Annales Univ. Sci. Budapest., Sect. Comp. 25 (2005) 113-130

SOME REMARKS ON THE φ AND ON THE σ FUNCTIONS

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Abstract. Some theorems are proved for the functions $n-\varphi(n)$, $\sigma(n)-$ $-n$, $\varphi_k(n)$, $\sigma_k(n)$, where $\varphi(n)$ is Euler's totient function, $\sigma(n)$ is the sum of divisors function, φ_k and σ_k are the k'th iterate of φ and σ.

1. Let $\varphi(n)$ be Euler's totient function, and $\sigma(n)$ be the sum of divisors of n.

We shall use the following notation: $\omega(n)$ = number of distinct prime factors of n; $\varphi_k(n) = k$-fold iterate of $\varphi(n)$, $\sigma_k(n) = k$-fold iterate of $\sigma(n)$;

$$\psi(n) := n - \varphi(n); \quad \rho(n) := \sigma(n) - n;$$

$\mathcal{P}$ = set of primes. p, q with or without suffixes always denote prime numbers.

Furthermore we shall write $x_1 := \log x$, $x_2 = \log x_1, \dots$.

W. Sierpinski asked in 1959 ([12], pp. 200-201) whether there exist infinitely many positive integers not of the form $\psi(n)$. J. Browkin and A. Schinzel [15] proved that none of the numbers $2^k \cdot 509203$ $(k = 1, 2, \dots)$ belong to $\psi(\mathbb{N})$. Erdős proved earlier in [14] that there are infinitely many integers not of the form $\rho(n)$.

The second named author asked whether are there infinitely many n for which $\omega(n) = k$ = fixed and $\psi(n)$ = prime, or not. If $n = p$, then $\psi(p) = 1$, if

The first author supported by the Hungarian National Foundation for Scientific Research under grant OTKA T046993 and the fund of the Applied Number Theory Research Group of the Hungarian Academy of Sciences and a grant from NSERC.

The second author supported by a grant from NSERC.

$n=p^2$, then $\psi(p^2)=p$. If $n=pq,\ p\neq q$, then $\psi(n)=p+q-1$ which can be prime for appropriate choices of p,q.

By using Vinogradov's method for the odd Goldbach problem one can get the following

Lemma 1. *Let* $\varepsilon_x\to 0$, *slowly*, $E_x:=e^{x_1^{\varepsilon_x}},\ E_x\le U<V<x,\ \Delta U=$ $=\dfrac{U}{(\log U)^\kappa},\ \Delta V=\dfrac{V}{(\log V)^\kappa}$, *where* κ *is a large constant, and let*

(1.1)
$$M[U,\Delta U;\ V,\Delta V]:=\#\{p+q-1\in\mathcal{P},\ p\in[U,U+\Delta U],\ q\in[V,V+\Delta V]\}.$$

Then

(1.2)
$$M(U,\Delta U;\ V,\Delta V)=\frac{c\pi([U,U+\Delta U])\pi([V,V+\Delta V])}{\log(U+V)}\left(1+O\left(\frac{1}{\log 4}\right)\right),$$

where c *is an absolute positive constant.*

We omit the proof.

Hence one can deduce the following

Theorem 1. *Let* $M(x)$ *be the number of those* $n\le x$, *for which* $\omega(n)=$ $=2,\ \psi(n)\in\mathcal{P}$ *holds. Then*

$$M(x)=c\frac{x}{x_1^2}x_2(1+o_x(1)). \tag{1.3}$$

Proof. Let $U_0=E_x,\ U_{j+1}=U_j+\Delta U_j,\ V_j=U_j$. Let us estimate those prime tuples p,q for which

$$pq<x,\ p<E_x,\ p+q-1\in\mathcal{P}. \tag{1.4}$$

By [1], Corollary 2.4.1, we obtain that for every fixed p, the number of those $q<\dfrac{x}{p}$ for which $q\in\mathcal{P},\ (p-1)+q\in\mathcal{P}$, is less than

$$c_i\frac{(p-1)}{\varphi(p-1)}\frac{x/p}{\log^2 x/p},$$

which by summing up to $p<x^{\varepsilon_x}$, is less than

$$c_2\frac{x}{x_1^2}\log\log E_x=O\left(\frac{x}{x_1^2}\varepsilon_x\cdot x_2\right). \tag{1.5}$$

