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A SPECIAL CYCLIC–WAITING QUEUEING SYSTEM WITH REFUSALS: THE DISCRETE TIME CASE

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Dedicated to Professor Imre Kátai on the occasion of his 65th birthday

1. Preliminaries and the problem

In [9] we have introduced a continuous cyclic-waiting system functioning in the following way: if the entering entity cannot be serviced upon arrival, it joins the queue in which it is cycling with a fixed cycle time, its further requests for service may be put only at the multiples of the cycle time. There we determined the ergodic distribution for such systems in case of Poisson arrivals and exponentially distributed service time. The model described in [9] received further development in several papers. It was investigated for the case of uniform service time distribution [10], in case of time-limited tasks [13]. Koba [3,5] found sufficient condition for the existence of equilibrium for the $GI/G/1$ system of this type, Koba and Mykhalevich [6,7,14] compared the FCFS and classical retrial service disciplines for it. Kovalenko [8] generalized some of Koba's results in case of light traffic. In [11] we studied the original model for the discrete time case, namely both the interarrival and service times were geometrically distributed.

In connection with the model from [9] Kovalenko proposed a generalization, to examine the case when customers of two types enter the system. In [12] we have investigated such problem, the entering customers constituted Poisson processes, the service time distributions were exponential. In the system only one customer of first type could be present, and could be accepted for service

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in case of free system. All other customers of the first type were refused and lost from the viewpoint of service. There was no restriction on the customers of second type. In [1,2] Kárász generalized this model for the cases when one or both service times were uniformly distributed.

In this paper we are going to consider the discrete time version of the problem described in [12]. Customers of two types may enter, the interarrival and service times are geometrically distributed. In the system only one customer of first type may be present and it can only be accepted for service in case of free system. All other customers of this type are refused, there is no restriction on the customers of second type.

2. Transition probabilities

For the description of the system we use an embedded Markov chain whose states are determined as the number of customers in the system at moments just before the start of service. We underline that actually this means the number of customers of second type, since before the service of the first type the system always is in free state.

If we consider the number of customers entering the system in the interval (t_{k-1}, t_k) (i.e. between the starting moments of two consecutive services), then the length to be taken into account is determined by a geometrically distributed random variable (it gives the possible starting moment of service after t_{k-1}) and the service time of the actual customer. By using the memoryless property of the geometrical distribution one can easily see that the numbers of entering customers on the intervals (t_{k-1}, t_k) are independent random variables, totally determined by the behaviour of the arrival process on them, so the numbers of present customers in the system at these moments constitute a Markov chain.

We find the transition probabilities of this chain, because of the service discipline we are interested in the number of entering customers of the second type.

We introduce the following probabilities:

a_{ji} - the probability of appearence of i customers of the second type at the service of j-th type (from the beginning of service of the j-th type till the beginning of service of the next one) if at the beginning of service there is only one customer in the system;

b_i - the probability os appearence of i customers of the second type at the service of second type (at the beginning of service there are at least two customers in the system);

c_i - the probability of appearence of i customers of the second type for the service after free state.

The corresponding generating functions are

$$A_j(z) = \sum_{i=0}^{\infty} a_{ji} z^i, \qquad B(z) = \sum_{i=0}^{\infty} b_i z^i, \qquad C(z) = \sum_{i=0}^{\infty} c_i z^i, \qquad j = 1, 2.$$

We shortly describe how to determine them. It is necessary to distinguish two cases: at the moment when the service of a customer begins the next one is present or not.

Let us consider the second possibility, i.e. determine a_{ji}. We divide the cycle time T into n equal parts and assume that for a time slice T/n a new customer can appear with probability r (for the sake of simplicity we will not denote the type of customer, only in the final formulae), and the service of the actual customer (if for this period it takes place) is continued with probability q and terminated with probability $1-q$. These two assumptions mean that both the interarrival and the service times have geometrical distributions. Let us denote the service time of a customer by u, the interarrival time by v. The probability of event $u - v = \ell$ ($\ell = 0, 1, 2, \ldots$) is equal to

$$(1) \qquad P\{u - v = \ell\} = \sum_{k=\ell+1}^{\infty} q^{k-1}(1-q)(1-r)^{k-\ell-1} r = \frac{r(1-q)q^{\ell}}{1-q(1-r)}.$$

