcal{D} = (\Omega, \mathcal{B}, C)$ *be a real decision mechanism, and* $\mathcal{K}_C$ *be the C-revealed extension operator. If* $R \subseteq S^d$ *is fulfilled for the C-revealed Richter- and Samuelson-relations, then*

$$\begin{aligned} C(X) &= C_R^{MAX}(\mathcal{K}_C(X)) = C_R^{MAX}(X) = \\ &= C_S^{ND}(\mathcal{K}_C(X)) = C_S^{ND}(X) \qquad \forall X \in \mathcal{B}. \end{aligned} \tag{6}$$

If $\mathcal{K}_C$ *is injective, then*

$$C(\mathcal{K}_C(X)) = C(X) \ \ \forall X \in \mathcal{B}.$$

Proof. First of all let us remark that on the basis of Proposition 2.2 we have that $C(X) = C_R^{MAX}(X) = C_S^{ND}(X)$ for all $X \in \mathcal{B}$.

Let $X \in \mathcal{B}$ and $x \in C(X)$ and let denote $\mathcal{Y}_X$ the set system which occurs in the definition of $\mathcal{K}_C(X)$, i.e.

$$\mathcal{Y}_X = \{Y \in \mathcal{B} : C(Y) = C(X)\}. \tag{7}$$

With this notation $x \in C(Y)$ for all $Y \in \mathcal{Y}_X$. Using the fact, that $C(X)$ and $C_R^{MAX}(X)$ coincide on the sets belonging to $\mathcal{B}$, we have xRy for all $y \in Y$.

Consequently, xRy for all $y \in \cup\{Y : Y \in \mathcal{Y}_X\} = \mathcal{K}_C(X)$. This means that $x \in C_R^{MAX}(\mathcal{K}_C(X))$, i.e. $C(X) \subseteq C_R^{MAX}(\mathcal{K}_C(X))$.

To prove the inverse inclusion, let $x \in C_R^{MAX}(\mathcal{K}_C(X))$ be chosen arbitrarily. Since $x \in \mathcal{K}_C(X)$, according to the construction of $\mathcal{K}_C(X)$, there exists $Y \in \in \mathcal{Y}_X$ such that $x \in Y$ (here $\mathcal{Y}_X$ is defined by (7)). Because of $x \in \in C_R^{MAX}(\mathcal{K}_C(X))$ we know that xRy for all $y \in \mathcal{K}_C(X)$. Consequently, if $Y \in \mathcal{Y}_X$ then xRy for all $y \in Y$. i.e. $x \in C_R^{MAX}(Y) = C(Y) = C(X) = = C_R^{MAX}(X)$. This proves that $C(X) \supseteq C_R^{MAX}(\mathcal{K}_C(X))$. Analogously can be obtain the second line of (6).

If $\mathcal{K}_C$ is injective, then $\mathcal{K}_C(X) \in \mathcal{B}$, so $C(\mathcal{K}_C(X)) = C_R^{MAX}(\mathcal{K}_C(X))$. $\diamond$

If the decision mechanism is not normal, then additional assumptions are needed to obtain similar proposition.

Proposition 3.3. *Let $\mathcal{D} = (\Omega, \mathcal{B}, C)$ be a real, but not necessary normal decision mechanism and let R be the C-revealed Richter-relation. Suppose that one of the following two conditions are satisfied for $X \in \mathcal{B}$:*

1. $\mathcal{K}_C(X) = X$;
2. $\mathcal{K}_C(X) \supset X$ *and*
   - 2a. *for all $y \in \mathcal{K}_C(X) \setminus X$ there exists $x \in C_R^{MAX}(X)$ such that $y\overline{R}x$;*
   - 2b. *moreover, if $C_R^{MAX}(X) \setminus C(X) \neq \emptyset$, then $x \in C_R^{MAX}(X) \setminus C(X)$ and $y \in \mathcal{K}_C(X) \setminus X$ implies xRy.*

Then

$$C_R^{MAX}(\mathcal{K}_C(X)) = C_R^{MAX}(X).$$

Proof. If $\mathcal{K}_C(X) = X$ then the statement is trivially valid.

Let now suppose, that $\mathcal{K}_C(x) \setminus X \neq \emptyset$. Let $x^* \in C_R^{MAX}(\mathcal{K}_C(X))$. Then $x^* \in \mathcal{K}_C(x)$ and x^*Rz for all $z \in \mathcal{K}_C(X)$. Since $X \subseteq \mathcal{K}_C(X)$ we have

$$x^*Rz \ \ \forall z \in X. \tag{8}$$

Let us assume that $x^* \in \mathcal{K}_C(X) \setminus X$. According to the assumption 2*a* there exists $z \in C_R^{MAX}(X) \subseteq X \subseteq \mathcal{K}_C(X)$ such that $x^*\overline{R}z$, which contradicts to the assumption $x^* \in C_R^{MAX}(\mathcal{K}_C(X))$. Thus $x^* \in X$, therefore from (8) follows that $x \in C_R^{MAX}(X)$, i.e. $C_R^{MAX}(\mathcal{K}_C(X)) \subseteq C_R^{MAX}(X)$.

To prove the contrary inclusion let us suppose indirectly that there exist $x^* \in C_R^{MAX}(X)$ such that $x^* \notin C_R^{MAX}(\mathcal{K}_C(X))$. The latter condition means that

$$\exists y^* \in \mathcal{K}_C(X): \ \ x^*\overline{R}y^*. \tag{9}$$

$y^* \notin X$, because in the contrary case (9) contradicts to the assumption $x^* \in C_R^{MAX}(\mathcal{K}_C(X))$. So, $y^* \in \mathcal{K}_C(X) \setminus X$.

