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ON THE AVERAGE OF $d(n)\omega(n)$ AND SIMILAR FUNCTIONS ON SHORT INTERVALS

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1. Introduction

For each integer $n \geq 2$, let $\omega(n)$, $\Omega(n)$ and $\tau(n)$ stand for the number of distinct prime divisors of n, the number of prime divisors of n counting their multiplicity and the number of positive divisors of n, respectively, with $\omega(1) = \Omega(1) = 0$ and $\tau(1) = 1$. Given an integer $n \geq 2$, let $\beta(n)$ be the sum of the distinct prime divisors of n. Moreover, given any positive integer k and any complex number z, let

$$\tau_k(n) = \#\{(d_1, d_2, \ldots, d_k) : d_1 d_2 \ldots d_k = n,\ d_i \in \mathbb{N}\}.$$

Finally, let $x_1 = \log x,\ x_{j+1} = \log x_j$ for each integer $j \geq 1$.

It was shown by De Koninck and Ivić [1], using analytic methods, that as $x \to \infty$,

$$(1) \qquad \sum_{n \leq x} \tau(n)\omega(n) = 2xx_1x_2 + Axx_1 + O(x),$$

where

$$A = 2\left(\sum_p \left(\log\left(1 - \frac{1}{p}\right) + \left(\frac{1}{p} + \frac{3}{2p^2} + \frac{4}{2p^3} + \ldots\right)\left(1 - \frac{1}{p}\right)^2\right) - \Gamma'(2)\right),$$

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where Γ stands for the Gamma function.

This result was later improved by Sitaramachandrarao [10] who showed that

$$\sum_{n\le x}\tau(n)\omega(n)=2xx_1x_2+Axx_1+Bxx_2+Cx+O\left(\frac{x}{x_1}\right),$$

with explicit constants $B\neq 0$ and C. Observe that this means that the term C_2xx_2 is missing in (1).

Later Ivić [4] investigated asymptotic formulas for sums of the type $\sum_{n\le x} f(n)g(n)$, where f (resp. g) belong to certain classes of multiplicative (resp. additive) functions. He did so by considering the generating function

$$\sum_{n=1}^{\infty}\frac{f(n)z^{g(n)}}{n^s}, \tag{2}$$

and then applying the method of A. Selberg in order to compute the asymptotic expansion of $\sum_{n\le x} f(n)z^{g(n)}$. He could carry over this argument when $\sum_{n=1}^{\infty}\frac{f(n)z^{g(n)}}{n^s}$ was a product of $\zeta(s)^w$ and $A(s,w)$, where $A(s,w)$ is a function which is regular in $|s-1|<\varepsilon$. In fact, Ivić proved the following two results.

Theorem A. *Let $k\ge 2$ be fixed and $N>k$ be an arbitrary but fixed integer. Then there exist computable constants $a_{k,j},b_{k,j},c_{k,j}$ $(a_{k,1}\neq 0)$ such that*

$$\sum_{n\le x} d_k(n)\omega(n)=x\sum_{j=1}^{k}(a_{k,j}x_2+b_{k,j})x_1^{k-j}+x\sum_{j=k+1}^{N}c_{k,j}x_1^{k-j}+O\left(xx_1^{k-N-1}\right).$$

Theorem B. *Let $m,N\ge 1$ and $k\ge 2$ be fixed integers. Then there exist polynomials $P_{k,m,j}(t)$ $(j=1,\dots,N)$ of degree m in t with computable coefficients such that*

$$\sum_{n\le x} d_k(n)\omega^m(n)=x\sum_{j=1}^{N}P_{k,m,j}(x_2)x_1^{k-j}+O\left(xx_1^{k-N-1}x_2^m\right).$$

We shall provide here short interval versions of similar theorems. In particular, we shall prove the following results.

