Annales Univ. Sci. Budapest., Sect. Comp. 25 (2005) 75-83

SUMS OF TWO SQUARES AND THE IRRATIONALITY OF A SERIES INVOLVING FIBONACCI NUMBERS

F. Luca (Morelia, Mexico)

1. Introduction

In [4] Erdős used only elementary properties of the divisor function of an integer to show that the sum of the series

$$\sum_{n\geq 1}\frac{1}{2^n-1} \tag{1}$$

is irrational. Since then, this has been generalized in various directions (see, for example, [2] and [5]).

Let $(F_n)_{n\geq 1}$ be the Fibonacci sequence given by $F_0=0$, $F_1=1$ and $F_{n+2}=F_{n+1}+F_n$ for $n\geq 0$. In 1989, André-Jeannin [1] used the theory of Padé approximations to prove that both series

$$\sum_{n\geq 1}\frac{1}{F_n}\quad\text{and}\quad\sum_{n\geq 1}\frac{(-1)^n}{F_n} \tag{2}$$

are irrational. The irrationality and transcendence of such types of sums was dealt with by several authors (see [8], [9] and the survey paper [3] and the references therein), mostly by employing either Padé approximations or deep tools from transcendental number theory.

In this paper our goal is to employ Erdős's elementary method for dealing with the irrationality of (1) to prove the following result:

Theorem 1. *The sum of the series*

$$\sum_{n\geq 0}\frac{1}{F_{2n+1}} \tag{3}$$

does not belong to $\mathbf{Q}[\sqrt{5}]$.

Since

$$F_{2n+1} = \frac{1}{\sqrt{5}} \cdot \left(\alpha^{2n+1} + \alpha^{-(2n+1)}\right), \tag{4}$$

where $\alpha := \frac{1+\sqrt{5}}{2}$, the fact that series (3) does not belong to $\mathbf{Q}[\sqrt{5}]$ is equivalent to the fact that

$$\sum_{n\geq 0} \frac{1}{\alpha^{2n+1} + \alpha^{-(2n+1)}}$$

does not belong to $\mathbf{Q}[\sqrt{5}]$. Our argument applies to a more general instance, namely the following

Theorem 2. *Let $\alpha > 1$ be a quadratic unit; i.e. an irrational real number satisfying an equation of the form $x^2 - rx + s = 0$, with r an integer and $s = \pm 1$, and let* $\mathbf{K}$ *be the quadratic field containing α. Then the sum of the series*

$$\sum_{n\geq 0} \frac{1}{\alpha^{2n+1} + \alpha^{-(2n+1)}} \tag{5}$$

does not belong to $\mathbf{K}$.

As we have already mentioned, our proof of Theorem 2 is elementary, although it does make use of some standard tools from analytic number theory, such as an estimate for the number of positive integers up to an upper bound x which can be written as a sum of two squares, as well as an elementary application of Brun's sieve.

2. The proofs

We proceed directly to the proof of Theorem 2.

The proof of Theorem 2. Throughout this proof we shall use $c_1, c_2, \ldots$ for computable constants which are either absolute, or depend only on α. We shall also use the Vinogradov symbols $\gg$ and $\ll$ as well as the Landau symbols O and o with the usual meanings. We shall assume that $\alpha > 1$ (otherwise, if $\alpha < 0$, we then replace α by $-\alpha$, and if $\alpha \in (0, 1)$, we then replace α by α^{-1}).

For a positive number x we use $\log x$ for the maximum between the natural logarithm of x and 1.