Let

$$A_1 = \sum_{\substack{i\le j\\ U_iV_j<x}} M(U_i, U_i+\Delta U_i;\ V_j, V_j+\Delta V_j), \tag{1.6}$$

$$A_2 = \sum_{\substack{i\le j\\ (U_i+\Delta U_i)(V_i+\Delta V_i)<x}} M(U_i, U_i+\Delta U_i;\ V_j+\Delta V_j). \tag{1.7}$$

The difference $A_1 - A_2$ is clearly less than the number of those $p, q \in \mathcal{P}$ for which $p+q-1 \in \mathcal{P}$, and

$$p \in [U_i, U_i+\Delta U_i], \quad q \in [V_j, V_j+\Delta V_j], \tag{1.8}$$

for such choices of i, j for which

$$U_iV_j < x,\ (U_i+\Delta U_i)(V_j+\Delta V_j) > x. \tag{1.9}$$

Let i, j be fixed, so that (1.9) is satisfied. If p, q is such a couple for which (1.8) holds, then

$$\frac{x - c_2x}{(\log U_i)^\kappa} < pq < x + \frac{c_2x}{(\log U_i)^\kappa}, \tag{1.10}$$

and we have to estimate those $q \in \mathcal{P}$, for which

$$\frac{x}{p} - \frac{c_2x}{p(\log U_i)^\kappa} < q < \frac{x}{p} + c_2\frac{x}{p(\log U_i)^\kappa},$$

$q+p-1 \in \mathcal{P}$ holds, then sum over $p \in (U_i, U_i+\Delta U_i)$.

Then, by sieve ([1], Corollary 2.4.1) this is less than

$$\sum_{p\in[U_i,U_i+\Delta U_i]} \frac{x}{\varphi(p)(\log U_i)^\kappa(\log x)^2} \ll \frac{x}{x_1^2}\cdot\frac{1}{(\log U_i)^{2\kappa}}. \tag{1.11}$$

Furthermore

$$\sum_{x>n>E_x} \frac{1}{n(\log n)^\kappa} \gg \sum_{n\in[U_i,U_i+\Delta U_i]} \frac{1}{U_i(\log U_i)^\kappa}(\Delta U_i) = \sum \frac{1}{(\log U_i)^{2\kappa}}, \tag{1.12}$$

and the right hand side is bounded in x, therefore

$$A_1 - A_2 = O\left(\frac{x}{x_1^2}\right).$$

To complete the proof of Theorem 1, it remains to apply Lemma 1 to (1.6). The proof is completed.

Similar theorem can be proved for $\rho(n)$.

2. Let m be such an integer for which $(m, \varphi(m)) = 1$. Let $p \in \mathcal{P}$, $(p, m) = 1$. Then

$$\psi(mp) = mp - (p-1)\varphi(m) = p(m - \varphi(m)) + \varphi(m). \tag{2.1}$$

Remark. According to the Hardy-Littlewood conjecture, if $(A, B) = 1$, $A, B > 0$, then in the set

$$\{Ap + B \mid p \in \mathcal{P}\},$$

there exist infinitely many primes.

Erdős [2] investigated the set of those m for which $(m, \varphi(m)) = 1$. He proved that the set $\{m \mid (m, \varphi(m)) = 1,\ m \le x\}$ is almost the same as $\{m \mid m \le x,\ p(m) > x_2\}$ and proved that

$$\#\{m \le x \mid (m, \varphi(m)) = 1\} = (1 + o_x(1))x \prod_{p < x_2}\left(1 - \frac{1}{p}\right).$$

One can prove easily that for every fixed $k \ge 2$ there exist infinitely many m with $\omega(m) = k$ for which $(m, \varphi(m)) = 1$. Consequently, if the Hardy-Littlewood conjecture holds, then for every fixed integer $k \ge 2$ there exist infinitely many n for which $\psi(n) =$ prime, $\omega(n) = k + 1$.

Similar assertion can be proved for $\sigma(n)$.

3. As we mentioned earlier, Erdős [2] proved that

$$x^{-1}\#\{n \le x,\ (n, \varphi(n)) = 1\} = (1 + o_x(1)) \prod_{p < x_2} (1 - 1/p).$$

We shall investigate the set

$$\mathcal{B}_{k+1} = \{n \le x,\ (n, \varphi_{k+1}(n)) = 1\}. \tag{3.1}$$

For $k \geq 1$ let $w_k(x) = \prod_{p<x_2^k} \left(1 - \frac{1}{p}\right)$.