Theorem 1. *Let $x^{7/12+\varepsilon} \le h(x) \le x$. Then, for each fixed integer k and suitable constants $B_0, B_1, E_1, E_2, \dots, E_k$,*

$$\frac{1}{h(x)} \sum_{x \le n \le x+h(x)} \tau(n)\omega(n) =$$

$$= B_0(x_1x_2 - x_1) + B_1(x_2+1) + \sum_{\nu=1}^{k} E_\nu x_1^{1-\nu} + O\left(x_1^{-k}x_2\right).$$

In [2] De Koninck and Ivić proved that if $f(n) = \sum_{p|n} p^\rho L(p)$ for some $\rho > 0$, where $L(x)$ is a slowly oscillating function, and if $x^{7/12}\log^{22} x \le h(x) \le x$, then

$$\sum_{x \le n \le x+h(x)} f(n) = (\zeta(1+\rho) + o(1)) \frac{h(x)x^\rho L(x)}{\log x},$$

from which it follows, in particular, that

$$\frac{1}{h(x)} \sum_{x \le n \le x+h(x)} \frac{\beta(n)}{n} = (\zeta(1+\rho) + o(1)) \frac{1}{\log x}.$$

Here we show the following stronger result.

Theorem 2. *Let $x^{7/12+\varepsilon} \le h(x) \le x$. Then, for each fixed integer k and suitable constants $D_1, D_2, \dots, D_k$,*

$$\frac{1}{h(x)} \sum_{x \le n \le x+h(x)} \frac{\beta(n)}{n} = \frac{D_1}{x_1} + \frac{D_2}{x_1^2} + \ldots + \frac{D_k}{x_1^k} + O\left(\frac{1}{x_1^{k+1}}\right).$$

2. The Hooley-Huxley contour and Ramachandra's theorem

In 1976 K. Ramachandra [8] obtained short interval mean value theorems for those arithmetical functions such that the corresponding Dirichlet series may be written as finite products of powers of L-functions multiplied by the product of finitely many $\log L(s,\chi)$ functions and a certain function regular in $\Re(s) > \frac{1}{2}$.

For our results, the main idea of the proof is to choose an appropriate line of integration in the Perron formula, namely the so-called Hooley-Huxley contour. To do so, first let S_1, S_2 and S_3 be the set of L-series, the set of their derivatives and the set of their logarithms, respectively. Observe that $\log L(s,\chi)$ is defined by analytic continuation from the halfplane $\sigma = \Re(s) > 1$; for each complex number z, we define

$$L(x,\chi)^z = \exp\{z \log L(s,\chi)\}.$$

Let $P_1(s)$ be any finite power product, with complex exponents, of functions of S_1, and let $P_2(s)$ (resp. $P_3(s)$) be any finite power product, with non-negative integer exponents, of functions of S_2 (resp. S_3). Moreover, let c_n be a sequence of complex numbers such that $|c_n| \ll n^\varepsilon$ for every $\varepsilon > 0$ and

$$\sum_{n=1}^{\infty} \frac{|c_n|}{n^\sigma} < +\infty \quad \text{for} \quad \sigma > \frac{1}{2}.$$

Let also $F_0(s) = \sum_{n=1}^{\infty} \frac{c_n}{n^s}$ and define the sequence $g_1, g_2, \ldots$ implicitly by

$$F_1(s) := P_1(s)P_2(s)P_3(s)F_0(s) = \sum_{n=1}^{\infty} \frac{g_n}{n^s}$$

and set

$$E(x) = \sum_{n \le x} g_n.$$

Given a positive number $r \le \frac{1}{2}$, we define the contour C_r by first considering the circle $\{s : |s-1| = r\}$, removing the point $1-r$, and proceeding on the remaining portion of the circle in the anticlockwise direction. Set $C_0 = C(r)$. Assume that r is small enough so that $F_1(s)$ has no singularities on the boundary and in the interior of C_0, except possibly at the point $s = 1$.