For any odd integer n let

$$\tau_1(n) := \sum_{d|n} (-1)^{(d-1)/2}$$

and let $\tau(n)$ stand for the total number of divisors of n. Write A for the sum appearing in (5) and notice that

$$A = \sum_{n\geq 0} \frac{1}{\alpha^{2n+1}} \cdot \frac{1}{1+\frac{1}{\alpha 2(2n+1)}} = \sum_{n\geq 0} \frac{1}{\alpha^{2n+1}} \sum_{k\geq 0} \frac{(-1)^k}{\alpha^{2k(2n+1)}} =$$

$$(6) \qquad = \sum_{\substack{n\geq 0\\ k\geq 0}} \frac{(-1)^k}{\alpha^{(2n+1)(2k+1)}} = \sum_{\substack{n\geq 1\\ n \text{ odd}}} \frac{\tau_1(n)}{\alpha^n}.$$

It is plain that the function $\tau_1(n)$ is multiplicative; i.e. if m and n are two coprime odd integers, then $\tau_1(mn) = \tau_1(m)\tau_1(n)$. Indeed, this can be verified by noticing that

$$\tau_1(m)\tau_1(n) = \left(\sum_{d_1|m} (-1)^{(d_1-1)/2}\right)\left(\sum_{d_2|n} (-1)^{(d_2-1)/2}\right) =$$

$$= \sum_{\substack{d_1|m\\ d_2|n}} (-1)^{(d_1-1)/2+(d_2-1)/2} = \sum_{\substack{d_1|m\\ d_2|n}} (-1)^{(d_1 d_2-1)/2} = \tau_1(mn),$$

where in the above equality we used the fact that

$$\frac{d_1 d_2 - 1}{2} - \frac{d_1 - 1}{2} - \frac{d_2 - 1}{2} = \frac{(d_1-1)(d_2-1)}{2}$$

is even for all odd integers d_1 and d_2. Moreover, by looking at the values of τ_1 in odd prime powers, it is easily seen that if p^a is an odd prime power, then

$$\tau_1(p^a) = (-1)^{(1-1)/2} + (-1)^{(p-1)/2} + \ldots + (-1)^{(p_a-1)/2} =$$

$$= \begin{cases} a+1, & \text{if } p \equiv 1 \pmod 4, \\ 1, & \text{if } p \equiv 3 \pmod 4,\ 2|a, \\ 0, & \text{if } p \equiv 3 \pmod 4,\ 2 \nmid a. \end{cases}$$

In particular, $\tau_1(n) \le \tau(n)$ and $\tau_1(n) = 0$ unless n can be represented as a sum of two squares. For any positive integer n let $f(n) := \log^{1/3} n$, $g(n) := \log^2 n$ and $h(n) := \log^4 n$. We shall need the following lemma.

Lemma. *There exist infinitely many positive integers m such that the following hold:*

1. m is an odd number which is a sum of two squares.

2. Every positive integer belonging to the interval $[m - g(m), m + g(m)]$ has the property that $\tau(n) < h(m)$.

3. If $m' > m$ is the smallest odd positive integer larger than m which is a sum of two squares, then $m' > m + f(m)$.

Assume for the moment that we have proved the above lemma. Pick a large number m satisfying the hypotheses of the lemma and write $A = A_m + R_m$, where

$$A_m := \sum_{\substack{n \le m \\ n \text{ odd}}} \frac{\tau_1(n)}{\alpha^n} \quad \text{and} \quad R_m := \sum_{\substack{n > m \\ n \text{ odd}}} \frac{\tau_1(n)}{\alpha^n}. \tag{7}$$

We first bound R_m. Clearly,

$$R_m = \sum_{\substack{n \in [m, m+g(m)] \\ n \text{ odd}}} \frac{\tau_1(n)}{\alpha^n} + \sum_{\substack{n > m+g(m) \\ n \text{ odd}}} \frac{\tau_1(n)}{\alpha^n}. \tag{8}$$

By the Lemma, $\tau_1(n) < \log^4 m$ if $m \le n \le m + g(m)$ and $\tau_1(n) = 0$ if $m < n < m'$. Thus,

$$\sum_{\substack{n \in [m, m+g(m)] \\ n \text{ odd}}} \frac{\tau_1(n)}{\alpha^n} < \frac{g(m) \log^4 m}{\alpha^{m+f(m)}} < \frac{\log^6 m}{\alpha^{m+f(m)}}. \tag{9}$$