For a fixed prime Q let $\kappa_0, \kappa_1, \dots$ be a sequence of completely additive functions defined for primes p as follows:

$$\kappa_0(p) = \begin{cases} 1 & \text{if } p = Q \\ \\ 0 & \text{if } p \neq Q \end{cases}, \quad \kappa_{j+1}(p) = \sum_{\substack{q\in\mathcal{P} \\ q|p-1}} \kappa_j(q).$$

Let

$$\rho_k(Q)(= \rho_k(Q|x)) = \prod_{\substack{p<x \\ \kappa_{k+1}(p)\neq 0 \\ p\in\mathcal{P}}} (1 - 1/p)\,. \tag{3.2}$$

To emphasize the value Q on which κ_j depend, we write $\kappa_j(p|Q)$ instead of $\kappa_j(p)$.

Let

$$N_k(Q|x) = \#\{n \leq x \mid Q \not| \ \varphi_{k+1}(n)\}. \tag{3.3}$$

In a paper of Indlekofer and Kátai [3] the following two theorems are proved, which will be quoted now as Lemma 2 and Lemma 3.

Lemma 2. *Let $x_3x_2 \leq Q \leq x_2^2$. Then*

$$N_1(Q|x) = x\rho_1(Q)\left(1 + O\left(\frac{x_2x_3}{Q}\right)\right),$$
$$\log\frac{1}{\rho_1(Q)} = \frac{x_2^2}{2Q} + O\left(\frac{x_2^3}{Q^2} + \frac{x_2\log Q}{Q}\right).$$

Lemma 3. *Let $\varepsilon > 0$, $k \geq 2$ be fixed, $x_2^{k+\varepsilon} \leq Q \leq x_2^{k+1-\varepsilon}$. Then*

$$N_k(Q|x) = \rho_k(Q)x\left(1 + O\left(\frac{1}{x_2}\right)\right),$$

and, moreover

$$\log\frac{1}{\rho_k(Q)} = A_{k+1}(x) + O\left(\frac{1}{Q}\right) + O\left(\frac{x_2^{2k+1}}{Q^2}\right),$$
$$A_{k+1}(x) = \frac{x_2^{k+1}}{(k+1)!(Q-1)} + O\left(\frac{x_2^{k+\varepsilon/2}}{Q}\right).$$

We shall say that $p_0, p_1, \dots, p_h$ is a chain of primes if $p_{j+1} - 1 \equiv 0 \pmod{p_j}$ $(j = 0, \dots, h-1)$ holds.

We shall give an upper estimate for $N_k(Q|x)$ for those Q which satisfy the conditions given in the Lemmas 2 and 3.

Let $M_k(Q, x)$ be the number of those $n \le x$, for which $p(n) = Q$, and $Q \nmid \varphi_{k+1}(n)$. Let $Q, q_0, \dots, q_k$ be an arbitrary chain of primes, i.e. $q_{k-1} | q_k - -1, \dots, q_0 \mid q_1 - 1,\ Q | q_0 - 1$. If $Q \nmid \varphi_{k+1}(n)$, then clearly $(q_k, n) = 1$. Hence we obtain that

$$M_k(Q, x) \le \frac{cx}{Q} \prod_{p<Q} \left(1 - \frac{1}{p}\right) \cdot \rho_k(Q), \tag{3.4}$$

where $\rho_k(Q)$ is defined in (3.2).

Let us estimate $\rho_k(Q)$. Let $\tilde{\kappa}_j$ be a truncation of κ_j, more exactly let $\kappa_0 = \tilde{\kappa}_0$, and

$$\tilde{\kappa}_{j+1}(p) = \sum_{\substack{q|p-1 \\ q<p^{1/6}}} \tilde{\kappa}_j(q). \tag{3.5}$$

Let

$$A_V = \sum_{q<V} \frac{\tilde{\kappa}_k(q)}{q-1}, \quad B_V^2 = \sum_{q<V} \frac{\tilde{\kappa}_k^2(q)}{q-1}.$$

From the Bombieri-Vinogradov inequality (see e.g. in [10]) one can deduce the following Turán-Kubilius type inequality:

$$\sum_{p \in [V, 2V]} (\kappa_{k+1}(p) - A_V)^2 \ll \frac{V}{\log V} \cdot B_V^2 + O\left(\frac{V}{(\log V)^D}\right), \tag{3.6}$$

where D is an arbitrary large constant.

