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ON THE ERGODICITY CONDITION OF A $GI/D/1$ RETRIAL QUEUEING SYSTEM WITH CONSTANT RETRIAL TIMES AND A DYNAMIC SERVICE PRIORITY

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The retrial queueing systems theory is one of the modern branches of queueing theory. Over the last two decades this theory has been developing greatly, see for instance [1-3]. In many practical situations retrial models can describe real-life processes more adequately than classical ones. The areas of application for retrial queueing systems are computer networks, telecommunication systems, landing processes in aviation, etc. At the same time, in some situations the retrial queueing model with some peculiarity would be more precise. Consider as example an aircraft landing process. In case of the airport runway (of the server, using the queueing systems terminology), being busy the arriving aircraft goes to the holding area (to the orbit, using the retrial queueing systems terminology), from where it can come back after some time. In principle, repeated returning from the orbit is possible, but the limited amount of aircraft fuel and some other factors make the airport flying control officer to control the landing process using the strategy of dynamic priorities, e.g. the strategy can be such that the incoming aircraft is admitted to land only if the runway is idle and no delayed aircrafts may return from holding area within some time interval. Therefore demands in the orbit have a higher priority comparing with the incoming demand, and for each delayed demand a time interval for it to be serviced is exactly determined at its arrival epoch. Let us call retrial queueing system with such a peculiarity a *retrial queueing system with dynamic service priority.*

In [5] a sufficient ergodicity condition was found for an $M/D/1$ queueing system with constant retrial times and a dynamic service priority. It was proved that if $T > \tau$ then $\lambda\tau < 1/2$ is a sufficient condition for steady-state regime existence (λ is an input flow rate). In the given paper it is proved that this result cannot be improved for the general case of a $GI/D/1$ queue. Also in [6] a more general case was considered, namely for the multi-channel $GI/D/m$ queueing system was proved that $\tau/a < m/2$ is sufficient ergodicity condition.

It should be mentioned that the retrial queueing systems with generally distributed orbit time, in particular constant orbit time, are comparatively little-investigated. At the same time, because of information and telecommunication technologies development, there is a lot of real-life systems which can be modeled with the help of retrial queueing systems with a constant orbit time [4, 8].

Consider a single-channel retrial queueing system with a constant orbit time, general independent interarrival time distribution, and a constant service time. Denote the interarrival time distribution function as $A(x)$, the constant service time as τ, and the orbit time as T. Suppose that demands in the orbit have a higher priority in comparison with a demand being arrived, i.e. the demand from the primary flow cannot change the service commencement times for the demands in the orbit.

Therefore the system being considered is a $GI/D/1$ retrial queueing system with a constant orbit time and a dynamic service priority. Service time interval will be set for each arriving demand; denoting the arrival epoch of the k-th demand as t_k this time interval is $(t_k + Tn_k,\ t_k + Tn_k + \tau)$, where n_k is a minimal positive integer for which this time interval does not intersect service intervals of preceding demands. Denote the system load due to primary input flow as ρ; $\rho = \tau/(\text{mean interarrival time})$.

Theorem. *For any $\varepsilon > 0$ a condition*

$$\rho < \frac{1}{2} + \varepsilon$$

is not sufficient ergodicity condition for GI/D/1 retrial queueing system with dynamic service priority.

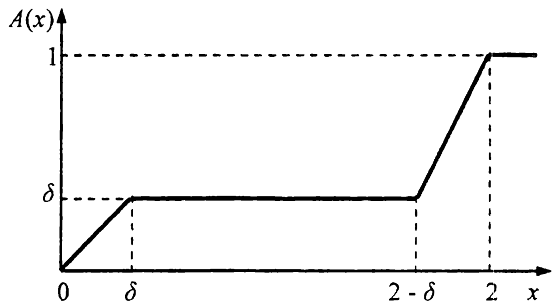

Fig.1. Interarrival time distribution function

Proof. Let the interarrival time d.f. be

$$A(x) = \begin{cases} x, & 0 \leq x < \delta, \\ \delta, & \delta \leq x < 2-\delta, \\ \dfrac{1-\delta}{\delta}x + \dfrac{3\delta - 2}{\delta}, & 2-\delta \leq x < 2, \\ 1, & x \geq 2, \end{cases}$$

where $0 < \delta < 1$. I.e. the interarrival time between k-th and $k+1$-st demands is either less than δ or greater than $2-\delta$ and less than 2.