Let $C_1 = C\left(\frac{1}{\log x}\right)$, and let L^-, L^+ be defined as the intervals on straight lines

$$L^- = \left[\left(1 - \frac{1}{r}\right)e^{-i\pi}, \ \left(1 - \frac{1}{\log x}\right)e^{-i\pi}\right],$$
$$L^+ = \left[\left(1 - \frac{1}{\log x}\right)e^{i\pi}, \ \left(1 - \frac{1}{r}\right)e^{i\pi}\right].$$

Let C^* be the contour going along L^- starting from $\left(1-\frac{1}{r}\right)e^{-i\pi}$, then on C_1, and finally on L^+.

Let B be the constant appearing in the well known density result

$$N_\chi(\alpha,T) := \#\{\rho=\beta+i\gamma : L(\rho,\chi)=0,\ \text{with } \beta\ge\alpha\ge 0 \text{ and } |\gamma|\le T\} = \\ = O\left(T^{B(1-\alpha)}\log^2 T\right),$$

which is valid for all characters χ occuring in P_1, P_2 and P_3. Letting $\varphi = 1-\frac{1}{B}-\varepsilon$, with an arbitrary $\varepsilon>0$, Ramachandra [8] proved the following result.

Theorem (Ramachandra). *Let x be a large number and $1\le h(x)\le x$. Set*

$$I(x,h(x)) = \frac{1}{2\pi i}\int_0^{h(x)}\left(\int_{C_0} F_1(s)(v+x)^{s-1}ds\right)dv.$$

Then

$$E(x+h(x))-E(x) = I(x,h(x)) + O_\varepsilon\left(h(x)\cdot\exp\{-(\log x)^{1/6}\cdot x^{\varphi}\}\right).$$

Remark. According to Huxley's result [3], the number φ may be replaced by any constant greater than 7/12.

Ramachandra used the Hooley-Huxley contour in order to prove his very general theorem. Later on, Kátai [5] applied Ramachandra's theorem to obtain that

$$\sum_{\substack{x\le n\le x+h(x)\\ \omega(n)=k}} 1 = (1+o(1))\frac{h(x)\cdot x_2^{k-1}}{(k-1)!x_1}$$

holds uniformly for $k\le x_2+c_x\sqrt{x_2}$, where c_x tends to $+\infty$ sufficiently slowly, and $h(x)\ge x^{\varphi+\varepsilon}$.

3. The proof of Theorem 1

Since

$$\sum_{m=1}^{\infty}\frac{z^{\omega(m)}}{m^s} = \zeta(s)^z G(s,z) \quad\text{and}\quad \sum_{m=1}^{\infty}\frac{z^{\Omega(m)}}{m^s} = \zeta(s)^z F(s,z),$$

where the functions $G(s,z)$ and $F(s,z)$ are regular in $\sigma > \frac{1}{2}$, it follows that the above Dirichlet series belong to the classes of functions satisfying Ramachandra's theorem. Kátai and Subbarao [6] used this to obtain asymptotic estimates of the expressions

$$(3) \qquad \sum_{x \le n \le x+h(x)} z^{\omega(n)}, \qquad \sum_{x \le n \le x+h(x)} z^{\omega(n)}|\mu(n)|, \qquad \sum_{x \le n \le x+h(x)} 1/t_k(n),$$

where $h(x) = x^{7/12+\varepsilon}$. More generally, they proved that, assuming that $F(s)$ satisfies the conditions of Ramachandra's theorem, that $r > 0$ and $\varepsilon > 0$ are small numbers, that $x^{\frac{7}{12}+\varepsilon} \le h(x) \le x^{\frac{2}{3}-\frac{2r}{3}}$, and that $E(x) = \sum_{n \le x} f(n)$ where $f(n) = z^{\omega(n)}$ or $z^{\omega(n)}|\mu(n)|$ or $1/\tau_k(n)$, then

$$\frac{E(x+h(x)) - E(x)}{h(x)} = \frac{1}{2\pi i} \int_{C^*} F(s) x^{s-1} ds + O\left(\exp\left\{-x_1^{1/6}\right\}\right),$$

where $C^* = \{s : |s-1| = 1/x_1,\ s \ne 1 - 1/x_1\}$.