Assume that $k = 1$. Then $A_V = B_V^2$, and so

$$\#\{p \in [V, 2V],\ \kappa_2(p) = 0\} \le c\frac{V}{(\log V)} \cdot \frac{1}{A_V} + O\left(\frac{V}{(\log V)^D A_V}\right). \tag{3.7}$$

Furthermore

$$A_V = \sum_{q<(2V)^{1/6}} \frac{\tilde{\kappa}_1(q)}{q-1} = \sum_{\substack{Q<q<(2V)^{1/6} \\ q-1\equiv 0 \pmod{Q}}} \frac{1}{q-1}.$$

Assume that $V^{1/6} \ge \exp(Q^{1/T})$, where T is an arbitrary fixed number. Then, from the Siegel-Walfisz theorem we obtain that

$$A_V \ge \frac{1}{(Q-1)} \int\limits_{\exp(Q^{1/T})}^{(2V)^{1/6}} \frac{1}{u \log u} du - c_1,$$

with an absolute constant c_1. Then

$$A_V \ge \frac{1}{(Q-1)} \left\{ \log\log(2V)^{1/6} - \frac{1}{T} \log Q \right\} - c_1. \tag{3.8}$$

Let $Q \le x_2/x_3$, V_0 be defined so that

$$\frac{\log\log(2V_0)^{1/6}}{Q-1} = \frac{1}{\varepsilon_1},$$

where ε_1 is an arbitrary (small) positive constant. Then

$$A_V \ge \frac{1}{2\varepsilon_1}, \tag{3.9}$$

if ε_1 is small enough.

Thus

$$\sum_{\substack{V_0 < p < x \\ \tilde{\kappa}_2(p) \ne 0}} 1/p = x_2 - \log\log V_0 - \sum_{\tilde{\kappa}_2(p)=0} 1/p + O(1),$$

and

$$\sum_{\substack{\tilde{\kappa}_2(p)=0 \\ V_0 < p < x}} 1/p \le 2c\,\varepsilon_1 (x_2 - \log\log V_0),$$

consequently

$$w_2(x) \le \exp(-(1-\varepsilon_2)x_2) \tag{3.10}$$

with an arbitrary $\varepsilon_2 > 0$.

We proved the following

Lemma 4. *Let $Q \le x_2/x_3$, ε_2 be an arbitrary positive number. Then*

$$M_1(Q,x) \le \frac{cx}{Q \log Q} \exp(-(1-\varepsilon_2)x_2). \tag{3.11}$$

Assume now that $k \geq 2$, and that $Q \leq x_2^{k-\varepsilon}$, where ε is an arbitrary small positive constant.

For integers l and m let

$$\delta(x,m,l) = \sum_{\substack{p \leq x \\ p \equiv l \pmod{m}}} 1/p.$$

Lemma 5. *For $l = 1$ or -1 and $m \leq x$, $x \geq 3$ we have*

$$\delta(x,m,l) \leq \frac{c_1 x_2}{\varphi(m)}, \tag{3.12}$$

where c_1 is an absolute constant.

For $l = 1$, this is Eq. (3.1) of [4]. The proof of (3.1) in $l = -1$ is the same, so we omit.

Let

$$A_j(y) = \sum_{p \leq y} \frac{\kappa_j(p)}{p}, \quad D_j^2(y) = \sum_{p \leq y} \frac{\kappa_j^2(p)}{p}. \tag{3.13}$$

In [4] the following assertion has been proved.

Lemma 6. *With some constant c_j, for $z > e^2$, we have*

$$A_j(z) < c_j \frac{(\log\log z)^j}{Q}. \tag{3.14}$$

Let m be an arbitrary positive integer, and let $y > e^{2^{j-2}Q^2}$. Then

$$A_{j+1}(y) = \frac{(\log\log y)^{j+1}}{(j+1)!(Q-1)} + O\left(\frac{(\log\log y)^j}{Q^{(m-1)/m}}\right). \tag{3.15}$$

The constants implied by the error terms may depend on j and m.

