A REMARK ON THE PRODUCT PARTITION OF INTEGERS INTO k PARTS

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Abstract. Let $f_k(n)$ be the number of solutions of the equation $n = m_1 m_2 \dots m_k$ in integers $(2 =)\, m_1 < m_2 < \dots < m_k$. The authors analyze the question how $\sum_{n \le x} f_k(n)$ can be estimated with good remainder terms by using known result.

1. Let $k \in \mathbb{Z}$ be an integer, $f_k(n)$ be the number of solutions of the equation

$$n = m_1 m_2 \dots m_k,$$

in integers $(2 \le) m_1 < m_2 < \dots < m_k$.

Let

$$F(t,s) = \prod_{n=2}^{\infty} \left(1 + \frac{t}{n^s}\right).$$

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Then

$$\log F(t,s) = \sum_{k=1}^{\infty} \frac{t^k}{k} \cdot (-1)^{k-1} \left(\sum_{n=2}^{\infty} \frac{1}{n^{ks}} \right) =$$
$$= \sum_{k=1}^{\infty} \frac{(-1)^{k-1} t^k}{k} (\zeta(ks) - 1) =$$
$$= t\zeta(s) - t + a(t,s),$$
$$a(t,s) = \sum_{k=2}^{\infty} \frac{(-1)^{k-1} t^k}{k} (\zeta(ks) - 1).$$

It is clear that for $|t| < \sqrt{2}$, the function $a(t,s)$ as a function of s is regular and bounded in $\text{Re } s > \frac{1}{2} + \delta$ where δ is an arbitrary positive constant.

Let

$$F_k(s) = \sum_{n=1}^{\infty} \frac{f_k(n)}{n^s}.$$

We have $e^{-t+a(t,s)} = b_0(s) + b_1(s)t + b_2(s)t^2 + \dots,\ b_0(s) = 1$, where $b_\nu(s)$ are bounded in $\sigma > \frac{1}{2} + \delta$, they are polynomials of $\zeta(2s), \dots, \zeta(\nu s)$ for every ν. The explicit form of them can be computed.

Let $x_1 = \log x,\ x_2 = \log x_1, \dots$.

Since

$$F(t,s) = 1 + \sum_{k=1}^{\infty} t^k F_k(s),$$
$$e^{t\zeta(s)} = \sum_{k=0}^{\infty} \frac{\zeta(s)^k}{k!} t^k,$$

therefore

$$F_k(s) = \sum_{\nu=0}^{k} \frac{b_\nu(s)}{(k-\nu)!} \zeta^{k-\nu}(s). \tag{1.1}$$

We would like to estimate

$$S_k(x) = \sum_{n \le x} f_k(n).$$

Since

$$e^{-t+a(t,s)} = \left(\sum \frac{(-t)^{\nu_0}}{\nu_0!}\right) \prod_{k=2}^{\infty} \left(\sum_{m=0}^{\infty} \frac{(-1)^{(k-1)m} t^{km}}{k^m} (\zeta(ks) - 1)^m\right),$$

therefore

$$\frac{b_\nu(s)}{(k-\nu)!} = E_1 \cdot G_1(s) + \ldots + E_p \cdot G_p(s), \tag{1.2}$$

where the general form of $G_l(s)$ can be written as

$$G(s) = \zeta^{m_1}(a_1 s) \ldots \zeta^{m_q}(a_q s) = \sum_{n=1}^{\infty} \frac{g(n)}{n^s}, \tag{1.3}$$

where $(2 \leq) a_1 < \ldots < a_q,\ m_1, \ldots, m_s$ are positive integers, and

$$m_1 a_1 + \ldots + m_q a_q \leq \nu.$$

Let

$$\begin{aligned} D(s) &= D(s|G) = \sum_{n=1}^{\infty} \frac{g(n)}{n^s}, \\ B(y) &= (B(y|G) =) \sum_{n \leq y} g(n). \end{aligned} \tag{1.4}$$

From (1.3) we have

$$B(y) = \sum_{n_1^{a_1} \ldots n_q^{a_q} \leq x} d_{m_1}(n_1) \ldots d_{m_q}(n_q). \tag{1.5}$$

Lemma 1. *We have*

$$B(y) \leq C x^{1/a_1} x_1^{m_1 - 1} \zeta\left(\frac{a_2}{a_1}\right) \ldots \zeta\left(\frac{a_q}{a_1}\right) \tag{1.6}$$

for $q \geq 2$, and

$$B(y) \leq C x^{1/a_1} x_1^{m_1 - 1} \quad if \quad q = 1. \tag{1.7}$$

Proof. Since

$$\sum_{n^a\le x} d_m(n) = \sum_{u_1\dots u_m\le x^{1/a}} 1 = \sum_{u_1\dots u_{m-1}\le x^{1/a}} \frac{x^{1/a}}{u_1\dots u_{m-1}} \le$$
$$\le x^{1/a}\left(\sum_{u<x^{1/a}}\frac{1}{u}\right)^{m-1} \le$$
$$\le x^{1/a}\left(\frac{1}{a}x_1+c\right)^{m-1},$$

therefore the assertion is true for $q=1$.

Let $q\ge 2$. Assume that the assertion is true for $q-1$.