If $x^* \in C(X)$, then from (9) follows that there exists $Y \in \mathcal{B}$ such that $x^* \in C(X) = C(Y) \subseteq C_R^{MAX}(Y)$ and $y^* \in Y$, but it contradicts to (9).

If $x^* C_R^{MAX}(X) \setminus C(X)$, then using the assumption $2b$ we obtain that $x^* R y^*$, which also contradicts to (9).

The last two contradictions indicate that $x^* \in C_R^{MAX}(\mathcal{K}_C(X))$. Consequently, $C_R^{MAX}(\mathcal{K}_C(X)) = C_R^{MAX}(X)$. ◊

Proposition 3.4. *Let $\mathcal{D} = (\Omega, \mathcal{B}, C)$ be a real decision mechanism and assume that $R \subseteq S^d$ holds between the C-revealed Richter- and Samuelson-relations. Then*

$$\begin{aligned}\mathcal{K}_C(X) &= \mathcal{K}_{C_R^{MAX}}(X) = \mathcal{K}_{C_R^{MAX}}(\mathcal{K}_{C_R^{MAX}}(X)) = \mathcal{K}_{C_R^{MAX}}(\mathcal{K}_C(X)) = \\ &= \mathcal{K}_{C_S^{ND}}(X) = \mathcal{K}_{C_S^{ND}}(\mathcal{K}_{C_S^{ND}}(X)) = \mathcal{K}_{C_S^{ND}}(\mathcal{K}_C(X)) \quad \forall X \in \mathcal{B}.\end{aligned} \tag{10}$$

If the C-revealed extension operator $\mathcal{K}_C$ is injective, then

$$\mathcal{K}_C(\mathcal{K}_C(X)) = \mathcal{K}_C(X).$$

Proof. On the basis of the Proposition 2.2 we have that

$$C(X) = C_R^{MAX}(X) = C_S^{ND}(X) \ \ \forall X \in \mathcal{B}.$$

Taking into consideration the Proposition 3.2 we obtain that

$$\begin{aligned}&\mathcal{K}_{C_R^{MAX}}(\mathcal{K}_C(X)) = \\ &\quad = \bigcup\{Y \in \mathcal{B} : C(Y) = C_R^{MAX}(Y) = C_R^{MAX}(\mathcal{K}_C(X)) = C(X)\} = \\ &\quad = \mathcal{K}_C(X).\end{aligned}$$

Analogously can be obtained the second line of (10).

The second statement follows from the second statement of Proposition 3.2. ◊

Corollary 3.1. *The set system $\mathcal{B}_{\mathcal{K}_C}$ defined by (5) is the smallest option set on which the C-revealed extension operator $\mathcal{K}_C$ is injective.*

The analog of the following proposition is used in paper [4] with the operator $\tilde{C}$ defined by (3) on the perfect decision mechanism.

Proposition 3.5. *Let $\mathcal{D} = (\Omega, \mathcal{B}, C)$ be a real decision mechanism and assume that $R \subseteq S^d$ holds between the C-revealed Richter- and Samuelson-relations. Then*

$$\mathcal{K}_{C_R^{MAX}}(C(X)) = \mathcal{K}_C(X) \qquad \forall X \in \mathcal{B}.$$

If C is injective, then $\mathcal{K}_C(C(X)) = \mathcal{K}_C(X)$.

Proof. Using the Propositions 2.2 and 2.3 for all $X \in \mathcal{B}$ we obtain that

$$\mathcal{K}_{C_R^{MAX}}(C(X)) = \bigcup \{Y \in \mathcal{B} : C_R^{MAX}(Y) = C(Y) = C_R^{MAX}(C(X)) = C(X)\} =$$
$$= \mathcal{K}_{C_R^{MAX}}(X) = \mathcal{K}_C(X).$$

If C is injective, then according to the Corollary 2.1 $C_R^{MAX}(C(X))$ can be replaced by $C(X)$, from which follows the second statement. ◇

In the following proposition we will execute the C-revealed extension operator in that situation, when for the given choice function the right inclusion is strict in (1) for some $\emptyset \neq X \in \mathcal{B}$, i.e. we do not assume about C the normality.

First of all let us execute the following example.

Example 3.1. Let $\Omega = \{a, b, c, d\}$ be the set of the possible alternatives. Here (and in the following examples, too) the first and second columns of the table describes the given choice function C. The further columns show the derived sets, named in the headline. Namely, here the third column shows the computed C-revealed choice function C_R^{MAX}, the fourth and sixth columns give the description of the C- and C_R^{MAX}-revealed extensions, respectively, and between them the fifth column shows the relation between the C- and C_R^{MAX}-revealed extension of the given sets.

X	$C(X)$	$C_R^{MAX}(X)$	$\mathcal{K}_C(X)$		$\mathcal{K}_{C_R^{MAX}}(X)$
c	c	c	c	$=$	c
a b	a	a b	a b c	$\subset$	a b c d
a c	a	a	a b c	$\supset$	a c
a d	d	a d	a c d	$=$	a c d
b c	b	b	a b c d	$\supset$	b c d
b d	b	b	a b c d	$\supset$	b c d
c d	d	d	a c d	$\supset$	c d
a b c	b	a b	a b c d	$=$	a b c d
a b d	b	a b	a b c d	$=$	a b c d
a c d	a d	a d	a c d	$=$	a c d
b c d	b	b	a b c d	$\supset$	b c d
a b c d	a b	a b	a b c d	$=$	a b c d

From the table the following C-revealed relation R can be read

R	a	b	c	d
a	1	1	1	1
b	1	1	1	1
c	0	0	1	0
d	1	0	1	1

It is seen that both directions of the inclusion and the equality also appear between the C- and C_R^{MAX}-revealed extensions. The following proposition explains this behavior.