We determine how much time passes till the beginning of service of the second customer from the moment of its arrival. It is equal to zero if the second customer appears on the last time slice of service of the first one, n if $u-v$ is contained in the interval $[1, n]$, $2n$ if $u - v \in [n+1, 2n]$, and, generally, in if $u - v \in [(i-1)n + 1, in]$. The corresponding probabilities from (1) are

$$\frac{r(1-q)}{1-q(1-r)}, \quad \frac{rq(1-q^n)}{1-q(1-r)}, \quad \frac{rq(q^n - q^{2n})}{1-q(1-r)}, \quad \ldots,$$

and generally

$$\sum_{\ell=(i-1)n+1}^{in} \frac{r(1-q)q^{\ell}}{1-q(1-r)} = \frac{rq}{1-q(1-r)} \left(q^{(i-1)n} - q^{in}\right).$$

The generating function of customers appearing for a time slice is $1-r+rz$, so the generating function of the number of customers entering during the waiting time is

$$\sum_{i=1}^{\infty} \frac{rq(1-q^n)}{1-q(1-r)} q^{(i-1)n}(1-r+rz)^{in} = \frac{rq(1-r+rz)^n(1-q^n)}{[1-q(1-r)][1-q^n(1-r+rz)^n]}.$$

Taking into account that the first customer obligatorily enters and the waiting time may be equal to zero, we get

$$A(z) = \frac{(1-r)(1-q)}{1-q(1-r)} + z\frac{r(1-q)}{1-q(1-r)} + z\frac{rq(1-r+rz)^n(1-q^n)}{[1-q(1-r)][1-q^n(1-r+rz)^n]},$$

where $\frac{(1-r))(1-q)}{1-q(1-r)}$ is the probability of event that during the service time a second one does not appear at all.

For all other states the transition probabilities can be found on the following way. In this case at the beginning of service of the first customer the second one is already present in the system. Let

$$x = u - \left[\frac{u-1}{n}\right] n$$

($[x]$ denotes the integer part of the number x), and y mean the deviation of interarrival time *mod* n $(1 \le y \le n)$. The duration of period between the starting moments of services of two consecutive customers will be

$$\left[\frac{u-1}{n}\right] n + y \quad \text{if} \quad x \le y \qquad \text{and} \qquad \left(\left[\frac{u-1}{n}\right] + 1\right) n + y \quad \text{if} \quad x > y.$$

Let us fix y and consider the cycle $[in+1, (i+1)n]$. If the service of the first customer is completed till y (including it), the waiting time is $in+y$ and the probability of this event

$$\sum_{j=in+1}^{in+y} q^{j-1}(1-q) = q^{in} - q^{in+y},$$

if $x > y$ the waiting time is $(i+1)n+y$ and the corresponding probability is

$$\sum_{j=in+y+1}^{(i+1)n} q^{j-1}(1-q) = q^{in+y} - q^{(i+1)n}.$$

i can take on any value from zero to infinity, at fixed y for the generating function of the number of entering customers during the investigated period we obtain the following expressions

$$\sum_{i=0}^{\infty}[q^{in}-q^{in+y}](1-r+rz)^{in+y}=\frac{(1-r+rz)^y-(1-r+rz)^y q^y}{1-q^n(1-r+rz)^n},$$

$$\sum_{i=0}^{\infty}[q^{in+y}-q^{in+n}](1-r+rz)^{in+n+y}=\frac{q^y(1-r+rz)^{n+y}-q^n(1-r+rz)^{n+y}}{1-q^n(1-r+rz)^n}.$$

y has truncated geometrical distribution

$$\frac{(1-r)^{k-1}r}{1-(1-r)^n},\qquad k=1,2,\ldots,n,$$

consequently the desired generating function of transition probabilities will be

$$B(z)=\sum_{i=0}^{\infty}b_i z^i=\frac{1-(1-r)^n(1-r+rz)^n}{1-(1-r)(1-r+rz)}\frac{r(1-r+rz)}{1-(1-r)^n}+$$

$$+\frac{1-q^n(1-r)^n(1-r+rz)^n}{1-q(1-r)(1-r+rz)}\frac{rq(1-r+rz)[(1-r+rz)^n-1]}{[1-(1-r)^n][1-q^n(1-r+rz)^n]}.$$

