ADDITIVE UNIQUENESS SETS FOR MULTIPLICATIVE FUNCTIONS

K.-H. Indlekofer (Paderborn, Germany)
Bui Minh Phong (Budapest, Hungary)

Abstract. We proved that if a multiplicative function F and a positive integer k satisfy $F(2)\neq 0$, $F(5)\neq 1$ and

$$F\left(n^2+m^2+k+1\right)=F(n^2+k)+F(m^2+1)\quad \text{for all}\quad n,m\in I\!N,$$

then $F(n)=n$ for all positive integers n, $(n,2)=1$.

1. Introduction

In this paper, let $I\!N$ and $\mathcal{P}$ stand for the set of positive integers and prime numbers, respectively. We denote by $\mathcal{M}$ the set of all multiplicative functions f such that $f(1)=1$. Furthermore, we deal with the set $\mathcal{B}$ of non-negative integers which can be represented as a sum of two squares of integers and with $\mathcal{S}$ the set of all squares of positive integers.

In the following subsets A and B of $I\!N$ are called additive uniqueness sets (AU-sets) for $\mathcal{M}$ if $f\in\mathcal{M}$ satisfies

$$f(a+b)=f(a)+f(b)\quad \text{for all}\quad a\in A\quad \text{and}\quad b\in B,$$

then $f(n)=n$ for all $n\in I\!N$. In 1992 C.Spiro [6] showed that $A=B=\mathcal{P}$ are AU-sets for $\mathcal{M}$. In [1] it is proved that $A=\mathcal{S}$ and $B=\mathcal{P}$ are also AU-sets for

The research was supported by the Applied Number Theory Research Group of the Hungarian Academy of Science and the Hungarian National Foundation for Scientific Research under grants OTKA T046993 and T043657. This work was also supported in part by a guest-professorship at the University of Paderborn.

$\mathcal{M}$. Recently in [5] (see also [4]) the second author showed that if $k \in I\!N$ and $f \in \mathcal{M}$ satisfy the conditions $f(4)f(9) \neq 0$ and

$$f\left(n^2+m^2+k\right) = f(n^2) + f(m^2+k) \quad \text{for all} \quad n, m \in I\!N,$$

then $f(n) = n$ for all $n \in I\!N$, $(n, 2k) = 1$. The proof of this result is used on the following theorem of K.-H. Indlekofer and N. M. Timofeev [3]:

Theorem (IT). *Let C be a non-zero integer and $A, B \in I\!N$ such that $(A, B) = 1$, $(AB, 2C) = 1$. Then there exists a positive constant $\theta = \theta(A, B, C)$ such that*

$$|\{n \leq x : \ A(n+C) = B(m+C), \ (A, n+C) = 1, n, m \in \mathcal{B}\}| > \theta \frac{x}{\log x}$$

holds for all $x \geq x_0(A, B, C)$.

We are interested in characterizing all multiplicative functions F satisfying the condition

$$(1) \qquad F(n^2+m^2+a+b) = F(n^2+a) + F(m^2+b) \ \text{ for all } \ n, \ m \in I\!N,$$

where a, b are given positive integers. Presently we are unable to determine all such solutions. Our purpose in this paper is to solve (1) for the case $\min(a, b) = 1$.

Theorem. *Assume that $k \in I\!N$ and $F \in \mathcal{M}$ satisfy the condition*

$$F\left(n^2+m^2+k+1\right) = F(n^2+1) + F(m^2+k) \quad \textit{for all} \quad n, m \in I\!N.$$

(a) *If $F(2) = 0$, then*

$$F\left(n^2+m^2+k+1\right) = F(n^2+1) = F(m^2+k) = 0 \quad \textit{for all} \quad n, m \in I\!N.$$

(b) *If $F(2) \neq 0$ and $F(5) = 1$, then $k \equiv 3 \pmod 4$ and*

$$\begin{cases} F(n^2+1) & = \chi_2(n) + 1, \\ F(n^2+k) & = \chi_2(n) - 1, \\ F(n^2+m^2+k+1) & = \chi_2(n) + \chi_2(m) \end{cases}$$

for all $n, m \in I\!N$, where χ_2 denotes the principal character (mod 2).