Proposition 3.6. *Let $\mathcal{D} = (\Omega, \mathcal{B}, C)$ be a real, but not necessary normal decision mechanism and let R be the C-revealed Richter-relation. Then one of the following statements will be satisfied for any $X \in \mathcal{B}$:*

1. *If $C_R^{MAX}(X) = C(X)$, then*

$$\mathcal{K}_{C_R^{MAX}}(X) \subseteq \mathcal{K}_C(X).$$

Moreover, if $\mathcal{K}_C(X) = X$ also holds, then

$$\mathcal{K}_{C_R^{MAX}}(X) = \mathcal{K}_C(X).$$

2. *If $C_R^{MAX}(X) \supset C(X)$ and $\mathcal{K}_C(X) = X$, then*

$$\mathcal{K}_{C_R^{MAX}}(X) \subseteq \mathcal{K}_C(X).$$

3. *If $C_R^{MAX}(X) \supset C(X)$ and $\mathcal{K}_C(X) \supset X$, furthermore, the following three conditions hold:*

   3a. $\mathcal{K}_C$ is injective on $\mathcal{B}$;

   3b. for all $y \in \mathcal{K}_C(X) \setminus X$ there exists $x \in C_R^{MAX}(X)$ such that $y\overline{R}x$;

   3c. $x \in C_R^{MAX}(X) \setminus C(X)$ and $y \in \mathcal{K}_C(X) \setminus X$ implies xRy.

   Then

$$\mathcal{K}_{C_R^{MAX}}(X) \supseteq \mathcal{K}_C(X).$$

Proof.

1. Let $C_R^{MAX}(X) = C(X)$. Then

$$\begin{aligned}\mathcal{K}_C(X) &= \bigcup\{Y \in \mathcal{B} : C(Y) = C(X)\} = \\ &= \left(\bigcup\{Y \in \mathcal{B} : C(Y) = C_R^{MAX}(Y) = C_R^{MAX}(X)\}\right) \bigcup \\ &\qquad \bigcup\left(\bigcup\{Y \in \mathcal{B} : C(Y) = C_R^{MAX}(X) \subset C_R^{MAX}(Y)\}\right) \supseteq \\ &\supseteq \left(\bigcup\{Y \in \mathcal{B} : C(Y) = C_R^{MAX}(Y) = C_R^{MAX}(X)\}\right) = \\ &= \mathcal{K}_{C_R^{MAX}}(X).\end{aligned}$$

Otherwise, if $\mathcal{K}_C(X) = X$, then from $X \subseteq \mathcal{K}_{C_R^{MAX}}(X)$ follows

$$\mathcal{K}_C(X) \subseteq \mathcal{K}_{C_R^{MAX}}(X),$$

consequently, in this case

$$\mathcal{K}_C(X) = \mathcal{K}_{C_R^{MAX}}(X).$$

2. It follows from the second part of the proof of the first statement, since there we did not used the equality $C(X) = C_R^{MAX}(X)$.

3. By the definition of $\mathcal{K}_{C_R^{MAX}}$ we have

$$\mathcal{K}_{C_R^{MAX}}(X) = \bigcup\left\{Y \in \mathcal{B} : C_R^{MAX}(Y) = C_R^{MAX}(X)\right\}.$$

According to the Proposition 3.2 the given assumptions guarantee, that $C_R^{MAX}(\mathcal{K}_C(X)) = C_R^{MAX}(X)$. Since $\mathcal{K}_C$ is injective, therefore $\mathcal{K}_C(X) \in \mathcal{B}$, so it appears between the sets deriving $\mathcal{K}_{C_R^{MAX}}(X)$, therefore $\mathcal{K}_{C_R^{MAX}}(X) \supseteq$ $\supseteq \mathcal{K}_C(X)$. ◊

Let us now analyse the Example 3.1 in the light of the Proposition 3.5.

First of all it is seen, that $\mathcal{K}_C$ is injective on $\mathcal{B}$.

If we choose

1. $X = \{a, c, d\}$, then
   $C(X) = C_R^{MAX}(X) = \{a, d\}$ and $\mathcal{K}_C(X) = X = \{a, c, d\}$, therefore $\mathcal{K}_C(X) = \mathcal{K}_{C_R^{MAX}}(X)$;
2. $X = \{a, c\}$, then $C(X) = C_R^{MAX}(X) = \{a\}$ and
   $\mathcal{K}_C(X) = \{a, b, c\} \supset X = \{a, c\}$, therefore $\mathcal{K}_C(X) \supseteq \mathcal{K}_{C_R^{MAX}}(X)$;

3. $X = \{b, c\}$, then $C(X) = \{b\} = C_R^{MAX}(X) = \{b, c\}$ and $\mathcal{K}_C(X) = X = \{b, c\}$, therefore $\mathcal{K}_C(X) \supseteq \mathcal{K}_{C_R^{MAX}}(X)$;
4. $X = \{a, b\}$, then $C(X) = \{a\} = C_R^{MAX}(X) = \{a, b\}$ and $\mathcal{K}_C(X) = \{a, b, c\} \supset X = \{a, b\}$.
   Here $b \in C_R^{MAX}(X) \setminus X$ and $c \in \mathcal{K}_C(X) \setminus X$ and aRc, i.e. the condition 3c fulfils. Moreover, $b \in C_R^{MAX}(X)$ but $c\overline{R}b$, so the condition 3b also fulfils. Therefore $\mathcal{K}_C(X) \subseteq \mathcal{K}_{C_R^{MAX}}(X)$.