Since

$$D_{k+1}^2(y) = \sum_{p \leq y} \left(\sum_{q \mid p-1} \kappa_k(q)\right)^2 =$$
$$= \sum_q \kappa_p^2(q)\delta(y,q,1) + \sum_{q_1 \neq q_2} \kappa_k(q_1)\kappa_k(q_2)\delta(y,q_1q_2,1),$$

by Lemma 5 and (3.13) we obtain that

$$
\begin{aligned}
D_{k+1}^2(y) &\le cD_k^2(y)\log\log y + c(\log\log y)A_k^2(y) \le \\
&\le cD_k^2(y)\log\log y + c\frac{(\log\log y)^{2k+1}}{Q^2}.
\end{aligned}
\tag{3.16}
$$

We have

$$
D_1^2(y) = A_1(y) \le c\frac{\log\log y}{Q},
$$

whence

$$
D_2^2(y) \ll \frac{(\log\log y)^2}{Q} + \frac{(\log\log y)^3}{Q^2},
$$

$$
D_3^2(y) \ll \frac{(\log\log y)^3}{Q} + \frac{(\log\log y)^5}{Q^2},
$$

and in general

$$
D_{k+1}^2(y) \le c\frac{(\log\log y)^{k+1}}{Q} + \frac{(\log\log y)^{2k+1}}{Q^2}.
\tag{3.17}
$$

From (3.6) we obtain that

$$
\begin{aligned}
&\#\{p\in[V,2V],\ \kappa_{k+1}(p)=0\} \ll \\
&\ll \frac{V}{\log V}\cdot\frac{B_V^2}{A_V^2} + O\left(\frac{V}{(\log V)^D A_V^2}\right)
\end{aligned}
\tag{3.18}
$$

we can substitute $A_V = A_{k+1}(V)$, $B_V^2 = D_{k+1}(V)$.

From (3.18), (3.17), (3.15) we can deduce that

$$
\begin{aligned}
&\frac{1}{\frac{V}{\log V}}\#\{p\in[V,2V],\ \kappa_{k+1}(p)=0\} \ll \\
&\qquad \ll \frac{Q}{(\log\log V)^{k+1}} + \frac{1}{\log\log V},
\end{aligned}
$$

whenever $V \ge V_0$ and $V_0 > e^{2^{j-2}Q^2}$ (see Lemma 6).

Let V_1 be so defined that

$$
\frac{Q}{(\log\log V_1)^{k+1}} = \varepsilon_1,
$$

where ε_1 is a small constant, i.e. $V_1 = \exp\left(\exp\left(\left(\frac{Q}{\varepsilon_1}\right)\right)^{1/k+1}\right)$.

Then, for $V > V_1$ we have

$$\#\{p \in [V, 2V],\ \kappa_{k+1}(p) \neq 0\} \geq (1 - 2\varepsilon_1)\frac{V}{\log V},$$

whence we can deduce that

$$\log w_k(x) = -\sum_{\kappa_{k+1}(p)\neq 0} \frac{1}{p} \leq -(1-2\varepsilon_1)\int\limits_{V_1}^{x} \frac{du}{u\log u} + O(1) =$$
$$= -(1-2\varepsilon_1)(\log\log x - \log\log V_1) + O(1) \leq (1-3\varepsilon_1)x_2.$$

Thus the following assertion is true.

Lemma 7. *Let $k \geq 2$, $Q \leq x_2^{k-\varepsilon}$, $\varepsilon > 0$ be fixed. Let $\varepsilon_1 > 0$ be another arbitrary small constant. Then*

$$M_k(Q, x) \leq \frac{cx}{Q\log Q}\exp(-(1-\varepsilon_1)x_2).$$

Lemma 8. *Let*

$$S(x|Q) := \sum_{\substack{n\leq x\\ p(n)=Q}} 1.$$

Then, uniformly in $1 \leq Q \leq x_1$ (say), we have

$$S(x|Q) = \frac{x}{Q}\prod_{p<Q}\left(1-\frac{1}{p}\right)(1+o_x(1)) =$$
$$= \frac{x}{Q}\frac{e^{-\gamma}}{(\log Q)}(1+o_x(1)),$$

see [1].

Now we shall prove the following

Theorem 2. *Let $k \geq 1$. Then*

$$\frac{1}{x}\#\{n \leq x \mid (n, \varphi_{k+1}(n)) = 1\} = \frac{(1+o_x(1))e^{-\gamma}}{(k+1)x_3}.$$

Proof. By using the notation (3.1), we have

$$\#\mathcal{B}_{k+1} = \sum_{Q<x_2^{k+1}} \#\left(\mathcal{B}_{k+1}^{(Q)}\right) + \#\left(\mathcal{B}_{k+1}^{*}\right) = \sum\nolimits_1 + \#\left(\mathcal{B}_{k+1}^{*}\right).$$

where

$$\mathcal{B}_{k+1}^{(Q)} = \{n \in \mathcal{B}_{k+1},\ p(n) = Q\}, \quad \mathcal{B}_{k+1}^{*} = \{n \in \mathcal{B}_{k+1},\ p(n) > x_2^{k+1}\}.$$