(c) *If $F(2) \neq 0$ and $F(5) \neq 1$, then $F(n) = n$ for all $n \in I\!N$, $(n, 2) = 1$.*

2. Lemmas concerning an arithmetical function satisfying (1)

In this section we assume that $a, b \in I\!N$ and F is an arithmetical function with condition (1). Let

$$S_j := F(j^2 + a) \text{ for all } j \in I\!N.$$

First we note from (1) that

$$F(n^2 + a) + F(m^2 + b) = F(m^2 + a) + F(n^2 + b),$$

consequently

$$(2) \qquad F(n^2 + b) = S_n + D \text{ and } F(n^2 + m^2 + a + b) = S_n + S_m + D$$

hold for all $n, m \in I\!N$, where $D := F(b + 1) - F(a + 1)$.

In the proof of our theorem, we shall use the following two lemmas.

Lemma 1. *We have*

$$(3) \qquad S_k + S_l = S_u + S_v \quad \text{if} \quad k^2 + l^2 = u^2 + v^2$$

and

$$(4) \qquad S_x + S_y = S_z \quad \text{if} \quad x^2 + y^2 + a = z^2.$$

Proof. Assume that positive integers k, l, u, v satisfy $k^2 + l^2 = u^2 + v^2$. Then by (2) we have

$$F(k^2 + l^2 + a + b) = S_k + S_l + D \text{ and } F(u^2 + v^2 + a + b) = S_u + S_v + D,$$

therefore (3) is true.

Now assume that positive integers x, y, z satisfy the equation $x^2 + y^2 + a = z^2$. Then we infer from (2) that

$$F(x^2 + y^2 + a + b) = F(z^2 + b)$$

and

$$F(x^2 + y^2 + a + b) = S_x + S_y + D, \quad F(z^2 + b) = S_z + D.$$

Thus (4) is true.

Lemma 2. *We have*

$$S_{n+12} = S_{n+9} + S_{n+8} + S_{n+7} - S_{n+5} - S_{n+4} - S_{n+3} + S_n \tag{5}$$

for all $n \in I\!N$ *and*

$$\begin{cases} S_7 & = 2S_5 - S_1, \\ S_8 & = 2S_5 + S_4 - 2S_1, \\ S_9 & = S_6 + 2S_5 - S_2 - S_1, \\ S_{10} & = S_6 + 3S_5 - S_3 - 2S_1, \\ S_{11} & = S_6 + 4S_5 - S_3 - S_2 - 2S_1, \\ S_{12} & = S_6 + 4S_5 + S_4 - S_2 - 4S_1. \end{cases} \tag{6}$$

Proof. This is Lemma 1 of [5].

3. The proof of the theorem

In this section we assume that $k \in I\!N$ and $F \in \mathcal{M}$ satisfy the condition

$$F\left(n^2 + m^2 + k + 1\right) = F(n^2 + 1) + F(m^2 + k) \quad \text{for all} \quad n, m \in I\!N.$$

We shall use the notations and results of Lemma 1-2 with $a = 1$ and $b = k$. Let $S_j := f(j^2 + 1)$. Then, by (2), we have

$$F\left(n^2 + m^2 + k + 1\right) = S_n + S_m + D \quad \text{and} \quad F(n^2 + k) = S_n + D \tag{7}$$

for all $n, m \in I\!N$,where $D = F(k + 1) - F(2)$.