4. The weakening operator $\mathcal{H}_P$ defined by C-revealed relation

J. Kortelainen in his paper [3] has discussed the so called *weakening operator* $\mathcal{H}_P : 2^\Omega \to 2^\Omega$ defined by a reflexive binary relation P on $\Omega \times \Omega$ with the following formula

$$\mathcal{H}_P(X) = \{x \in \Omega : \exists y \in X \text{ such that } yPx\}.$$

This operator is defined for all subsets from 2^Ω.

We have to mention that the weakening operator $\mathcal{H}_P$ is, in reality, a set-extension operator.

Under the assumption of the reflexivity and transitivity of the relation P it is also proved in [3] that $\mathcal{H}_P$ is a closure operator, i.e. the conditions of the Definition 3.1 are fulfilled by $\mathcal{H}_P$, namely

1. $X \subseteq \mathcal{H}_P(X) \;\; \forall X \in 2^\Omega \setminus \emptyset$;
2. $\mathcal{H}_P(\mathcal{H}_P(X)) = \mathcal{H}_P(X) \;\; \forall X \in 2^\Omega \setminus \emptyset$;
3. $X \subseteq Y$ implies $\mathcal{H}_P(X) \subseteq \mathcal{H}_P(Y)$ if $X, Y \in 2^\Omega \setminus \emptyset$.

However, we have to mention, that the conditions 1 and 3 are valid without transitivity of P, as it is seen from the given proof in [3].

In the further part of this section we will discuss the $\mathcal{H}_P$ operator with particularly chosen relation P. Namely, let be given a real decision mechanism $\mathcal{D} = (\Omega, \mathcal{B}, C)$ and let R and S be the C-revealed Richter- and Samuelson-relations, respectively. We will examine the behavior of the weakening operators defined by $P = R$ and $P = S^d$. In the following we will refer to $\mathcal{H}_R(X)$ and $\mathcal{H}_{S^d}(X)$ as R- or *S^d-revealed extensions of X*.

Proposition 4.1. *Let $\mathcal{D} = (\Omega, \mathcal{B}, C)$ be a real decision mechanism and assume that $R \subseteq S^d$ holds between the C-revealed Richter- and Samuelson-relations, and R is reflexive on $\Omega \times \Omega$. Then*

$$X \cap C_R^{MAX}(\mathcal{H}_R(X)) \subseteq C(X)$$

holds for all $X \in \mathcal{B}$.

If the relation R *is transitive, too, then the inclusion*

$$C(X) \subseteq C_R^{MAX}(\mathcal{H}_R(X))$$

also holds for all $X \in \mathcal{B}$.

Proof. Let $x^* \in X \cap C(\mathcal{H}_R(X))$. It means that $x^* \in X$ and $x^* R y$ for all $y \in \mathcal{H}_R(X)$. So $x^* \in X$ and $x^* R y$ for all $y \in X \subseteq \mathcal{H}_R(X)$. Therefore $x^* \in C(X)$, consequently, $X \cap C_R^{MAX}(\mathcal{H}_R(X)) \subseteq C(X)$.

Let $x^* \in C(X)$ and let y^* be an arbitrary point of $\mathcal{H}_R(X)$. According to the definition of $\mathcal{H}_R(X)$ there exists $\overline{x} \in X$ such that $\overline{x} R y^*$. Otherwise, $x^* \in C(X)$ implies $x^* R \overline{x}$. From the two last relations using the transitivity of R follows the relation $x^* R y^*$. Since $y^* \in \mathcal{H}_R(X)$ has been chosen arbitrarily, therefore $x^* \in C_R^{MAX}(\mathcal{H}_R(X))$, consequently, $C(X) \subseteq C_R^{MAX}(\mathcal{H}_R(X))$. ◇

Proposition 4.2. *Let* $\mathcal{D} = (\Omega, \mathcal{B}, C)$ *be a real decision mechanism and assume that* R *and* S^d *are reflexive on* $\Omega \times \Omega$ *and between them* $R \subseteq S^d$ *holds. Then*

$$\mathcal{H}_R(X) \subseteq \mathcal{H}_{S^d}(X).$$

Proof. Let $x^* \in \mathcal{H}_R(X)$. According to the definition $\mathcal{H}_R(X)$ there exists $y^* \in X$ such that $y^* R x^*$. From $R \subseteq S^d$ follows $y^* S^d x^*$. Since $y^* \in X$, therefore $x^* \in \mathcal{H}_{S^d}(X)$. ◇

5. Connections between C- and R-revealed set-extensions

In this section we will compare the C- and R-revealed set-extension with respect to the inclusion.