Assume that $k \geq 2$. We split $\sum_1 = \sum^{(1)} + \sum^{(2)} + \sum^{(3)} + \sum^{(4)}$, where in $\sum^{(1)}$, $Q \leq x_2^{k-\varepsilon}$, in $\sum^{(2)}$ $x_2^{k-\varepsilon} \leq Q \leq x_2^{k+\varepsilon}$, in $\sum^{(3)}$ $x_2^{k+\varepsilon} \leq Q \leq x_2^{k+1-\varepsilon}$, and in $\sum^{(4)}$, $x_2^{k+1-\varepsilon} \leq Q \leq x_2^{k+1+\varepsilon}$.

From Lemma 8 we obtain that

$$\sum\nolimits^{(2)} \ll x \sum_{x_2^{k-\varepsilon} \leq Q \leq x_2^{k+\varepsilon}} \frac{1}{Q \log Q} \ll \varepsilon \frac{x}{x_3},$$

and similarly that

$$\sum\nolimits^{(4)} \ll \varepsilon \frac{x}{x_3}.$$

From Lemma 7 we have $\sum^{(1)} \ll x \exp\left(-\frac{x_2}{2}\right)$, say, and by Lemma 3, that

$$\sum\nolimits^{(3)} \ll x/x_2^2.$$

Thus

$$\#(\mathcal{B}_{k+1}) \leq \#(\mathcal{B}_{k+1}^{*}) + O\left(\frac{\varepsilon x}{x_3}\right).$$

Finally we shall prove that

$$p(n) > x_2^{k+1+\varepsilon}, \quad n \leq x$$

implies that $n \in \mathcal{B}_{k+1}$, for all but a small percentage of the integers.

If $n \in \mathcal{B}_{k+1}$, $p(n) > x_2^{k+1}$ and $(n, \varphi_{k+1}(n)) \neq 1$, then there is a prime number Q for which $Q|n$, and $Q|\varphi_{k+1}(n)$.

Thus either $Q^2|\varphi_k(n)$ or there is some $q_0 \equiv 1 (\mathrm{mod}\ Q)$ for which $q_0|\varphi_k(n)$. In the first case $Q|\varphi_k(n)$ obviously holds. Thus always exists a chain of primes $Q \to q_0 \to \ldots \to q_j$, such that $n = Qm$, $q_j|m$.

Thus

$$E := \#\left\{n \le x,\ p(n) > x_2^{k+1+\varepsilon},\ (n, \varphi_{k+1}(n)) \ne 1\right\} \le$$
$$\le c \sum_{Q > x_2^{k+1+\varepsilon}} \sum_{j=0}^{k} \#\left\{n = Qm \le x,\ p(m) > x_2^{k+1+\varepsilon},\ q_j | m\right\},$$

where q_j is the final term in the chain $Q \to q_0 \to \ldots \to q_j$.

Let

$$E_Q^{(j)} := \{n = Qm \le x,\ p(m) > x_2^{k+1+\varepsilon},\ q_j \mid m\}.$$

Then

$$E_Q^{(j)} \ll \frac{x}{Q} \prod_{p<Q} \left(1 - \frac{1}{p}\right) \cdot \sum \frac{1}{q_j}.$$

Since

$$\sum \frac{1}{q_j} \le c x_2 \sum \frac{1}{q_{j-1}} \le \ldots \le \frac{c x_2^{j+1}}{Q},$$

therefore

$$E_Q^{(j)} \ll \frac{x}{Q^2} \frac{x_2^{j+1}}{\log Q},$$

and so

$$\sum_{Q > x_2^{k+1}} \sum_{j=0}^{k} E_Q^{(j)} \ll \frac{x}{x_2^{\varepsilon}} \sum \frac{1}{Q \log Q} \ll \frac{x}{x_2^{\varepsilon}}.$$

Thus

$$E = O\left(\frac{x}{x_2^{\varepsilon}}\right).$$

We proved that

$$\#(\mathcal{B}_{k+1}) = \#\left\{n \le x \mid p(n) > x_2^{k+1}\right\} + o(1) \frac{x}{x_3}.$$

Hence the theorem readily follows.

The case $k = 1$ is similar, somewhat easier.