First we note from Lemma 1 that if $x^2 + y^2 + 1 = z^2$, then $S_x + S_y = S_z$. Since $2^2 + 2^2 + 1 = 3^2$ and $4^2 + 8^2 + 1 = 9^2$, we have

$$\begin{cases} S_3 & = 2S_2, \\ \\ S_9 & = S_4 + S_8. \end{cases} \tag{8}$$

On the other hand, by using the facts

$$F(10) = F(2)F(5) \quad \text{and} \quad F(2)F(65) = F(5)F(26),$$

we have

$$S_3 = S_1 S_2 \quad \text{and} \quad S_1 S_8 = S_2 S_5. \tag{9}$$

Therefore, we infer from (6), (8) and (9) that $S_4 = S_9 - S_8 = S_6 - S_4 - S_2 + S_1$, consequently

$$(10)\qquad \begin{cases} S_6 &= 2S_4 + S_2 - S_1,\\ S_7 &= 2S_5 - S_1,\\ S_8 &= 2S_5 + S_4 - 2S_1,\\ S_9 &= 2S_5 + 2S_4 - 2S_1,\\ S_{10} &= 3S_5 + 2S_4 - S_2 - 3S_1,\\ S_{11} &= 4S_5 + 2S_4 - 2S_2 - 3S_1,\\ S_{12} &= 4S_5 + 3S_4 - 5S_1. \end{cases}$$

Lemma 3. *If $S_2 = 0$, then $S_1 = 0$.*

Proof. In the following we assume that $S_2 = F(5) = 0$. Hence we obtain from (8)-(10) that

$$S_3 = 0 \quad \text{and} \quad S_8 = S_{12} = 0,$$

because $S_8 = F(65) = F(5)F(13) = 0$ and $S_{12} = F(145) = F(5)F(29) = 0$. An application of the formula of S_8 and S_{12} in (10) shows that $S_4 = S_1$ and $S_5 = \frac{1}{2}S_1$. Consequently, one can check from (10) the following results:

$$\begin{cases} S_2 = S_3 = S_7 = S_8 = S_{12} = 0,\\ S_1 = S_4 = S_6 = S_9 = S_{11},\\ S_5 = S_{10} = \frac{1}{2}S_1. \end{cases}$$

Hence, it follows from (5) that the sequence $\{S_n\}_{n=1}^{\infty}$ is also periodic, namely

$$(11)\qquad S_n = S_m \ \text{ if } \ n \equiv m \pmod 5 \ \text{ and } \ S_j \in \left\{S_1, 0, 0, S_1, \frac{1}{2}S_1\right\}$$

for all $j \in I\!N$.

We shall prove that $S_1 = 0$. Assume that $S_1 \neq 0$. Then, by using (11), we have

$$\frac{1}{2}S_1 = S_5 = F(2)F(13) = S_1F(13),$$

$$S_1 = S_{21} = F(21^2 + 1) = F(2)F(13)F(17) = F(13)S_1S_4 = F(13)S_1^2$$

and

$$S_1 = S_4 = S_{34} = F(34^2 + 1) = F(13)F(89),$$

which imply that

$$F(13) = \frac{1}{2},\ S_1 = 2 \ \text{ and } \ F(89) = 4.$$

Hence

$$1 = \frac{1}{2}S_1 = S_{55} = F(55^2+1) = F(2)F(17)F(89) = S_1^2F(89) = 2^2.4 = 16,$$

which is impossible.

Thus, $S_1 = 0$ and the proof of Lemma 3 is complete.

Lemma 4. *If $S_1 = 0$, then*

$$F\left(n^2+m^2+k+1\right) = F(n^2+1) = F(m^2+k) = 0 \quad \textit{for all} \quad n, m \in I\!N.$$

Proof. We assume that $S_1 = F(2) = 0$. In this case, we have

$$S_{2t+1} = F\left((2t+1)^2+1\right) = F(2)F(2t^2+2t+1) = 0$$

for all $t \in I\!N$. Hence

$$S_1 = S_3 = S_5 = S_7 = S_9 = S_{11} = 0.$$

It is easy to check from the formula of S_3 and S_9 in (8)-(10) that

$$S_2 = S_4 = 0.$$

Hence, (10) gives

$$S_6 = S_8 = S_{10} = S_{12} = 0.$$

Thus we have proved that $S_i = 0$ for all $i \in I\!N$, $1 \le i \le 12$, which with (5) show that $S_n = F(n^2+1) = 0$ for all $n \in I\!N$.