Proposition 5.1. *Let* $\mathcal{D} = (\Omega, \mathcal{B}, C)$ *be a real decision mechanism and assume that* $R \subseteq S^d$ *holds between the* C*-revealed Richter- and Samuelson-relations and* R *is reflexive on* $\Omega \times \Omega$. *Then for all* $X \in \mathcal{B}$

$$\mathcal{K}_C(X) \subseteq \mathcal{H}_R(C(X)) \subseteq \mathcal{H}_R(X).$$

Proof. Let $x^* \in \mathcal{K}_C(X)$. According to the definition of $\mathcal{K}_C(X)$ there exists $Y \in \mathcal{B}$ for which $C(Y) = C(X)$ and $x^* \in Y$. Since $C(X) \neq \emptyset$, therefore there exists $y^* \in C(Y) = C(X) \subseteq X$ for which y^*Rx^*. By the definition of $\mathcal{H}_R(C(X))$ we have that $x^* \in \mathcal{H}_R(C(X))$.

The second inclusion follows from the fact, that $C(X) \subseteq X$ implies $\mathcal{H}_R(C(X)) \subseteq \mathcal{H}_R(X)$. ◊

In the next proposition we will give a necessary and sufficient condition which guarantees the strict implication between $\mathcal{K}_C(X)$ and $\mathcal{H}_R(X)$ in the previous proposition for some $X \in \mathcal{B}$.

Proposition 5.2. *Let $\mathcal{D} = (\Omega, \mathcal{B}, C)$ be a real decision mechanism and assume that $R \subseteq S^d$ holds between the C-revealed Richter- and Samuelson-relations, and R is reflexive on $\Omega \times \Omega$. Then*

$$\mathcal{K}_C(X) \subset \mathcal{H}_R(X)$$

for some $X \in \mathcal{B}$ if and only if there exists $Y \in \mathcal{B}$ such that

$$Y \setminus \mathcal{K}_C(X) \neq \emptyset \text{ and } X \cap C(Y) \neq \emptyset. \tag{11}$$

Proof. *Sufficiency:* Let us assume that there exists $Y \in \mathcal{B}$ which satisfies the condition (11), i.e. there exist $x^* \in Y \setminus \mathcal{K}_C(X)$ and $y^* \in X \cap C(Y)$. It is obvious that $x^* \neq y^*$ because $y^* \in X \subseteq \mathcal{K}_C(X)$, but

$$x^* \notin \mathcal{K}_C(X). \tag{12}$$

Since $y^* \in X \cap C(Y)$, hence $y^* \in X$ and y^*Rx for all $x \in Y$. Otherwise, $x^* \in Y \setminus \mathcal{K}_C(X)$, from which $x^* \in Y$. Therefore y^*Rx^*. It means that

$$x^* \in \mathcal{H}_R(X). \tag{13}$$

(12) and (13) imply $\mathcal{K}_C(X) \subset \mathcal{H}_R(X)$.

Necessity: Let us assume that $\mathcal{K}_C(X) \subset \mathcal{H}_R(X)$. So there exists $x^* \in \Omega$ such that $x^* \notin \mathcal{K}_C(X)$ and $x^* \in \mathcal{H}_R(X)$. From the definition of $\mathcal{H}_R(X)$ follows that there exists $y^* \in X$ such that y^*Rx^*. If we use the definition of Richter relation we have that there exists $Y \in \mathcal{B}$ such that $y^* \in C(Y) \cap X$ and $x^* \in Y \setminus \mathcal{K}_C(X)$. Therefore (11) holds. ◊

The following example illustrates the necessity of the condition (11).

Example 5.1. Let the choice mechanism and the derived set-extensions be given by the following table:

X	$C(X)$	$\mathcal{K}_C(X)$	$\mathcal{H}_R(X)$
c	c	c	$c\ d$
d	d	d	$c\ d$
$a\ b$	a	$a\ b\ c$	$a\ b\ c$
$a\ c$	a	$a\ b\ c$	$a\ b\ c\ d$
$b\ c$	b	$b\ c$	$b\ c\ d$
$c\ d$	$c\ d$	$c\ d$	$c\ d$
$a\ b\ c$	a	$a\ b\ c$	$a\ b\ c\ d$

From the table we can read that the following matrices describe the C-revealed relations R and S^d, and they satisfy the $R \subseteq S^d$ rule.

R	a	b	c	d
a	1	1	1	0
b	0	1	1	0
c	0	0	1	1
d	0	0	1	1

S^d	a	b	c	d
a	1	1	1	1
b	0	1	1	1
c	0	0	1	1
d	1	1	1	1

1. Let $X = \{a, b, c\}$. Then (11) fulfils with the choice $Y = \{b, c\}$, since $C(Y) = \{b\}$, $Y \setminus \mathcal{K}_C(X) = \{d\}$ and $X \cap C(Y) = \{b\}$. Therefore the inclusion between $\mathcal{K}_C(X)$ and $\mathcal{H}_R(X)$ is strict.
2. Let $X = \{a, b\}$. Then $\{a, b\} \cap Y \neq \emptyset$ satisfies if Y is chosen from $\{a, b\}$, $\{abc\}$, $\{a, b, c\}$. But for all these sets
$$Y \setminus \mathcal{K}_C(\{a, b\}) = Y \setminus \{a, b, c\} = \emptyset,$$
therefore the inclusion between $\mathcal{K}_C(X)$ and $\mathcal{H}_R(X)$ is not strict.