Remark. Observe that the reason for the upper bound $x^{\frac{2}{3}-\frac{2r}{3}}$ on $h(x)$ is only due to a technical condition used in the proof; indeed one can show that the result is in fact valid for $x^{\frac{7}{12}+\varepsilon} \le h(x) \le x$.

Returning to the proof of Theorem 1, we let

$$F(s) = \sum_{n=1}^{\infty} \frac{\tau(n)\omega(n)}{n^s}.$$

One easily verifies that

$$F(s) = \sum_p \sum_{\alpha=1}^{\infty} \frac{\tau(p^\alpha)}{p^{\alpha s}} \sum_{\substack{m=1 \\ (m,p)=1}}^{\infty} \frac{\tau(m)}{m^s} = \zeta^2(s) \sum_p \left(1 - \frac{1}{p^s}\right)^2 \sum_{\alpha=1}^{\infty} \frac{\alpha+1}{p^{\alpha s}} =$$

$$= \zeta^2(s) \sum_p \left(1 - \frac{1}{p^s}\right)^2 \left\{\frac{1}{(1-1/p^s)^2} - 1\right\} = \zeta^2(s) \sum_p \left\{\frac{2}{p^s} - \frac{1}{p^{2s}}\right\}.$$

Since $\sum_p \frac{1}{p^s} = \log\zeta(s) - \sum_{r=2}^{\infty} \frac{1}{r} \sum_p \frac{1}{p^{rs}}$, it follows that

$$2\sum_p \frac{1}{p^s} - \sum_p \frac{1}{p^{2s}} = 2\log\zeta(s) - 2\sum_p \frac{1}{p^{2s}} - \sum_{r=3}^{\infty} \frac{2}{r} \sum_p \frac{1}{p^{rs}} = 2\log\zeta(s) - U(s),$$

say, where $U(s)$ is regular for $\Re(s) > \frac{1}{2}$, so that we may write

$$F(s) = F_1(s) - F_2(s), \quad \text{with } F_1(s) = \zeta^2(s) \cdot 2\log\zeta(s) \text{ and } F_2(s) = \zeta^2(s)U(s).$$

Now define $\alpha_1(n)$ and $\alpha_2(n)$ implicitly by the representations

$$F_1(s) = \sum_{n=1}^{\infty} \frac{\alpha_1(n)}{n^s} \quad \text{and} \quad F_2(s) = \sum_{n=1}^{\infty} \frac{\alpha_2(n)}{n^s}.$$

Clearly both $F_1(s)$ and $F_2(s)$ belong to Ramachandra's class of functions. It follows that

$$\frac{1}{h(x)} \sum_{x \le n \le x+h(x)} \alpha_1(n) = \frac{1}{2\pi i} \int_{C^*} F_1(s) x^{s-1} ds + O\left(\exp\left\{-x_1^{1/6}\right\}\right).$$

Now we can write

$$F_1(s) = (s-1)^2\zeta^2(s) \cdot \frac{1}{(s-1)^2} \cdot 2\left\{\log((s-1)\zeta(s-1)) + \log\frac{1}{s-1}\right\} =$$
$$= \frac{2\log\frac{1}{s-1}}{(s-1)^2}\big((s-1)^2\zeta(s-1)\big) + \frac{2}{(s-1)^2}\big((s-1)^2\zeta(s)\big)\log((s-1)\zeta(s)).$$

But since $(s-1)\zeta(s) \to 1$ as $s \to 1$, it follows that $(s-1)\zeta(s)$ and $\log((s-1)\zeta(s))$ are regular in the neighbourhood of 1.