Now we use Theorem (IT) to show that $D = F(k+1) - F(2) = 0$. Assume that $D \neq 0$. Then, from (7) and from our assumptions, we have

$$F(\nu + C) = D \quad \text{for all} \quad \nu \in \mathcal{B}, \tag{12}$$

where $C = k+1$. Let $A := (2C)^2 + 1$, $B = 1$. Then $(A, B) = (AB, 2C) = 1$, therefore Theorem (IT) implies that there are positive integers $\nu, \mu \in \mathcal{B}$ such that

$$A(\nu + C) = \mu + C \quad \text{and} \quad (A, \nu + C) = 1.$$

From (12) we obtain

$$F(A)D = F(A)F(\nu + C) = F\left(A(\nu + C)\right) = F(\mu + C) = D,$$

which with $D \neq 0$ implies $F(A) = 1$. This is impossible, because

$$F(A) = F((2C)^2 + 1) = S_{2C} = 0.$$

The part (a) of our theorem is proved.

Lemma 5. *If $F(2) \neq 0$, then $S_1 = F(2) = 2$ and $S_2 = F(5) \in \{1, 5\}$.*

Proof. Assume that $S_1 = F(2) \neq 0$. First we note from Lemma 3 and Lemma 4 that $S_1 \neq 0$ and $S_2 \neq 0$ are satisfied. It follows from (8) and (9) that $S_3 = S_1 S_2 = 2S_2$, consequently $S_1 = 2$.

Next we prove that

$$S_5 = S_4 + 2S_2 - 1. \tag{13}$$

Indeed, using the following relations

$$17^2 + 1 = 2.(12^2 + 1), \quad 1^2 + 13^2 = 7^2 + 11^2 \quad \text{and} \quad 1^2 + 17^2 = 11^2 + 13^2,$$

which with (3), using the fact $S_1 = F(2) = 2$ and the multiplicativity of F, imply

$$S_{17} = 2S_{12}, \; 2 + S_{13} = S_7 + S_{11} \quad \text{and} \quad 2 + S_{17} = S_{11} + S_{13}. \tag{14}$$

Thus, we have established, in view of (10), that

$$S_7 = 2 + S_{13} - S_{11} = 4 + S_{17} - 2S_{11} = 4 + 2S_{12} - 2S_{11}$$

and so

$$S_7 - 4 = 2(S_{12} - S_{11}) = 2(S_4 + 2S_2 - 4).$$

Therefore (13) is true, because it is known from (10) that $S_7 - 4 = 2(S_5 - 3)$.

Now we prove that

$$S_2 \in \{1, 5\}. \tag{15}$$

We obtain from (10) and (13)-(14) that

$$\begin{aligned} S_{13} &= S_7 + S_{11} - 2 = (2S_5 - 2) + (4S_5 + 2S_4 - 2S_2 - 6) - 2 = \\ &= 6S_5 + 2S_4 - 2S_2 - 10 = 6(S_4 + 2S_2 - 1) + 2S_4 - 2S_2 - 10 = \\ &= 8S_4 + 10S_2 - 16 \end{aligned}$$

and

$$S_{13} = F(13^2 + 1) = F(2)F(5)F(17) = 2S_2S_4,$$

which show that

$$(S_2-4)S_4=5S_2-8. \tag{16}$$

Let $x:=S_2$. It is obvious from the above relation that $x-4\neq 0$, therefore $S_4=(5x-8)/(x-4)$.