Then

$$\sum_{n_1^{a_1}\dots n_q^{a_q}\le x} d_{m_1}(n_1)\dots d_{m_q}(n_q) = \sum_{n_2^{a_2}\dots n_q^{a_q}\le x} d_{m_2}(n_2)\dots d_{m_q}(n_q)\sum\nolimits_{n_2,\dots n_q},$$

$$\sum\nolimits_{n_2,\dots,n_q} = \sum_{n_1\le\left(\frac{x}{n_2^{a_2}\dots n_q^{a_q}}\right)^{1/a_1}} d_{m_1}(n_1)\le cx^{1/a_1}x_1^{m_1-1}\cdot\frac{1}{n_2^{a_2/a_1}\dots n_q^{a_q/a_1}},$$

and so

$$B(y)\le cx^{1/a_1}x_1^{m_1-1}\zeta\left(\frac{a_2}{a_1}\right)^{m_2-1}\dots\ \zeta\left(\frac{a_q}{a_1}\right)^{m_q-1}. \tag{1.8}$$

Lemma 2. *Let*

$$D_k(x) := \sum_{n\le x} d_k(n) = xQ_{k-1}(\log x)+\Delta_k(x),$$
$$Q_{k-1}(\log x) := \operatorname*{Res}_{s=1} x^{s-1}\zeta^k(s)s^{-1}.$$

Let $\alpha_2\le\alpha_3\le\dots$ be such a sequence for which $\alpha_2>\frac{1}{2}$, and for each $\varepsilon>0$,

$$\Delta_k(x)=O\left(x^{\alpha_k+\varepsilon}\right)$$

holds. Possible value of α_k can be found in [1], Theorem 13.2.

2. Let us estimate

$$E(x \mid G, k, \nu) = \sum_{mn \le x} g(m) d_{k-\nu}(n), \tag{2.1}$$

where G is a function of form (1.3).

Then

$$\begin{aligned} E(x \mid G, k, \nu) &= \sum_{m \le x} g(m) D_{k-\nu}(x/m) = \\ &= \sum_{m \le x} g(m) \left\{ \frac{x}{m} Q_{k-\nu-1}\left(\log \frac{x}{m}\right) + O\left(\left(\frac{x}{m}\right)^{\alpha_k+\varepsilon}\right)\right\} = \\ &= \sum_{m \le x} \frac{g(m)}{m} Q_{k-\nu-1}\left(\log \frac{x}{m}\right) + O\left(x^{\alpha_k+\varepsilon} \sum \frac{g(m)}{m^{\alpha_k+\varepsilon}}\right). \end{aligned}$$

Since $\alpha_{k-\nu} > 1/2$, therefore

$$\sum \frac{g(m)}{m^{\alpha_{k-\nu}+\varepsilon}} < \infty.$$

Let

$$Q_{k-\nu-1}(y) = \sum_{\mu=0}^{k-\nu-1} e_\mu y^\mu.$$

Therefore

$$\sum_{m \le x} \frac{g(m)}{m} \sum_{\mu=0}^{k-\nu-1} e_\mu (x_1 - (\log m))^\mu = \sum_{h=0}^{k-\nu-1} x_1^h U_h(x),$$

$$U_h(x) = \sum_{h=0}^{k-\nu-l-1} d_{h,l} \sum_{m \le x} \frac{g(m)}{m} (\log m)^l.$$

Let

$$\eta_l := \sum \frac{g(m)}{m} (\log m)^l. \tag{2.2}$$

Since

$$\begin{aligned} \sum_{m \ge x} g(m) \frac{(\log m)^l}{m} &\ll \sum_{t=0}^{\infty} \frac{[\log(2^t x)]^l}{2^t x} \sum_{2^t x < m < 2^{t+1} x} g(m) \ll \\ &\ll \sum_{t=0}^{\infty} \frac{(\log 2^t x)^l}{\sqrt{2^t x}} (\log x)^{m_1 - 1} \zeta(2)^\nu \ll \frac{(\log x)^K}{\sqrt{x}} \end{aligned}$$

holds with a suitable large K, therefore

$$U_h(x) = \sum_{h=0}^{k-\nu-l-1} d_{h,l}\eta_l + O\left(x^{-1/2+\varepsilon}\right),$$

$\varepsilon > 0$ is an arbitrary small positive integer. Let us observe furthermore that in (1.1) $b_0(s) = 1$.

We proved the following

Lemma 3. *For* (2.1) *we have*

$$E(x \mid G, k, \nu) = x\tilde{Q}_{k-\nu-1}(\log x) + O\left(x^{\alpha_k+\varepsilon}\right).$$

From Lemma 3 the following assertion follows.

Theorem 1. *Let $k \geq 2$ be an arbitrary integer. Then*

$$S_k(x) = x\tilde{P}_{k-1}(\log x) + O\left(x^{\alpha_k+\varepsilon}\right),$$

where $\tilde{P}_{k-1}(y) = \pi_{k-1}\ y^{k-1} + \ldots + \pi_0$ is a polynomial, the leading coefficient π_{k-1} satisfies $\pi_{k-1} = \dfrac{1}{k!}$.

Remarks.

1. A.F. Lavrik [2] counted the coefficients of the polynomials Q_{k-1} in Lemma 2. By his method and by counting the coefficients of the expansions $b_\nu(s) = b_\nu^{(0)} + b_\nu^{(1)}(s-1) + \ldots$, one can determine the coefficients of $\tilde{P}_{k-1}$ in Theorem 1.

2. A.A. Karacuba [3] proved, by using the method of I.M. Vinogradov, that

$$\sum_{n \leq x} d_k(n) = xP_{k-1}(x_1) + O\left(x^{1-\frac{c}{k^{2/3}}+\varepsilon}\right)$$

uniformly as $k/x_2 \leq \varepsilon_x$, where $\varepsilon_x \to 0$ arbitrarily, $\varepsilon > 0$, $c > 0$, the constant implied by the error term is absolute, P_{k-1} is a polynomial of degree $k-1$, the leading term of which is 1.

By using his theorem we can deduce that Theorem 1 remains valid uniformly as $\dfrac{k}{x_2} \leq \varepsilon_x$, with $\alpha_k = 1 - \dfrac{c}{k^{2/3}}$.

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