Now define the constants $B_0, B_1, \ldots, B_k,\ C_0, C_1, \ldots, C_k$ and the functions $U_k(s)$ and $V_k(s)$ implicitly by the relations

$$2(\zeta(s)(s-1))^2 = B_0 + B_1(s-1) + \ldots + B_k(s-1)^k + U_k(s)(s-1)^{k+1},$$

$$2(\zeta(s)(s-1))^2 \log(\zeta(s)(s-1)) =$$
$$= C_0 + C_1(s-1) + \ldots + C_k(s-1)^k + V_k(s)(s-1)^{k+1},$$

so that $U_k(s)$ and $V_k(s)$ are regular and bounded for $|s-1| \le 1/x_1$. Thus

$$\frac{1}{2\pi i} \int_{C^*} F_1(s) x^{s-1} ds = \sum_{\nu=0}^{k} B_\nu \cdot \mathcal{I}_\nu + L_1 + \sum_{\nu=0}^{k} C_\nu \mathcal{J}_\nu + L_2,$$

where

$$\mathcal{I}_\nu = \frac{1}{2\pi i}\int\limits_{C^*} \frac{x^{s-1}}{(s-1)^{2-\nu}} \log\frac{1}{s-1} ds \quad (0\le \nu\le k),$$

$$\mathcal{J}_\nu = \frac{1}{2\pi i}\int\limits_{C^*} \frac{x^{s-1}}{(s-1)^{2-\nu}} ds \qquad (0\le \nu\le k),$$

$$L_1 = \frac{1}{2\pi i}\int\limits_{C^*} \frac{x^{s-1}}{(s-1)^{2-(k+1)}} U_k(s)\log(s-1) ds,$$

$$L_2 = \frac{1}{2\pi i}\int\limits_{C^*} \frac{x^{s-1}}{(s-1)^{2-(k+1)}} V_k(s)\log(s-1) ds.$$

Since $s = 1+\dfrac{e^{i\theta}}{x_1}$, $ds = \dfrac{1}{x_1} i e^{i\theta}$, $\log(s-1) = \log(1/x_1) + i\theta$ and $\log\dfrac{1}{s-1} = $ $= x_2 - i\theta$, it follows that, taking into account that $x^{e^{i\theta}\cdot x_1^{-1}} = e^{e^{i\theta}}$, we have

$$\mathcal{I}_\nu = \frac{x_1^{2-\nu}}{2\pi x_1}\int\limits_{-\pi}^{\pi} e^{e^{i\theta}} e^{(\nu-2)i\theta}(x_2 - i\theta)e^{i\theta} d\theta =$$

$$= \frac{x_1^{1-\nu}\cdot x_2}{2\pi}\int\limits_{-\pi}^{\pi} e^{e^{i\theta}}\cdot e^{i(\nu-1)\theta} d\theta - \frac{i x_1^{1-\nu}}{2\pi}\int\limits_{-\pi}^{\pi} e^{e^{i\theta}}\cdot e^{i(\nu-1)\theta}\theta d\theta.$$

Define $\eta_0 = 0$ and observe that, for $h\ne 0$, we have

$$\eta_h := \frac{1}{2\pi}\int\limits_{-\pi}^{\pi} e^{ih\theta}\theta d\theta = \frac{1}{2\pi}\int\limits_{-\pi}^{\pi}\left(\frac{e^{ih\theta}}{ih}\right)'\theta d\theta =$$

$$= \frac{1}{2\pi}\left[\frac{e^{ih\theta}}{ih}\theta\right]_{-\pi}^{\pi} - \frac{1}{2\pi i h}\int\limits_{-\pi}^{\pi} e^{ih\theta} d\theta =$$

$$= \frac{1}{2\pi i h}\left[\pi e^{ih\theta} + \pi e^{-ih\theta}\right] =$$

$$= \frac{(-1)^h}{ih}.$$

Therefore, using the representation $e^{e^{i\theta}} = \sum_{t=0}^{\infty}\dfrac{1}{t!}e^{it\theta}$, it follows from this that

$$\frac{1}{2\pi}\int\limits_{-\pi}^{\pi} e^{e^{i\theta}} e^{i(\nu-1)\theta} d\theta = \begin{cases} 0 & \text{if } \nu\ge 2,\\ 1 & \text{if } \nu = 0,1,\end{cases} \tag{4}$$