On the other hand, we infer from (9), (10) and (13) that

$$2S_8=S_2S_5=S_2(S_4+2S_2-1)=x\left(\frac{5x-8}{x-4}+2x-1\right)$$

and

$$2S_8=2(2S_5+S_4-2S_1)=4(S_4+2S_2-1)+2S_4-8=6\left(\frac{5x-8}{x-4}\right)+8x-12.$$

After simplifications of these relations of $2S_8$, we obtain the equation of the form

$$x(x-1)(x-5)=0,$$

consequently $x=S_2\in\{0,1,5\}$. Thus (15) is proved, since $x=S_2\neq 0$. The Lemma 5 is proved.

Lemma 6. *If $F(2)\neq 0$ and $F(5)=1$, then $k\equiv 3 \pmod 4$, $D=-2$ and $S_n=\chi_2(n)+1$ for all $n\in I\!N$.*

Proof. Assume that $F(2)\neq 0$ and $F(5)=1$. Then it follows from (8), (13) and (16) that $S_3=S_5=2$ and $S_4=1$. Consequently, we infer from (10) that

$$S_n=S_{(n,2)} \quad \text{for} \quad n\in I\!N, \quad 1\leq n\leq 12,$$

which by (5) implies that the sequence $\{S_n\}_{n=1}^{\infty}$ is periodic, namely

$$S_n=S_{(n,2)}=\chi_2(n)+1 \quad \text{for all} \quad n\in I\!N, \tag{17}$$

where χ_2 denotes the principal character (mod 2).

In order to prove that k is odd, we note from (7) and (17) that

$$F(k+25+1)=F(3^2+4^2+k+1)=S_3+S_4+D=3+D,$$

$$F(k+26+1)=F(1^2+5^2+k+1)=S_1+S_5+D=4+D$$

and

$$F(k+26)F(k+27)=F\left((k+26)^2+5^2+k+1\right)=S_{k+26}+2+D,$$

which imply

$$S_k = \chi_2(k) + 1 = D^2 + 6D + 10. \tag{18}$$

Hence $\chi_2(k) = (D+3)^2 \in \{0,1\}$, consequently

$$D \in \{-4,-3,-2\}. \tag{19}$$

Assume that k is even. Then (18) implies that $D=-3$. By

$$\left(\frac{k}{2}\right)^2 + 1 + k + 1 = \left(\frac{k}{2}+1\right)^2 + 1,$$

we have

$$S_{\frac{k}{2}+1} = S_{\frac{k}{2}} + S_1 + D = S_{\frac{k}{2}} - 1,$$

which gives

$$1 = \chi_2\left(\frac{k}{2}\right) - \chi_2\left(\frac{k}{2}+1\right).$$

This is true for the case when $k \equiv 2 \pmod 4$.

Since $F(n^2+k) = S_n + D = \chi_2(n) - 2$ and $F(2) = S_1 = 2$, we have

$$F\left(2\ell^2 + \frac{k}{2}\right) = -1 \quad \text{for all} \quad \ell \in I\!N.$$

Let $p = 4Q+1 \in \mathcal{P}$ and $p > k$. Then

$$\left(2Q^2 + \frac{k}{2}, (2Q+1)^2 + k\right) = \left(2Q^2 + \frac{k}{2}, 4Q+1\right) =$$

$$= (4Q^2 + k, p) = (k - Q, p) = 1$$

and

$$\left(2Q^2 + \frac{k}{2}\right)\left[(2Q+1)^2 + k\right] = 2\left(Q(2Q+1) + \frac{k}{2}\right)^2 + \frac{k}{2}$$

imply that

$$F\left(2Q^2 + \frac{k}{2}\right) F\left[(2Q+1)^2 + k\right] = F\left[2\left(Q(2Q+1) + \frac{k}{2}\right)^2 + \frac{k}{2}\right].$$

Hence we infer that

$$S_{2Q+1} + D = S_{2Q+1} - 3 = 1, \quad \text{i.e.} \quad S_{2Q+1} = 4.$$

This is impossible. Therefore k is odd.