Proposition 5.3. *Let $\mathcal{D} = (\Omega, \mathcal{B}, C)$ be a real decision mechanism and assume that $R \subseteq S^d$ holds between the C-revealed Richter- and Samuelson-relations, and S^d is reflexive on $\Omega \times \Omega$. Then*

$$\mathcal{K}_C(X) \subseteq \mathcal{H}_{S^d}(X).$$

Proof. Let $x^* \in \mathcal{K}_C(X)$. According to the definition $\mathcal{K}_C(X)$ there exists $Y \in \mathcal{B}$ such that $C(Y) = C(X)$ and $x^* \in Y$. From the second assumption $C(X) = C_S^{ND}(X) \neq \emptyset$. By the definition of Samuelson relation we have that $x^* \overline{S} y$ for all $y \in C(X) = C(Y)$. Hence $y \overline{S^{-1}} x^*$ for all $y \in C(X) \subseteq X$. From the definition of $\mathcal{H}_{S^d}(X)$ follow that $x^* \in \mathcal{H}_{S^d}(X)$. ◊

The Propositions 4.3 and 5.3 together give the inclusions

$$\mathcal{K}_C(X) \subseteq \mathcal{H}_R(X) \subseteq \mathcal{H}_{S^d}(X).$$

Let us now discuss the connection between the $\mathcal{H}_P$-extensions of a set and its $\mathcal{K}_C$-extension.

Proposition 5.4. *Let $\mathcal{D} = (\Omega, \mathcal{B}, C)$ be a normal real decision mechanism with a reflexive choice-representing relation P. Then*

$$\mathcal{H}_P(X) \subseteq \mathcal{H}_P(\mathcal{K}_C(X)).$$

Proof. Let $x^* \in \mathcal{H}_P(X)$. According to the definition $\mathcal{H}_P(X)$ there exists $y^* \in X$ such that $y^* P x^*$. Hence $y^* \in X \subseteq \mathcal{K}_C(X)$ such that $y^* P x^*$. By the definition of $\mathcal{H}_P(\mathcal{K}_C(X))$ we have that $x^* \in \mathcal{H}_P(\mathcal{K}_C(X))$. $\diamondsuit$

The following question is, the inclusion in the Proposition 5.5. under which condition will be strict?

Proposition 5.5. *Let $\mathcal{D} = (\Omega, \mathcal{B}, C)$ be a normal real decision mechanism with a reflexive choice-representing relation P. If for some $X \in \mathcal{B}$ there exist $x^* \in \mathcal{K}_C(X) \setminus X$ and $y^* \in \Omega$ such that $x^* P y^*$ and $x \overline{P} y^*$ for all $x \in X$ then*

$$\mathcal{H}_P(X) \subset \mathcal{H}_P(\mathcal{K}_C(X)). \tag{14}$$

Proof. Since $x \overline{P} y^*$ for all $x \in X$ therefore there does not exist $x \in X$ such that $x P y^*$. Consequently

$$y^* \notin \mathcal{H}_P(X). \tag{15}$$

Otherwise, there exist $x^* \in \mathcal{K}_C(X) \setminus X$ and $y^* \in \Omega$ such that $x^* P y^*$. From this follows that

$$y^* \in \mathcal{H}_P(\mathcal{K}_C(X)). \tag{16}$$

(15) and (16) together imply that (14). $\diamondsuit$

In the next proposition a necessary and sufficient condition will be given for the strict inclusion in (14) if $P = R$.

Proposition 5.6. *Let $\mathcal{D} = (\Omega, \mathcal{B}, C)$ be a real decision mechanism and assume that $R \subseteq S^d$ holds between the C-revealed Richter- and Samuelson-relations, and R is reflexive on $\Omega \times \Omega$. Then*

$$\mathcal{H}_R(X) \subset \mathcal{H}_R(\mathcal{K}_C(X)) \tag{17}$$

holds for some $X \in \mathcal{B}$ *if and only if there exists* $Y \in \mathcal{B}$ *such that*

$$(\mathcal{K}_C(X) \setminus X) \cap C(Y) \neq \emptyset \tag{18}$$

and

$$Y \setminus \bigcup \{A \in \mathcal{B} : C(A) \cap X \neq \emptyset\} \neq \emptyset. \tag{19}$$

Proof. Under the assumptions we have that $C_R^{MAX}(X) = C(X)$ for all $X \in \mathcal{B}$.

Sufficiency: Let (18) and (19) be fulfilled. It means that there exist

$$y^* \in (\mathcal{K}_C(X) \setminus X) \cap C(Y)$$

and

$$x^* \in Y \setminus \bigcup \{A \in \mathcal{B} : C(A) \cap X \neq \emptyset\}.$$

Because of $y^* \in \mathcal{K}_C(X) \cap C(Y)$ and $x^* \in Y$ we obtain that

$$y^* \in \mathcal{K}_C(X) \cap C(Y) \subset \mathcal{K}_C(X) \quad \text{and} \quad y^* R x^*.$$

By the definition of $\mathcal{H}_R(\mathcal{K}_C(X))$ therefore we have that

$$x^* \in \mathcal{H}_R(\mathcal{K}_C(X)). \tag{20}$$

Otherwise $x^* \notin \bigcup \{A \in \mathcal{B} : C(A) \cap X \neq \emptyset\}$. If $A \in \mathcal{B}$ such that $C(A) \cap X \neq \emptyset$, then $x^* \notin A$. From this for all $y \in X$ does not exist $A \in \mathcal{B}$ such that $y \in C(A)$ and $x^* \in A$. So for all $y \in X$ the relation $y\overline{R}x^*$ is satisfied. Therefore,

$$x^* \notin \mathcal{H}_R(X). \tag{21}$$

From (20) and (21) follows (17).