The interarrival time p.d.f. be

$$a(x) = \begin{cases} 1, & 0 \leq x < \delta, \\ 0, & \delta \leq x < 2-\delta, \\ \dfrac{1-\delta}{\delta}, & 2-\delta \leq x < 2, \\ 0, & x \geq 2. \end{cases}$$

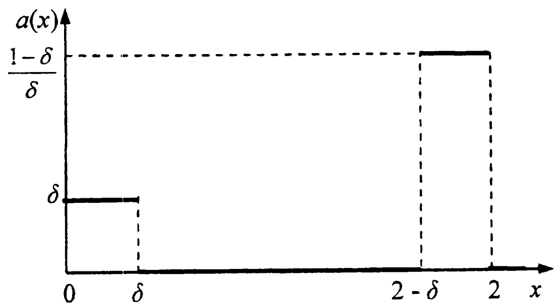

Fig.2. Interarrival time probability density function

The mean interarrival time

$$a = \int_0^\infty x dA(x) = 2 - \delta\left(\frac{5}{2} - \delta\right).$$

Let the service time be $\tau = 1 + \delta$ and the constant orbit time T be equal to 2. Then

$$\rho = \frac{\tau}{a} = \frac{1+\delta}{2-\delta(5/2-\delta)},$$

i.e. $\rho \underset{\delta \to 0}{\to} \frac{1}{2}$. Denote server idle periods between the servicing of k-th and $k+1$-st demands as ξ_k.

Lemma.

$$1 - 2\delta < \xi_k < 1.$$

Proof. Denote the arrival epoch of the k-th demand as t_k; its service commencement epoch as s_k. Then

$$s_k = t_k + Tn_k, \tag{1}$$

since the queueing system is the system with constant retrial times. Here n_k is the number of retrials for the k-th demand.

Next we apply the method of mathematical induction. Firstly, we check the correctness of the statement for $k = 1$. Suppose that $t_1 = s_1 = 0$ without loss of generality. Denote the interarrival intervals $t_{k+1} - t_k$, $k \geq 1$ as γ_k. We have two cases:

a) $0 < \gamma_1 < \delta$

From the Fig. 3 it follows that

$$\xi_1 = s_2 - 1 - \delta = \gamma_1 + 1 - \delta.$$

Hence $1 - \delta < \xi_1 < 1$.

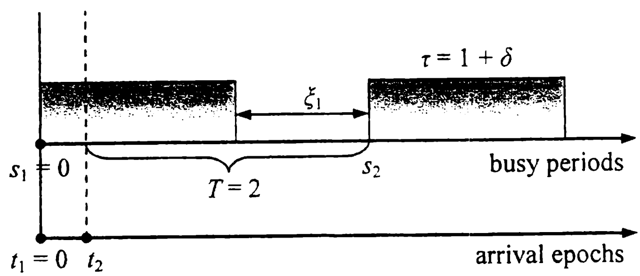

Fig. 3. First and second demands: arrival epochs and busy periods, case 1

b) $2 - \delta < \gamma_1 < 2$

From Fig. 4 it follows that

$$\xi_1 = s_2 - 1 - \delta = \gamma_1 - 1 - \delta.$$

Hence $1 - 2\delta < \xi_1 < 1 - \delta$.

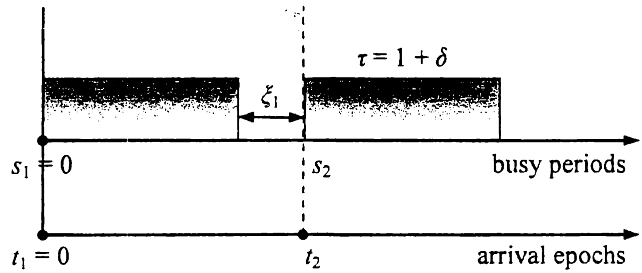

Fig. 4. First and second demands: arrival epochs and busy periods, case 2

Therefore,

$$1 - 2\delta < \xi_1 < 1.$$

Suppose that the statement of lemma is true for $\xi_1, \xi_2, \ldots, \xi_{k-1}$. Therefore demands have been serviced in turn. We prove the statement for ξ_k.