The bound the tail of the sum appearing in (8), let $n > m + g(m)$ and write $n = m + \lfloor g(m) \rfloor + k$ for some positive integer k. We show that the inequality

$$\frac{\tau_1(n)}{\alpha^n} < \frac{1}{\alpha^{m+f(m)+k/2}} \tag{10}$$

holds for large enough values of m. Indeed, since $\tau_1(n) \le \tau(n)$ and since there exists an absolute constant c_1 so that $\tau_1 < e^{\frac{c_1 \log n}{\log\log n}}$ holds for all positive integers $n \ge 3$, it follows that if we set $c_2 := \dfrac{c_1}{\log \alpha}$, in order to prove (10) it suffices to show that

$$n - c_2 \frac{\log n}{\log\log n} > m + f(m) + \frac{k}{2},$$

which is equivalent to

$$\lfloor \log^2 m \rfloor - \log^{1/3} m + \frac{k}{2} > c_2 \frac{\log(m + \lfloor \log^2 m \rfloor + k)}{\log\log(m + \lfloor \log^2 m \rfloor + k)}. \tag{11}$$

Clearly, for large m, inequality (11) is implied by

$$\log^2 m + k > 2c_2 \log(2m + k). \tag{12}$$

If $k < 2m$, then (12) is implied by $\log^2 m > 2c_2 \log(4m)$, which holds for m large enough, while if $k \geq 2m$ then (12) is implied by $k \geq 2c_2 \log(2k)$, which holds for k large enough; hence, for m large enough. Having proved (10), it follows that

$$\sum_{\substack{n > m+g(m) \\ n \text{ odd}}} \frac{\tau_1(n)}{\alpha^n} < \frac{1}{\alpha^{m+f(m)}} \sum_{k \geq 1} \frac{1}{\alpha^{k/2}} < \frac{c_3}{\alpha^{m+f(m)}}, \tag{13}$$

where c_3 is a constant. Thus, with (9) and (13), we get that the inequality

$$R_m < \frac{\log^6 m}{\alpha^{m+f(m)}} + \frac{c_3}{\alpha^{m+f(m)}} < \frac{2\log^6 m}{\alpha^{m+f(m)}} \tag{14}$$

holds for m sufficiently large satisfying conditions of the Lemma. Thus, we have shown that

$$|A - A_m| < \frac{2\log^6 m}{\alpha^{m+f(m)}}. \tag{15}$$

Assume that $A \in \mathbf{K}$, write $\mathbf{K} := \mathbf{Q}[\sqrt{d}]$ for some positive square-free integer $d > 1$ and write $A := \dfrac{u + v\sqrt{d}}{w}$, where $u,\ v,\ w$ are integers with $w \geq 1$. Write also $M := \max(|u| + |v|\sqrt{d},\ w)$. We may rewrite (15) as

$$|u + v\sqrt{d} - A_m w| < \frac{2M\log^6 m}{\alpha^{m+f(m)}}. \tag{16}$$

Notice now that A_m is an algebraic integer (because α is a quadratic unit), therefore the number appearing inside the absolute value in the left hand side inequality (16) is an algebraic integer in $\mathbf{K}$. Write σ for the unique non-trivial Galois automorphism of $\mathbf{K}$ over $\mathbf{Q}$. Then,

$$|\sigma(u + v\sqrt{d} - A_m w)| = |u + v\sqrt{d} - \sigma(A_m)w| <$$

$$(17) \qquad < |u| + |v|\sqrt{d} + w\left(\sum_{\substack{n\le m\\ n\ \text{odd}}} \tau_1(n)\alpha^n\right) \le 2M \sum_{\substack{n\le m\\ n\ \text{odd}}} \tau_1(n)\alpha^n.$$