Necessity: Let us assume that $\mathcal{H}_R(X) \subset \mathcal{H}_R(\mathcal{K}_C(X))$. Then there exists $x^* \in \Omega$ such that $x^* \notin \mathcal{H}_R(X)$ and $x^* \in \mathcal{H}_R(\mathcal{K}_C(X))$. $x^* \notin \mathcal{H}_R(X)$ means, that for all $y \in X$ we have $y\overline{R}x^*$.

Let us introduce the set-system

$$\mathcal{A}(x^*, y) = \{Y \in \mathcal{B} : x^* \in Y, y \in C(Y)\}.$$

By the definition of the Richter-relation $y\overline{R}x^*$ implies

$$\mathcal{A}(x^*, y) = \emptyset \ \ \forall y \in X. \tag{22}$$

Otherwise, $x^* \in \mathcal{H}_R(\mathcal{K}_C(X))$ means that there exists $y^* \in \mathcal{K}_C(X)$ such that $y^* R x^*$. From this follows that

$$\mathcal{A}(x^*, y^*) \neq \emptyset. \tag{23}$$

From (22) and (23) we obtain that $y^* \notin X$, i.e. $y^* \in \mathcal{K}_C(X) \setminus X$, but $y^* \in C(Y)$ for $Y \in \mathcal{A}(x^*, y^*)$, consequently (20) fulfils.

Furthermore, (22) means that $y \in C(A)$ for some $A \in \mathcal{B}$ implies $x^* \notin A$, i.e.

$$x^* \notin \bigcup \{A \in \mathcal{B} : C(A) \cap X \neq \emptyset\}.$$

From (23) follows that there exists $Y \in \mathcal{A}(x^*, y^*)$ such that $x^* \in Y$, therefore

$$x^* \in Y \setminus \bigcup \{A \in \mathcal{B} : C(A) \cap X \neq \emptyset\}. \qquad \diamondsuit$$

The sufficiency and necessity of (18) and (19) will be illustrated in the following example:

Example 5.2. Let the choice mechanism and the derived set-extensions given by the following table:

X	$C(X)$	$\mathcal{K}_C(X)$	$\mathcal{H}_R(X)$	$\mathcal{H}_R(\mathcal{K}_C(X))$
c	c	c	$c\ d$	$c\ d$
$a\ b$	a	$a\ b\ c$	$a\ b\ c$	$a\ b\ c\ d$
$a\ c$	a	$a\ b\ c$	$a\ b\ c\ d$	$a\ b\ c\ d$
$b\ c$	b	$b\ c$	$b\ c\ d$	$b\ c\ d$
$c\ d$	$c\ d$	$c\ d$	$c\ d$	$c\ d$
$a\ b\ c$	a	$a\ b\ c$	$a\ b\ c\ d$	$a\ b\ c\ d$

The following two matrices show the derived C-revealed relations R and S^d, and it is seen that they satisfy the $R \subseteq S^d$ rule.

R	a	b	c	d
a	1	1	1	0
b	0	1	1	0
c	0	0	1	1
d	0	0	1	1

S^d	a	b	c	d
a	1	1	1	1
b	0	1	1	1
c	0	0	1	1
d	1	1	1	1

1. Let $X = \{a, b\}$. Choose $Y = \{c, d\}$. Then

$$(\mathcal{K}_C(X) \setminus X) \cap C(Y) = \{c\} \cap \{c, d\} = \{c\} \neq \emptyset.$$

Otherwise, the following subsets satisfy the condition $C(A) \cap X \neq \emptyset$: $\{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}$. Therefore,

$$Y \setminus (\cup A) = \{c,d\} \setminus \{a,b,c\} = \{d\} \neq \emptyset.$$

So, the conditions (18) and (19) are satisfied, and the inclusion between $\mathcal{H}_R(X)$ and $\mathcal{H}_R(\mathcal{K}_C(X))$ is strict, indeed.

2. Let $X = \{a,c\}$. The following subsets satisfy the condition $C(A) \cap X \neq \emptyset$: $\{c\}, \{a,b\}, \{a,c\}, \{c,d\}, \{a,b,c\}$, so $\cup A = \{a,b,c,d\}$.
   Otherwise, $(\mathcal{K}_C(X) \setminus X) \cap C(Y) \neq \emptyset$ will be satisfied if Y is one of the following sets: $\{c\}, \{b,c\}, \{c,d\}$. But choosing any of them as Y, we obtain that $Y \setminus (\cup A) = \emptyset$, i.e. the condition of the proposition is not fulfilled, and between $\mathcal{H}_R(X)$ and $\mathcal{H}_R(\mathcal{K}_C(X))$ also the equality holds.
3. Let $X = \{b,c\}$. In this case the assumption (18) trivially does not fulfil, and between $\mathcal{H}_R(X)$ and $\mathcal{H}_R(\mathcal{K}_C(X))$ the equality holds.

Finally, we will be interested in finding condition for the structure of the option set $\mathcal{B}$, which guarantees the equality of $\mathcal{H}_R(X)$ and $\mathcal{H}_R(\mathcal{K}_C(X))$.

Proposition 5.7. *Let $\mathcal{D} = (\Omega, \mathcal{B}, C)$ be a real decision structure, where the option set $\mathcal{B}$ contains all two-point subsets of 2^Ω. Moreover, let us assume that $R \subseteq S^d$ holds between the C-revealed Richter- and Samuelson-relations, and R is reflexive on $\Omega \times \Omega$. Then*

$$\mathcal{H}_R(X) = \mathcal{H}_R(\mathcal{K}_C(X)) \tag{24}$$

fulfils for some $X \in \mathcal{B}$ if and only if for all $x \in \mathcal{H}_R(\mathcal{K}_C(X)) \setminus X$ there exists $y \in X$ such that $y \in C(\{x,y\})$.