On the basis of these formulae $A_1(z)$, $A_2(z)$ and $B(z)$ will be

(2)
$$A_1(z)=\frac{(1-r_2)(1-q_1)}{1-q_1(1-r_2)}+$$

$$+z\frac{r_2(1-q_1)}{1-q_1(1-r_2)}+z\frac{r_2q_1(1-r_2+r_2z)^n(1-q_1^n)}{[1-q_1(1-r_2)][1-q_1^n(1-r_2+r_2z)^n]},$$

(3)
$$A_2(z)=\frac{(1-r_2)(1-q_2)}{1-q_2(1-r_2)}+$$

$$+z\frac{r_2(1-q_2)}{1-q_2(1-r_2)}+z\frac{r_2q_2(1-r_2+r_2z)^n(1-q_2^n)}{[1-q_2(1-r_2)][1-q_2^n(1-r_2+r_2z)^n]},$$

(4)
$$B(z)=\frac{1-(1-r_2)^n(1-r_2+r_2z)^n}{1-(1-r_2)(1-r_2+r_2z)}\frac{r_2(1-r_2+r_2z)}{1-(1-r_2)^n}+$$

$$+\frac{1-q_2^n(1-r_2)^n(1-r_2+r_2z)^n}{1-q_2(1-r_2)(1-r_2+r_2z)}\frac{r_2q_2(1-r_2+r_2z)[(1-r_2+r_2z)^n-1]}{[1-(1-r_2)^n][1-q_2^n(1-r_2+r_2z)^n]}.$$

$C(z)$ can be determined from $A_1(z)$ and $A_2(z)$. For a time slice the customers of two types may appear with probabilities r_1 and r_2, respectively, even there is possible the entry of both of them, this situation is known as collision. In the case of collision we assume that the customer of second type is refused and lost. For a time slice the probability of appearence of at least one customer

$$1-(1-r_1)(1-r_2)=r_1+r_2-r_1r_2.$$

The service (the busy period) begins with a customer of first type with probability

$$\lambda_1=\frac{r_1}{r_1+r_2-r_1r_2},$$

with a customer of second type with probability

$$\lambda_2=\frac{r_2(1-r_1)}{r_1+r_2-r_1r_2}$$

(there appeared a customer of second type, but a one of first type not). So

$$C(z)=\frac{r_1}{r_1+r_2-r_1r_2}A_1(z)+\frac{r_2(1-r_1)}{r_1+r_2-r_1r_2}A_2(z).$$

3. Equilibrium distribution

Theorem. *Let us consider a discrete queueing system with two types of customers in which both the interarrival and service time distributions are geometrical with parameters r_i and q_i, respectively. The service of a customer may start upon arrival or (in case of busy server or queue) at moments differing from it by the multiples of cycle time T equal to n time units. There is no restriction on the customers of the second type, customers of the first type may only join a free system, all other ones of this type are refused. We define a Markov chain whose states correspond to the number of customers in the system at moments just before starting their service. Then the generating function of equilibrium distribution is given by*

$$P(z)=\frac{p_0\left[C(z)-\frac{B(z)}{z}\right]+p_1[A_2(z)-B(z)]}{1-\frac{B(z)}{z}},$$

where $A_1(z)$, $A_2(z)$, $B(z)$ and $C(z)$ are determined by (2-4), and p_0 and p_1 are the first two probabilities from the desired distribution. They are connected with the relation

$$p_1 = \frac{1-c_0}{a_{20}} p_0;$$

p_0 is determined by the condition $P(1) = 1$, and it is equal to

$$p_0 = \frac{1 - B'(1)}{1 - B'(1) + C'(1) + \dfrac{1-c_0}{a_{20}}\left[A_2'(1) - B'(1)\right]}.$$

The condition of existence of an ergodic distribution is the fulfilment of inequality

$$\frac{r_2 q_2}{1-q_2^n} \frac{1 - q_2^n(1-r_2)^n}{1 - q_2(1-r_2)} < (1-r_2)^n. \tag{5}$$