We now bound the last sum appearing in (17). Clearly,

$$\sum_{\substack{n\le m\\ n\ \text{odd}}} \tau_1(n)\alpha^n = \sum_{n\in[m-g(m),m]} \tau(n)\alpha^n + \sum_{1\le n\le m-g(m)} \tau(n)\alpha^n <$$

$$< \alpha^m \log^6 m + \alpha^{m-g(m)} \sum_{1\le n\le m-g(m)} \tau(n) = \alpha^m \log^6 m + O(m\alpha^{m-g(m)}\log m) =$$

$$(18) \qquad = \alpha^m \log^6 m + O\left(\alpha^m \cdot \frac{m\log m}{\alpha^{\log^2 m}}\right) < 2\alpha^m \log^6 m,$$

where in the above inequality we used the well-known fact that the estimate

$$(19) \qquad \sum_{n<x} \tau(n) = O(x\log x)$$

holds for every sufficiently large real number x. From (16)-(18), we get

$$(20) \qquad N_{\mathbf{K}}(u + v\sqrt{d} - A_m w) < \frac{8M^2\log^{12} m}{\alpha^{\log^{1/3} m}},$$

and since the number appearing in the left hand side of (20) is an integer, it follows that it must be zero for m sufficiently large. This shows that $A = A_m$ must hold for m sufficiently large, which is impossible because $R_m > 0$ always.

It remains therefore to prove the Lemma.

The proof of the Lemma. We choose a large real number x.

We first deal with conditions 1 and 3. By result of Landau (see [6]), there exists a positive constant c_4 such that the number of odd numbers m in $[1, x]$ which can be written as a sum of two squares is $c_4 x(1+o(1))/\log^{1/2} x$. Assume now that $m \le x$ is an odd positive integer satisfying condition 1, but failing condition 3. Then there exists a positive integer $k < \log^{1/3} x$ (necessarily even) such that both m and $m+k$ are sums of two squares. Fix the number k. Let $\mathcal{A}_{1,k}(x)$ be the set of all positive integers $m < x$ such that either m or $m+k$ is divisible by p^2 for some prime $p \equiv 3 \pmod 4$ with $p > \log x$, and let $\mathcal{A}_{2,k}(x)$ be the set of all positive integers $m < x$ such that neither m nor $m+k$ is divisible by any prime $p > \log x$ with $p \equiv 3 \pmod 4$. It is clear that if $m < x$ is odd and satisfies 1 but not 3, then $m \in \mathcal{A}_{1,k}(x) \cup \mathcal{A}_{2,k}(x)$ holds with some even $k < \log^{1/3} x$. We first estimate $|\mathcal{A}_{1,k}(x)|$. Assume that $p > \log x$ is a fixed prime with p^2 dividing either m or $m+k$. Clearly, $p^2 \le m+k < 2x$, and the number of such numbers m is

$$\left\lfloor \frac{x}{p^2} \right\rfloor + \left\lfloor \frac{x+k}{p^2} \right\rfloor + 1 \ll \frac{x}{p^2}.$$

Summing up the above inequality over all the possible values of p, we get

$$|\mathcal{A}_{1,k}(x)| \ll \sum_{p>\log x} \frac{x}{p^2} \ll \frac{x}{\log x}. \tag{21}$$

To estimate $|\mathcal{A}_{2,k}(x)|$, we note that since $k < \log^{1/3} x$, it follows that k is coprime to all primes $p > \log x$. Thus, by Brun's method (see, for example, Theorem 2.2 on page 68 of [7]), it follows that the number of positive integers $m < x$ such that $m(m+k)$ is coprime to all primes $p \equiv 3 \pmod 4$ with $p > \log x$ is

$$\begin{aligned} |\mathcal{A}_{2,k}(x)| \ll x \prod_{\substack{\log x < p < x \\ p \equiv 3 \pmod 4}} \left(1 - \frac{2}{p}\right) = \\ = x\exp\left(-2 \sum_{\substack{\log x < p < x \\ p \equiv 3 \pmod 4}} \frac{1}{p} + O\left(\sum_{p>\log x} \frac{1}{p^2}\right)\right) \le \\ \le x\exp(-\log\log x + \log\log\log x + o(1)) \ll \frac{x \log\log x}{\log x}, \end{aligned} \tag{22}$$