and that

$$\frac{1}{2\pi}\int\limits_{-\pi}^{\pi} e^{e^{i\theta}} e^{i(\nu-1)\theta}\theta d\theta = \sum_{t=0}^{\infty}\frac{1}{t!}\frac{1}{2\pi}\int\limits_{-\pi}^{\pi} e^{i(t+\nu-1)\theta}\theta d\theta = \frac{1}{i}\sum_{\substack{t=0\\ t\neq 1-\nu}}^{\infty}\frac{1}{t!}\frac{(-1)^{t+\nu-1}}{t+\nu-1}.$$

Gathering these estimates, we obtain that

$$\mathcal{I}_0 = x_1x_2 - x_1, \quad \mathcal{I}_1 = x_2+1, \quad \mathcal{I}_\nu = -x_1^{1-\nu}\mathcal{D}_\nu \quad \text{for } \nu\geq 2,$$

where each $\mathcal{D}_\nu$ is a computable constant.

On the other hand, it follows from (4) that

$$\mathcal{I}_\nu = \frac{x_1^{1-\nu}}{2\pi}\int\limits_{-\pi}^{\pi} e^{e^{i\theta}} e^{i(\nu-1)} d\theta = \begin{cases} x_1 & \text{if } \nu=0,\\ 1 & \text{if } \nu=1,\\ 0 & \text{if } \nu\geq 2.\end{cases}$$

Moreover, we have

$$|L_1| \leq x_1^{-k}\int\limits_{-\pi}^{\pi}\left|U_k\left(1+\frac{1}{x_1}e^{i\theta}\right)\right|\cdot(x_2+|\theta|)d\theta \ll x_2\cdot x_1^{-k}$$

and one can also easily establish that

$$|L_2| \ll x_1^{-k}.$$

Combining these estimates, the proof of Theorem 1 is thus complete.

Remark 1. From the Ramachandra's theorem, it follows that under the assumption

$$N(\sigma,T) \ll T^{(2+\varepsilon)(1-\sigma)},$$

which is somewhat weaker than the Riemann hypothesis, Theorem 1 holds for the shorter interval $x^{\frac{1}{2}+\varepsilon} \leq h(x) \leq x$.

Remark 2. It is clear from the proof of Theorem 1 that similar estimates can be obtained for the following sums:

$$\sum_{x\leq n\leq x+h(x)}\tau_k(n)\omega(n), \qquad \sum_{x\leq n\leq x+h(x)}\tau_k(n)\Omega(n), \qquad \sum_{x\leq n\leq x+h(x)} r(n)\omega(n),$$

where $r(n) = \#\{(u,v) : n = u^2+v^2\}$.

4. The proof of Theorem 2

We begin by writing

$$F(s) := \sum_{n=1}^{\infty} \frac{\beta(n)}{n} = \left[\sum_p \left\{\sum_{\alpha=1}^{\infty} \frac{p}{p^{\alpha s}}\right\}\left(1-\frac{1}{p^s}\right)\right]\zeta(s) =$$

$$= \zeta(s)\left\{\sum_p \frac{1}{p^{s-1}}\right\} = \{\log\zeta(s-1)+u(s)\}\zeta(s),$$

say, with $u(s)$ bounded and regular for $\Re(s) > 2$. Thus

$$F(s) = \left(\log\frac{1}{s-2}\right)\zeta(s) + \zeta(s)[\log((s-1)\zeta(s-1)) + u(s)] =$$

$$= \left(\log\frac{1}{s-2}\right)\zeta(s) + \zeta(s)v(s),$$

say, with $v(s)$ regular for $\Re(s) > 2$. It follows that

$$F(s+1) = \sum_{n=1}^{\infty} \frac{\beta(n)/n}{n^s} = \zeta(s+1)\log\frac{1}{s-1} + \zeta(s+1)v(s+1).$$