Proof. We consider the embedded Markov chain describing the functioning of the system. The ergodic probabilities denoted by p_i, $i = 0, 1, 2, \ldots$ satisfy the equations

$$p_j = \sum_{k=2}^{j+1} p_k b_{j-k+1} + p_0 c_j + p_1 a_{2j}, \qquad j \geq 1,$$

$$p_0 = p_0 c_0 + p_1 c_{20}.$$

It leads to the equation

$$P(z)\left[1 - \frac{B(z)}{z}\right] = p_0 \left[C(z) - \frac{B(z)}{z}\right] + p_1[A_2(z) - B(z)].$$

From the equation for p_0

$$p_1 = \frac{1-c_0}{a_{20}} p_0.$$

In order to determine p_0 we use the condition $P(1) = 1$. From it have

$$p_0 = \frac{1 - B'(1)}{1 - B'(1) + C'(1) + \dfrac{1-c_0}{a_{20}}\left[A_2'(1) - B'(1)\right]}.$$

We show that

$$a_{20} C'(1) + (1-c_0)[A_2'(1) - B'(1)] > 0. \tag{6}$$

By using (6) it is equal to

$$\lambda_1 a_{20} A_1'(1) + (1 - \lambda_1 a_{10}) A_2'(1) - (1 - \lambda_1 a_{10} - \lambda_2 a_{20}) B'(1).$$

i.e.

$$\frac{r_1}{r_1 + r_2 - r_1 r_2} \frac{(1-r_2)(1-q_2)}{1 - q_2(1-r_2)} A_1'(1) +$$

$$+ \left[1 - \frac{r_1}{r_1 + r_2 - r_1 r_2} \frac{(1-r_2)(1-q_1)}{1 - q_1(1-r_2)} \right] A_2'(1) -$$

$$- \left[\frac{r_1}{r_1 + r_2 - r_1 r_2} \frac{r_2}{1 - q_1(1-r_2)} + \frac{r_2(1-r_1)}{r_1 + r_2 - r_1 r_2} \frac{r_2}{1 - q_2(1-r_2)} \right] B'(1).$$

We have

$$A_1'(1) = \frac{r_2}{1 - q_1(1-r_2)} + \frac{n r_2^2 q_1}{[1 - q_1(1-r_2)](1 - q_1^n)},$$

$$A_2'(1) = \frac{r_2}{1 - q_2(1-r_2)} + \frac{n r_2^2 q_2}{[1 - q_2(1-r_2)](1 - q_2^n)},$$

$$B'(1) = 1 - \frac{n r_2 (1-r_2)^n}{1 - (1-r_2)^n} + \frac{n r_2^2 q_2 [1 - q_2^n (1-r_2)^n]}{[1 - q_2(1-r_2)][1 - (1-r_2)^n](1 - q_2^n)}.$$

Substituting these values after some arithmetics for the value of (6) we obtain

$$(7) \quad \frac{r_1}{r_1 + r_2 - r_1 r_2} \frac{(1-r_2)(1-q_2)}{1 - q_2(1-r_2)} \frac{n r_2^2 q_1}{[1 - q_1(1-r_2)](1 - q_1^n)} +$$

$$(8) \quad + \frac{n r_2^2 q_2}{[1 - q_2(1-r_2)](1 - q_2^n)} -$$

$$(9) \quad - \frac{r_1}{r_1 + r_2 - r_1 r_2} \frac{(1-r_2)(1-q_1)}{1 - q_1(1-r_2)} \frac{n r_2^2 q_2}{[1 - q_2(1-r_2)](1 - q_2^n)} +$$

$$(10) \quad + \frac{r_1}{r_1 + r_2 - r_1 r_2} \frac{r_2}{1 - q_1(1-r_2)} \frac{n r_2 (1-r_2)^n}{1 - (1-r_2)^n} +$$

$$(11) \quad + \frac{r_2(1-r_1)}{r_1 + r_2 - r_1 r_2} \frac{r_2}{1 - q_2(1-r_2)} \frac{n r_2 (1-r_2)^n}{1 - (1-r_2)^n} -$$