where in the last inequality in (22) above we used the well-known fact that the estimate

$$\sum_{\substack{\log p < y \\ p \equiv 3 \pmod 4}} \frac{1}{p} = \frac{1}{2} \log\log y + c_5 + o\left(\frac{1}{\log y}\right)$$

holds for large values of the positive integer y with some absolute constant c_5, estimate which in turn is a well-known corollary of the Prime Number Theorem in arithmetic progressions. Summing up both (21) and (22) over all the allowable values of k, we get that

$$\sum_{k<\log^{1/3} x} |\mathcal{A}_{1,k}(x) \cup \mathcal{A}_{2,k}(x)| \ll$$

$$\ll \frac{x\log^{1/3}x\log\log x}{\log x} = \frac{x\log\log x}{\log^{2/3}x} = o\left(\frac{x}{\log^{1/2}x}\right).$$

The above argument shows that most odd numbers $m < x$ satisfying 1 satisfy 3 as well. In particular, the number of odd numbers m in the interval $[x/2, x]$ satisfying both 1 and 3 is $c_6(1+o(1))x/\log^{1/2}x$, where $c_6 = c_4/2$.

We now find an upper bound on the number of numbers $m \in [x/2, x]$ which fail condition 2. Clearly, since $(\log^4 x)/2 < \log^4 m$ holds for all $m \in [x/2, x]$ when x is large, it suffices to find an upper bound on the number of numbers $m \in [x/2, x]$ for which there exists $n \in [m-\log^2 x,\ m+\log^2 x]$ and with $\tau(n) > (\log^4 x)/2$. By estimate (19), it follows that there are at most $O(x/\log^3 x)$ such positive integers $n \le x$. If m has the property that there exists such an n with $|m-n| < \log^2 x$, this puts m in an interval of length $2\log^2 x$ around n, so m can take at most $2\log^2 x$ values for a fixed n. Since we have at most $O(x/\log^3 x)$ such n, it follows that the number of numbers $m \in [x/2, x]$ failing condition 2 is at most $O(x/\log x) = o(x/\log^{1/2}x)$. This completes the proofs of both the Lemma and Theorem 2.

References

[1] **André-Jeannin R.,** Irrationalité de la somme des inverses de certaines suites récurrentes, *C.R. Acad. Sci. Paris, Sér. I. Math.,* **308** (1989), 539-541.

[2] **Borwein P.,** On the irrationality of $\sum(1/(q^n+r))$, *J. Number Theory,* **37** (1991), 253-259.

[3] **Duverney D., Nishioka Ke., Nishioka Ku. and Shiokawa I.,** Transcendence of Rogers-Ramanujan continued fraction and reciprocal sums of Fibonacci numbers, *Number Theory and its Applications* (Japanese), (Kyoto, 1997), *Sūrikaisekikenkyūsho Kōkyūroku,* **1060** (1998), 91-100.

[4] **Erdős P.,** On arithmetical properties of Lambert series, *Proc. Mat. Inst. of Sciences of India,* **13** (1947), 63-78.

[5] **Erdős P.,** On the irrationality of certain series, *Math. Student,* **36** (1968), 222-226.

[6] **Halberstam H. and Richert H.-E.,** *Sieve methods,* Academic Press, 1974.

[7] **Landau E.,** *Handbuch der Lehre von der Verteilung der Primzahlen,* 3rd edition, Chelsea Publ. Co., 1974, 663-669.

[8] **Matala-aho T. and Prévost M.,** Irrationality measures for the series of reciprocals from recurrence sequences, *J. Number Theory,* **96** (2002), 275-292.

[9] **Prévost M.,** On the irrationality of $\sum \frac{t^n}{A\alpha^n + B\beta^n}$, *J. Number Theory,* **73** (1998), 139-161.

(Received June 10, 2004)

F. Luca
Mathematical Institute, UNAM
Ap. Postal 61-3 (Xangari)
CP 58 089
Morelia, Michoacán
Mexico
fluca@matmor.unam.mx