Assume that k is odd. Then the relation

$$F\left[\left(\frac{k+3}{2}\right)^2 + 2^2 + k + 1\right] = F\left[\left(\frac{k+5}{2}\right)^2 + 1\right],$$

with (7) and (18)-(19) implies

$$D = S_{\frac{k+5}{2}} - S_{\frac{k+3}{2}} - S_2 = S_{\frac{k+5}{2}} - S_{\frac{k+3}{2}} - 1 \in \{-4, -3, -2\}.$$

This is true for the case when $k \equiv 3 \pmod 4$ and $D = -2$.

Lemma 6 and the part (b) of our theorem is proved.

Lemma 7. *If $F(2) \neq 0$ and $F(5) \neq 1$, then*

$$\begin{cases} F(n^2+1) & = n^2+1, \\ F(n^2+k) & = n^2+k, \\ F(n^2+m^2+k+1) & = n^2+m^2+k+1 \end{cases} \tag{20}$$

are satisfied for all $n, m \in I\!N$.

Proof. First we note from Lemma 5 that $F(2) \neq 0$ and $F(5) \neq 1$ imply that $S_1 = F(2) = 2$ and $S_2 = F(5) = 5$. Hence we infer from (8) and (16) that $S_3 = 2S_2 = 10$ and

$$S_4 = \frac{(5S_2 - 8)}{S_2 - 4} = 17.$$

These with (10) and (13) imply that $S_n = F(n^2+1) = n^2+1$ for $1 \leq n \leq 12$. Hence we infer from (5) that the first relation of (20) is true.

Next we prove that

$$D = k - 1. \tag{21}$$

Indeed, if k is odd, then

$$F\left[\left(\frac{k+3}{2}\right)^2 + 2^2 + k + 1\right] = F\left[\left(\frac{k+5}{2}\right)^2 + 1\right],$$

with the first relation of (20) implies

$$D = S_{\frac{k+5}{2}} - S_{\frac{k+3}{2}} - S_2 = \left(\frac{k+5}{2}\right)^2 + 1 - \left(\frac{k+3}{2}\right)^2 - 1 - 5 = k - 1.$$

If k is even, then by

$$\left(\frac{k}{2}\right)^2 + 1 + k + 1 = \left(\frac{k}{2} + 1\right)^2 + 1,$$

we also obtain from the first relation of (20) that

$$D = S_{\frac{k}{2}+1} - S_{\frac{k}{2}} - S_1 = \left(\frac{k}{2} + 1\right)^2 + 1 - \left(\frac{k}{2}\right)^2 - 1 - 2 = k - 1.$$

Thus (21) is proved, therefore all relations of (20) follow from (7). The proof of Lemma 7 is complete.

In order to complete the proof of our theorem, it remains to deduce from (20) the following result.

Lemma 8. *If* (20) *holds, then*

$$F(n) = n \quad \textit{for all} \quad (n, 2) = 1. \tag{22}$$

Proof. First we prove (22) for $n \in I\!N$ with the condition $(n, 2(k+1)) = 1$. Assume that $n \in I\!N$, $(n, 2(k+1)) = 1$. Then the Theorem (IT) implies that there are positive integers μ and ν such that

$$n(\mu + k + 1) = \nu + k + 1 \quad \text{and} \quad (n, \mu + k + 1) = 1.$$

Thus, from (20) we have

$$n(\mu + k + 1) = \nu + k + 1 = F(\nu + k + 1) = F[n(\mu + k + 1)] =$$

$$= F(n)F(\mu + k + 1) = F(n)(\mu + k + 1),$$

which proves (22) under the condition $(n, 2(k+1)) = 1$.