4. By using the above method, one can prove the assertions formulated in the following

Theorem 3. *Let $k \geq 1,\ l,h \neq 0$ be fixed integers,*

$$\begin{aligned} S_k^{(l)}(x) &= \#\{n \leq x,\ (n, \varphi_k(n+l)) = 1\}, \\ T_k^{(l)}(x) &= \#\{n \leq x,\ (n, \sigma_k(n+l)) = 1\}, \\ R_k^{(l,h)}(x) &= \#\{p \leq x,\ (p+h,\ \sigma_k(p+l)) = 1\}, \\ Q_k^{(l+h)}(x) &= \#\{p \leq x,\ (p+h,\ \varphi_k(p+l)) = 1\}. \end{aligned}$$

Then

$$\begin{aligned} \frac{S_k^{(l)}(x)}{x} &= (1+o_x(1))\frac{e^{-\gamma}}{kx_3}, & \frac{T_k^{(l)}(x)}{x} &= (1+o_x(1))\frac{e^{-\gamma}}{kx_3}, \\ \frac{R_k^{(l,h)}(x)}{\operatorname{li} x} &= (1+o_x(1))\frac{e^{-\gamma}}{kx_3}, & \frac{Q_k^{(l,h)}(x)}{\operatorname{li} x} &= (1+o_x(1))\frac{e^{-\gamma}}{kx_3}, \end{aligned}$$

$\gamma =$ *Euler's constant.*

5. Let

$$N(x) := \#\{n \mid \varphi(n) \leq x\}.$$

Erdős proved in [5] that $\dfrac{N(x)}{x} \to A\ (\neq 0)$, and in [6] he noted:

Let $\alpha(n)$ be a nonnegative multiplicative function, and assume that there exists its density function $f(x)$. Let

$$M(x) = \#\{n \mid n\alpha(n) \leq x\}.$$

Then

$$\frac{M(x)}{x} \to \int_0^\infty f(u)du.$$

Bateman [7] proved that

$$\begin{aligned} N(x) &= Ax + O\left(x \cdot \exp\left(-c\sqrt{x_1 \cdot x_2}\right)\right), \\ A &= \frac{\zeta(2)\zeta(3)}{\zeta(6)}, \end{aligned}$$

with some $c > 0$, by using analytic method. Later Balazard and Smati [8] deduced the same result with elementary method.

We can see that, for $k \ge 1$,

$$\frac{\varphi_{k+1}(n)}{\varphi_k(n)} \sim w_k(x) = \prod_{p<x_2^k} (1-1/p)\,,$$

for almost all $n \le x$. This was observed first by Erdős [11].

Let us write

$$\frac{\varphi_{k+1}(n)}{\varphi_k(n)} = w_k(x)\Gamma_k(n), \tag{5.1}$$

$$\Gamma_k(n) = \prod_{\substack{p\mid\varphi_k(n)\\ p>x_2^k}} (1-1/p) \cdot \prod_{\substack{p\nmid\varphi_k(n)\\ p\le x_2^k}} \frac{1}{1-1/p}. \tag{5.2}$$

Then

$$\varphi_{k+1}(n) = w_k(x)\dots w_1(x)\Gamma_k(n)\dots\Gamma_1(n)\varphi(n), \tag{5.3}$$

and so $\varphi_{k+1}(n) \le x$ holds if and only if

$$\Gamma_k(n)\dots\Gamma_1(n)\varphi(n) \le \frac{x}{w_k(x)\dots w_1(x)} = Y, \tag{5.4}$$

where

$$Y = (1+o_x(1))k!x_3^k x. \tag{5.5}$$

If $\varphi_{k+1}(n) \le x$, then $n \le c\,x\,x_2^{k+1}$, which directly follows from the inequality

$$\varphi(n) \ge \frac{cn}{\log\log n}.$$

Thus $\varphi_{k+1}(n) \le x$ implies that $n \le Z = c\,x\,x_2^{k+1}$ for a suitable c.

Let $\varepsilon > 0$ be an arbitrary constant. We shall prove that

$$\frac{1}{Y}\#\{n<Z \;\big|\; |\Gamma_k(n)\dots\Gamma_1(n)-1|>\varepsilon\} \to 0, \tag{5.6}$$

as $x \to \infty$.