Proof. *Sufficiency.* According to the Proposition 3.1 $X \subseteq \mathcal{K}_C(X)$. Since $\mathcal{H}_R$ satisfies the third property of a closure operator defined in Definition 3.1, therefore we have that

$$\mathcal{H}_R(X) \subseteq \mathcal{H}_R(\mathcal{K}_C(X)).$$

Let now $x \in \mathcal{H}_R(\mathcal{K}_C(X))$. If $x \in X$ then $X \subseteq \mathcal{H}_R(X)$ implies $x \in \mathcal{H}_R(X)$. Let now $x \in \mathcal{H}_R(\mathcal{K}_C(X)) \setminus X$. According to the assumptions there exists $y \in X$ such that $\{x,y\} \in \mathcal{B}$ and $y \in C(\{x,y\})$. Since under the condition $R \subseteq S^d$ we know that $C(X) = C_R^{MAX}(X)$ for all $X \in \mathcal{B}$, we have that yRx, i.e. $x \in \mathcal{H}_R(X)$, consequently,

$$\mathcal{H}_R(X) \supseteq \mathcal{H}_R(\mathcal{K}_C(X)).$$

Necessity. Let us assume that (24) is fulfilled for some $X \in \mathcal{B}$.

Choose an arbitrary $x \in \mathcal{H}_R(\mathcal{K}_C(X)) \setminus X$. Since $x \in \mathcal{H}_R(X)$ also holds, therefore there exists $y \in X$ such that yRx, i.e. for this y there exists $Y \in \mathcal{B}$

such that $y \in C(Y)$ and $x \in Y$. It means that $y \in C(Y) \cap X$, consequently, yRy and yRx. Therefore $y \in C(\{x, y\})$. $\diamondsuit$

To illustrate the conditions of the proposition let us consider the following example.

Example 5.3. Let the choice mechanism and the derived set-extensions given by the following table:

X	$C(X)$	$\mathcal{K}_C(X)$	$\mathcal{H}_R(X)$	$\mathcal{H}_R(\mathcal{K}_C(X))$
c	c	c	c	c
a b	b	a b c	a b c d	a b c d
a c	a	a c d	a c d	a b c d
a d	a	a c d	a b c d	a b c d
b c	b	a b c	a b c d	a b c d
b d	b d	b d	a b c d	a b c d
c d	d	c d	b c d	b c d
a c d	a	a c d	a b c d	a b c d

The following two matrices show the derived C-revealed relations R and S^d, and it is seen that they satisfy the $R \subseteq S^d$ rule.

R	a	b	c	d
a	1	0	1	1
b	1	1	1	1
c	0	0	1	0
d	0	1	1	1

S^d	a	b	c	d
a	1	0	1	1
b	1	1	1	1
c	0	0	1	0
d	0	1	1	1

If we choose, for example,

1. $X = \{a, c\}$, then $\mathcal{H}_R(\mathcal{K}_C(X)) \backslash X = \{b, d\}$ and $a \notin C(\{a, b\})$, $c \notin C(\{b, c\})$, therefore the condition is injured. The equality between $\mathcal{H}_R(X)$ and $\mathcal{H}_R(\mathcal{K}_C(X))$ is not valid, indeed.
2. $X = \{c, d\}$, then $\mathcal{H}_R(\mathcal{K}_C(X)) \setminus X = \{b\}$ and $d \in C(\{b, d\})$, therefore the condition is satisfied. The equality between $\mathcal{H}_R(X)$ and $\mathcal{H}_R(\mathcal{K}_C(X))$ is valid, indeed.
3. $X = \{c\}$, then $\mathcal{H}_R(\mathcal{K}_C(X)) \setminus X = \emptyset$, so the condition of the proposition fulfils trivially. The equality between $\mathcal{H}_R(X)$ and $\mathcal{H}_R(\mathcal{K}_C(X))$ is valid, indeed.

References

[1] **Aizerman M. and Aleskerov F.**, *Theory of choice*, North Holland, 1995

[2] **Arrow K. J.**, Rational choice functions and orderings, *Economica*, **26** (1959), 121-127.

[3] **Kortelainen J.**, On the relationship between modified sets, topological spaces and rough sets, *Fuzzy Sets and Systems*, **61** (1994), 91-95.

[4] **Koshevoy G.A.**, Choice functions and abstract convex geometry, *Mathematical Social Sciences*, **38** (1999), 35-44.

[5] **Magyarkúti Gy.**, Note on generated choice and axioms of revealed preferences, *Central European Journal of Operation Research*, **8** (2000), 57-62.

[6] **Plott C.**, Path independence, rationality and social choice, *Econometrica*, **41** (1973), 1075-1091.

[7] **Richter M.**, Revealed preference theory, *Econometrica*, **34** (1966), 635-645.

[8] **Samuelson P.**, *Foundation of economic analysis*, Harvard University Press, Cambridge, USA, 1947,

[9] **Sen A.**, Choice function and revealed preference, *Review of Economic Studies*, **38** (1974), 307-409.

(Received February 23, 2006)

B. Bodó
Department of Operations Research
Eötvös Loránd University
Pázmány Péter sét. 1/C
H-1117 Budapest, Hungary

Mailing address:
Bodó Beáta
Temető u. 631
Gabčíkovo
93005 Slovakia