Next we prove that

$$F(p^\alpha) = p^\alpha \quad \text{for} \quad p \in \mathcal{P}, p > 2 \quad \text{and} \quad p \mid k + 1. \tag{23}$$

Let $2, p_1, \dots, p_r$ be all distinct prime divisors of $2(k+1)$. Let $p \in \{p_1, \dots, p_r\}$ and $\alpha \in I\!N$. We consider the equation

$$x^2 + k = p^\alpha y. \tag{24}$$

Since $k \equiv -1 \pmod p$, therefore there are $x_\alpha, y_\alpha \in I\!N$ such that

$$x_\alpha^2 + k = y_\alpha p^\alpha \quad \text{and} \quad (p^\alpha - x_\alpha)^2 + k = (p^\alpha - 2x_\alpha + y_\alpha)\, p^\alpha.$$

It is obvious that one of y_α and $p^\alpha - 2x_\alpha + y_\alpha$ is coprime to p. Assume that $x_\alpha, y_\alpha \in I\!N$ satisfy (24) and $(y_\alpha, p) = 1$. Let $x = p^\alpha t + x_\alpha$ and $y = p^\alpha t^2 + 2x_\alpha t + y_\alpha$. Then (x, y) is also a solution of (24).

Since $p^\alpha t^2 + 2x_\alpha t + y_\alpha \equiv 0 \pmod{p_i}$ has at most two solutions and $p_i \geq 3$ for all $1 \leq i \leq r$, consequently there is $t_i \in I\!N$ such that

$$(p^\alpha t_i^2 + 2x_\alpha t_i + y_\alpha, p_i) = 1 \quad \text{for} \quad i = 1, \dots, r.$$

On the other hand, it is easy to see that

$$(p^\alpha (y_\alpha + 1)^2 + 2x_\alpha (y_\alpha + 1) + y_\alpha, 2) = (y_\alpha + 1 + y_\alpha, 2) = 1.$$

Hence an application of the Chinese Remainder Theorem shows that there is $t_0 \in I\!N$ for which

$$\big(p^\alpha t_0^2 + 2x_\alpha t_0 + y_\alpha, 2(k+1)\big) = 1.$$

Thus we have proved that

$$(x_0, y_0) = (p^\alpha t_0 + x_\alpha, p^\alpha t_0^2 + 2x_\alpha t_0 + y_\alpha)$$

is a solution of (24) with the condition $(y_0, 2(k+1)) = 1$.

Finally, we infer from (20) and (22) that

$$p^\alpha y_0 = x_0^2 + k = F\left(x_0^2 + k\right) = F\left(p^\alpha y_0\right) = F\left(p^\alpha\right) F(y_0) = F\left(p^\alpha\right) y_0,$$

which proves (23). The proof of Lemma 8 is complete.

The theorem is proved.

Acknowledgement. The second author is deeply grateful to the University of Paderborn for the opportunity and support to carry out this research.

References

[1] **De Koninck J.-M., Kátai I. and Phong B. M.,** A new characteristic of the identity function, *J. Number Theory,* **63** (1997), 325-338.

[2] **De Koninck J.-M. and Kátai I.,** On the multiplicative group generated by shifted binary quadratic forms (submitted)

[3] **Indlekofer K.-H. and Timofeev N.M.,** Shifted B-numbers as a set of uniqueness for additive and multiplicative functions, *Acta Arith.,* **116** (2005), 295-313.

[4] **Phong B.M.,** A characterization of the identity function, *Acta Acad. Paed. Agriensis Sect. Math.,* (1997), 1-9.

[5] **Phong B.M.,** On sets characterizing the identity function, *Annales Univ. Sci. Budapest. Sect. Comp.,* **24** (2004), 295-306.

[6] **Spiro C.,** Additive uniqueness set for arithmetic functions, *J. Number Theory,* **42** (1992), 232-246.

(Received April 11, 2005)

K.-H. Indlekofer
Faculty of Computer Science, Electrical
Engineering and Mathematics
University of Paderborn
Warburger Str. 100
D-33098 Paderborn, Germany
k-heinz@uni-paderborn.de

Bui Minh Phong
Department of
Computer Algebra
Eötvös Loránd University
Pázmány Péter Sét. 1/C
H-1117 Budapest, Hungary
bui@compalg.inf.elte.hu