$$(12) \quad - \frac{r_1}{r_1 + r_2 - r_1 r_2} \frac{r_2}{1 - q_1(1-r_2)} \frac{n r_2^2 q_2 [1 - q_2^n (1-r_2)^n]}{[1 - q_2(1-r_2)][1 - (1-r_2)^n](1 - q_2^n)} -$$

$$(13) \quad - \frac{r_2(1-r_1)}{r_1 + r_2 - r_1 r_2} \frac{r_2}{1 - q_2(1-r_2)} \frac{n r_2^2 q_2 [1 - q_2^n (1-r_2)^n]}{[1 - q_2(1-r_2)][1 - (1-r_2)^n](1 - q_2^n)}.$$

(8) may be represented in the form

(14)
$$\frac{nr_2^2q_2}{[1-q_2(1-r_2)](1-q_2^n)}\left[\frac{r_1}{r_1+r_2-r_1r_2}+\frac{r_2(1-r_1)}{r_1+r_2-r_1r_2}\right].$$

From the first term of (14) and (12) we get
(15)
$$\frac{r_1}{r_1+r_2-r_1r_2}\frac{nr_2^2q_2}{[1-q_2(1-r_2)](1-q_2^n)}\left\{1-\frac{r_2}{1-q_1(1-r_2)}\frac{1-q_2^n(1-r_2)^n}{1-(1-r_2)^n}\right\}.$$

Coming to the common denominator for the expression in the brackets, the numerator equals

$$[1-q_1(1-r_2)][1-(1-r_2)^n]-r_2[1-q_2(1-r_2)^n]=$$

$$=1-q_1(1-r_2)-(1-r_2)^n+q_1(1-r_2)(1-r_2)^n-r_2+r_2q_2^n(1-r_2)^n=$$

$$=1-q_1(1-r_2)-r_2-(1-r_2)^n[1-q_1(1-r_2)-r_2]-r_2(1-r_2)^n+r_2q_2^n(1-r_2)^n=$$

$$=[1-q_1(1-r_2)-r_2][1-(1-r_2)^n]-r_2(1-r_2)^n(1-q_2^n)=$$

$$=(1-r_2)(1-q_1)[1-(1-r_2)^n]-r_2(1-r_2)^n(1-q_2^n),$$

i.e. (15) equals

$$\frac{r_1}{r_1+r_2-r_1r_2}\frac{(1-r_2)(1-q_1)[1-(1-r_2)^n]}{[1-q_1(1-r_2)][1-(1-r_2)^n]}\frac{nr_2^2q_2}{[1-q_2(1-r_2)](1-q_2^n)}-$$

$$-\frac{r_1}{r_1+r_2-r_1r_2}\frac{r_2(1-r_2)^n(1-q_2^n)}{[1-q_1(1-r_2)][1-(1-r_2)^n]}\frac{nr_2^2q_2}{[1-q_2(1-r_2)](1-q_2^n)}.$$

Here the first member compensates (9), (10) and the second term give together

$$\frac{r_1}{r_1+r_2-r_1r_2}\frac{nr_2^2}{[1-q_1(1-r_2)][1-(1-r_2)^n]}\times$$

$$\times\left\{(1-r_2)^n-\frac{r_2q_2(1-r_2)^n(1-q_2^n)}{[1-q_2(1-r_2)](1-q_2^n)}\right\}=$$

$$=\frac{r_1}{r_1+r_2-r_1r_2}\frac{nr_2^2(1-r_2)^n}{[1-q_1(1-r_2)][(1-(1-r_2)^n)]}\frac{1-q_2}{1-q_2(1-r_2)}>0.$$

The same may be done for the second term of (14), (13) and (11), so the inequality (6) is proved. Consequently, for the existence of ergodic distribution the inequality $1 - B'(1)$ must be fulfilled. This leads to the inequality

$$\frac{nr_2(1-r_2)^n}{1-(1-r_2)^n} - \frac{nr_2^2[1-q_2^n(1-r_2)^n]}{(1-q_2^n)[1-(1-r_2)^n][1-q_2(1-r_2)]} > 0,$$

from which we obtain the condition of existence of the ergodic distribution (5). The theorem is proved.

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