Hence, by the analogon of Erdős's theorem directly follows

Theorem 4. *Let $k \geq 1$ be an arbitrary fixed integer. Then*

$$\#\{n \,|\, \varphi_{k+1}(n) \leq x\} = (1 + o_x(1))\#\left\{\varphi(n) \leq k! x_3^k x\right\}.$$

Proof. It remains to prove (5.6). If

$$|\Gamma_k(n) \ldots \Gamma_1(n) - 1| > \varepsilon,$$

then, for some $j \in \{1, \ldots, k\}$.

$$|\Gamma_j(n) - 1| > \delta, \quad \text{where } 1 + \delta = (1 + \varepsilon)^{1/k}. \tag{5.7}$$

Let $\sqrt{x} \leq U \leq x^2$, and count those integers $n \in [U, 2U]$ for which (5.7) holds. Let $\varepsilon_1 > 0$ be a small constant. We have

$$\sum_{x_2^{j-\varepsilon_1} < p < x_2^{j+\varepsilon_1}} \frac{1}{p} \leq \log \frac{j + \varepsilon_1}{j - \varepsilon_1} + O\left(\frac{1}{x_3}\right) \leq \frac{3}{j}\varepsilon_1,$$

whenever x is large enough. Thus

$$\Gamma_j(n) = e^{-\beta_j(n)} \cdot e^{\gamma_j(n)} \cdot e^{0(\varepsilon_1)}, \tag{5.8}$$

where

$$\beta_j(n) = -\sum_{\substack{p > x_2^{j+\varepsilon_1} \\ p \mid \varphi_j(n)}} \log(1 - 1/p), \tag{5.9}$$

$$\gamma_j(n) = \sum_{\substack{p \nmid \varphi_j(n) \\ p \leq x_2^{j+\varepsilon_1}}} \log \frac{1}{1 - 1/p}. \tag{5.10}$$

Let

$$B_j(U) = \sum_{U \leq n \leq 2U} \beta_j(n), \tag{5.11}$$

$$C_{j,r}(U) = \sum_{U \leq n \leq 2U} \gamma_j^r(n). \tag{5.12}$$

By sieve method one can prove that

$$\sum_{\substack{p \nmid \varphi_j(n) \\ n\in[U,2U]}} 1 \ll \frac{U}{(\log U)^c} \quad \text{if} \quad p \le (\log\log u)^{j-\varepsilon_1},$$

whith a suitable constant $c>0$ (see (4.2) and Theorem 3.4 in [9]), whence

$$B_j(U) \ll \frac{Ux_3}{x_1^c} \tag{5.13}$$

follows.

Let us estimate (5.12). We have

$$\begin{aligned} C_{j,r}(U) \ll & \sum_{x_2^{j+1+\varepsilon_1}<q_1,\dots,q_r} \frac{1}{q_1\dots q_r} \sum_{\substack{q_1\dots q_r \mid \varphi_j(n) \\ n\in[U,2U]}} 1+ \\ & + \sum_{s=1}^{r-1} \sum_{x_2^{j+1+\varepsilon_1}<q_1<\dots<q_s} \frac{1}{q_1^{a_1}\dots q_s^{a_s}} \sum_{\substack{q_1\dots q_s \mid \varphi_j(n) \\ n\in[U,2U]}} 1 = \\ & = \sum\nolimits_1 + \sum\nolimits_2 . \end{aligned}$$

The main contribution of $\sum_{\substack{q_1\dots q_r \mid \varphi_j(n) \\ n\in[U,2U]}} 1$ is smaller than

$$\sum \frac{U}{\pi_1\dots\pi_r},$$

where π_ν is the tail of the chain of primes

$$q_\nu \to p_\nu^{(1)} \to p_\nu^{(2)} \to \dots \to p_\nu^{(j-1)} \to \pi_\nu$$

for every ν, thus

$$\sum \frac{1}{\pi_\nu} \le \frac{(\log\log U)^j}{q_\nu} \ll \frac{x_2^j}{q_j},$$

consequently

$$C_{j,r}(U) \ll U x_2^{jr} \left(\sum \frac{1}{q^2} \right)^r .$$

Since

$$\sum_{q>x_2^{j+\varepsilon_1}} \frac{1}{q^2} \ll \frac{1}{x_2^{j+\varepsilon_1}\cdot x_3},$$

we obtain that

$$C_{j,r}(U) \ll U \cdot x_2^{-r\varepsilon_1},$$

and so

$$\sum_{n\le Z} \gamma_j^r(n) \ll \frac{Z}{x_2^{r\varepsilon_1}}. \tag{5.14}$$

Since r can be arbitrary large, therefore

$$\#\{n \le Z \mid \gamma_j(n) > \delta\} \ll \frac{Z}{x_2^{2k+1}}.$$

Hence the theorem is straightforward.

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(Received November 22, 2004)

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