a) $0 < \gamma_k < \delta$

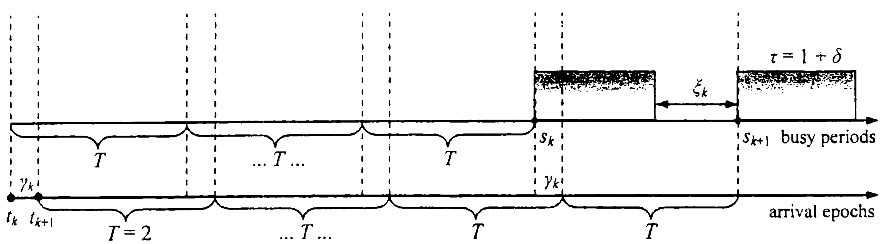

Fig.5. k-th and $k+1$-th demands: arrival epochs and busy periods, case 1

$$\xi_k = 2 + \gamma_k - 1 - \delta = \gamma_k + 1 - \delta,$$

see Fig. 5. Hence $1 - \delta < \xi_k < 1$.

b) $2 - \delta < \gamma_k < 2$

$$\xi_k = \gamma_k - 1 - \delta,$$

see Fig. 6.

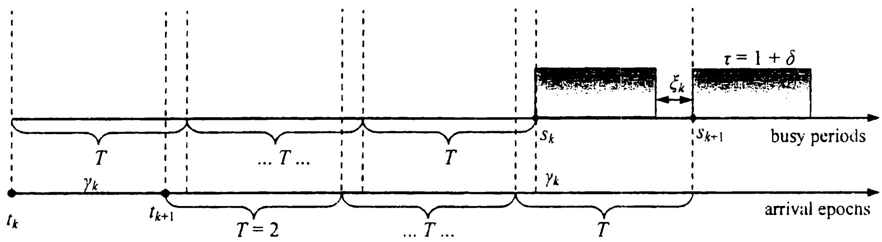

Fig.6. k-th and $k+1$-th demands: arrival epochs and busy periods, case 2

Hence $1 - 2\delta < \xi_k < 1 - \delta$.

Therefore

$$1 - 2\delta < \xi_k < 1.$$

Corollary. Demands in the system being considered are scheduled for service in the order of arrival.

From the corollary and equation (1) we have

$$n_{k+1} = \begin{cases} n_k + 1, & 0 < \gamma_k < \delta, \\ n_k, & 2 - \delta < \gamma_k < 2. \end{cases}$$

Denote the indicator of the event $\{\gamma_k < \delta\}$ as I_k,

$$I_k = \begin{cases} 1, & \gamma_k < \delta, \\ 0, & \text{otherwise}. \end{cases}$$

Therefore

$$s_k = t_k + 2(I_1 + \ldots + I_{k-1}).$$

Here $\sum_{j=1}^{k-1} I_j$ has a binomial distribution with parameters $k-1$ and δ. From the strong law of large numbers it follows that

$$P\left\{\frac{I_1 + \ldots + I_{k-1}}{k-1} \underset{k\to\infty}{\to} E(I_k)\right\} = 1.$$

Hence $s_k - t_k \underset{k\to\infty}{\to} \infty$. Therefore the demands will be accumulated in the system. At the same time

$$\rho = \frac{1+\delta}{a} = \frac{1}{2} + o(1), \qquad \delta \to 0.$$

Therefore for $\rho = \frac{1}{2} + \varepsilon$, $\varepsilon > 0$ the queueing system will not be stable.

Simulation results

N	Remaining, realization 1	Remaining, realization 2	Remaining, realization 3	Remaining, average	Avg. remain. / N, %
$\delta = 0.05$ ($\rho = 0.559$)					
1000	54	47	52	51.00	5.10
3000	159	173	149	160.33	5.34
5000	252	262	243	252.33	5.05
$\delta = 0.1$ ($\rho = 0.625$)					
1000	111	105	110	108.67	10.87
3000	294	323	313	310.00	10.33
5000	510	513	532	518.33	10.37
$\delta = 0.2$ ($\rho = 0.779$)					
1000	201	224	190	205.00	20.50
3000	609	642	636	629.00	20.97
5000	1062	991	1015	1022.67	20.45

Table 1. Simulation results

The system being considered has been modeled with the help of direct Monte-Carlo simulation method. Denote the number of demands having arrived in the system as N. We are interested in the number of demands which remain in the system at the arriving epoch of N-th demand. The simulation results are listed in the table above.

From the Table 1 it follows that the number of remaining demands in the system increases approximately linearly with time.

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