Now observe that

$$\frac{1}{h(x)} \sum_{x\le n\le x+h(x)} \frac{\beta(n)}{n} =$$

$$= \frac{1}{x}\cdot\frac{1}{h(x)} \sum_{x\le n\le x+h(x)} \beta(n) - \frac{1}{h(x)} \sum_{x\le n\le x+h(x)} \beta(n)\left(\frac{1}{n}-\frac{1}{x}\right) =$$

$$= \frac{1}{x}\cdot\frac{1}{h(x)} \sum_{x\le n\le x+h(x)} \beta(n) + O\left(\frac{1}{x}\sum_{x\le n\le x+h(n)} \frac{\beta(n)}{n}\right).$$

Hence, proceeding as in the proof of Theorem 1, we get that

$$\frac{1}{h(x)} \sum_{x\le n\le x+h(x)} \frac{\beta(n)}{n} = \frac{1}{2\pi i}\int_{C^*} F(s+1)x^{s-1}ds + O\left(\exp\left\{-x_1^{1/6}\right)\right. =$$

$$= \frac{1}{2\pi i}\int_{C^*} x^{s-1}\zeta(s+1)\log\frac{1}{s-1}ds =$$

$$= \sum_{\nu=0}^{\infty} a_\nu T_\nu,$$

where the a_ν's are defined implicitly by

$$\zeta(s+1) = \zeta(2) + \zeta'(2)(s-1) + \ldots = a_0 + a_1(s-1) + \ldots$$

and where the T_ν's can be written as

$$T_\nu = \frac{1}{2\pi i} \int_{C^*} (s-1)^\nu x^{s-1} \log \frac{1}{s-1} ds =$$

$$= \frac{1}{2\pi} \frac{1}{x_1^{\nu+1}} \int_{-\pi}^{\pi} e^{i(\nu+1)\theta} e^{e^{i\theta}} (x_2 - i\theta) d\theta.$$

Collecting the above estimates completes the proof of Theorem 2.

References

[1] **DeKoninck J.-M. and Ivić A.,**, *Topics in arithmetical functions*, North-Holland, Amsterdam, 1980.

[2] **DeKoninck J.-M. and Ivić A.,** The average prime divisor of an integer in short intervals, *Arch. Math.,* **52** (1989), 440-448.

[3] **Huxley M.N.,** On the difference between consecutive primes, *Invent. Math.,* **15** (1972), 164-170.

[4] **Ivić A.,** Sums of products of certain arithmetical functions, *Publ. Inst. Math. (Beograd) N.S.,* **41** (55) (1987), 31-41.

[5] **Kátai I.,** A remark on a paper of K. Ramachandra, *Number theory. Proc. Ootacamund,* ed. K. Alladi, Lecture Notes in Math. **1122**, Springer, 1984, 147-152.

[6] **Kátai I. and Subbarao M.V.,** Some remarks on a paper of Ramachandra, *Lietuvos mat. rink.,* **43** (4) (2003), 497-506.

[7] **Левин Б.В. и Тимофеев Н.М.,** Распределение арифметических функций в среднем по прогрессиям (теоремы типа Виноградова-Бомбьери), *Мат. сборник,* **125** (**167**) (1984) (4) (12), 558-572. (**Levin B.V. and Timofeev N.M.,** Distribution of arithmetical functions in the mean over progressions (theorems of A.I. Vinogradov - E. Bombieri type), *Math. USSR Sbornik,* **53** (1986), 563-579. (English version))

[8] **Ramachandra K.,** Some problems of analytic number theory, *Acta Arith.,* **31** (1976), 313-324.

[9] **Selberg A.,** Note on a paper by L.G. Sathe, *J. Indian Math. Soc.,* **18** (1954), 83-87.

[10] **Sitaramachandrarao R.,** *Letter to Jean-Marie De Koninck dated March 3, 1986.*

[11] **Wolke D.,** Über die mittlere Verteilung der Werte Zahlentheoretischer Funktionen auf Restklassen I., *Math. Ann.,* **202** (1973), 1-25.

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