Annales Univ. Sci. Budapest., Sect. Comp. 24 (2004) 307-322

NON–NORMAL LIMIT THEOREM FOR A NEW TAIL INDEX ESTIMATION

L. Szeidl (Pécs, Hungary)

Dedicated to Professor Imre Kátai on the occasion of his 65th birthday

Abstract. The recent work deals with a new tail index estimation based on empirical power processes (Szeidl [14], Szeidl and Zolotarev [16]). It is proved that this estimation converges to the tail index with probability 1, it converges in mean square and the limit distribution of linearly normalized estimation is non-Gaussian law.

Let $X_1, X_2, \dots$ be a sequence of independent, nonnegative and identically distributed random variables with common distribution function $F(x)$. We suppose that the asymptotic condition

$$\overline{F}(x) = 1 - F(x) = x^{-\alpha} L(x), \quad x \to \infty \tag{1}$$

holds, where $\alpha > 0$ is constant and $L(x)$ is a slowly varying function at infinity. The condition (1) means that the tail distribution function regularly varies at $+\infty$ with index α, i.e.

$$\lim_{t\to\infty} (1 - F(tx))/(1 - F(t)) = x^{-\alpha}, \quad x > 0.$$

In the last decades several mathematicians have been dealing with the estimation of the tail index α (see, for example, the papers of Hill [8], De Haan and Resnick [5], Hall [7], Csörgő, Deheuvels and Mason [3], Csörgő and Viharos [4], Viharos [17], Resnick and Stărică [11] and others). The known estimations

---

The research was supported by the Hungarian National Foundation for Scientific Research under grant OTKA T038423.

are essentially based on the usage of ordered samples. The paper of Csörgő and Viharos [4], partly gives a good survey on the statistical behavior of the estimation of this type. Below, we are going to introduce and analyze the asymptotic behavior of a new estimation for the tail index that comes up when investigating the empirical power processes (Szeidl [14], Szeidl and Zolotarev [16]).

Consider the empirical power processes

$$Z_n(t) = \sum_{j=1}^{n} X_j^t, \quad 0 \le t < \infty, \quad n = 1, 2, \dots .$$

Let $A > 1$ and $s > 1$ be given constant numbers and let us define the sequence of random moments $t_{n,s}$, $n = 1, 2, \dots$ with the help of the empirical power processes $Z_n(t)$, for which $t_{1,s} = 0$ and in the case $n \ge 2$

$$t_{n,s} = \begin{cases} \min\{t : Z_n(t) = n^s\}, & \text{if } \max\{X_j : 1 \le j \le n\} > A, \\ 0, & \text{if } \max\{X_j : 1 \le j \le n\} \le A. \end{cases}$$

Let us introduce the sequence of statistics

$$\hat{\alpha}_{n,s} = \frac{1}{s} t_{n,s}, \quad n \ge 1$$

with the help of random moments $t_{n,s}$. This paper deals with the statistical properties of the estimates $\hat{\alpha}_{n,s}$, $n = 1, 2, \dots$.

By choosing appropriate (deterministic) functions $a_n(t)$, $b_n(t) \ne 0$, $n = 1, 2, \dots$ the process $\overline{Z}_n(t) = (Z_n(t) - a_n(t))/b_n(t)$ on the interval $0 \le t \le \alpha/2$ can be approximated by Gaussian process (Csörgő, M. et al. [2], Szeidl [13, 14], while on the interval $\alpha/2 < t < \infty$ the limit process is continuous with probability I and has stable marginal distributions (Szeidl [13, 14]), Szeidl and Zolotarev [16]). In the special case, on the interval $\alpha < t < \infty$ choosing the deterministic functions as follows

$$\begin{aligned} a_n(t) &= 0, \quad n = 1, 2, \dots \\ b_n(t) &= D_n^t, \quad \text{where} \quad D_n = \sup\{u : n \ge u^\alpha L^{-1}(u),\ u > 0\}, \end{aligned}$$

the finite dimensional distributions of the process $\overline{Z}_n(t) = Z_n(t)/b_n(t)$ converge to the suitable finite dimensional distributions of the process $\zeta(t/\alpha)$, $\alpha < t < \infty$, where the integral in the definition of the process

$$\zeta(t) = t \int_0^\infty N(u) u^{-1-t} du, \quad 1 < t < \infty$$

is defined in the sense of quadratic mean and $\{N(u),\ u \geq 0\}$ denotes a homogeneous Poisson process with intensity 1. The random variables $\zeta(t)$ defined above take only nonnegative values with probability 1, their distribution functions $G_t(x)$ are stable and the characteristic functions can be determined explicitly, see Szeidl [14], and its canonical form (see Zolotarev [18] p. 17.) is the following

$$g_t(\lambda) = \int_{-\infty}^{\infty} e^{i\lambda x} dG_t(x) =$$
$$= \exp\{-|\lambda|^{1/t}\Gamma(1-1/t)e^{-i(\pi/2t)\mathrm{sgn}\lambda}\}, \quad \lambda \in \mathbb{R}^1, \quad 1 < t < \infty.$$

So for the given values x the distribution function of random variables $\zeta(t)$ can be numerically determined.

Note that the convergence of one-dimensional distributions means special case of the classical limit theorems (see Gnedenko, Kolmogorov [6]).

It is clear from the definition of D_n that it can be expressed in form of $D_n = n^{1/\alpha}(h(n))^{1/\alpha}$ (see, for example, Ibragimov and Linnik [9]), where $h(n)$ is a slowly varying function at $+\infty$. From this, it immediately follows that for all fixed constant $s > 1$ the quotient of the coefficients of random variables $Z_n(s\alpha)$ and $Z_n((s+\varepsilon)\alpha)$ in the definition of $\overline{Z}_n(s\alpha)$ and $\overline{Z}_n((s+\varepsilon)\alpha)$ for arbitrary constant $0 < |\varepsilon| < s-1$ is

$$n^{\varepsilon}(h(n))^{\varepsilon}.$$

It means that the exponents of the coefficients differ from each other in a positive value. This observation makes possible the estimation of the tail index α using empirical power processes. The next theorems deal with the asymptotic properties of the sequence of estimates $\hat{\alpha}_{n,s}$, $n = 1, 2, \ldots$.

Theorem 1. *If the condition (1) holds, then $\hat{\alpha}_{n,s} \to \alpha$, $n \to \infty$ with probability 1.*

Theorem 2. *Suppose that the condition (1) holds, then for all real numbers $x > 0$ the following statement is true*

$$(2) \qquad \lim_{n\to\infty} P\left(\left|\frac{\hat{\alpha}_{n,s}}{\alpha} - 1 + \frac{\log h(n)}{\log n}\right| > \frac{x}{\log n}\right) = G_s(e^{-sx}) + 1 - G_s(e^{sx}).$$

Theorem 3. *Suppose that the condition (1) holds. Let $p > 0$ be an arbitrary positive constant, then the following relation holds*

$$(3) \qquad E|\hat{\alpha}_{n,s} - \alpha|^p \leq \alpha^{-p}(\varepsilon_n + o(1/\log n)), \quad n \to \infty,$$

where

$$\varepsilon_n = \inf\left\{\varepsilon : \frac{\log\log n + \log\delta_n(\varepsilon)}{\log n} < \varepsilon\right\}, \quad n = 2, 3, \dots,$$

and

$$\delta_n(\varepsilon) = \max[L(n^{1/\alpha(1-\varepsilon)})], \ 1/L(n^{1/\alpha(1+\varepsilon)})], \quad n = 1, 2, \dots .$$

Corollary 1. If there exists a finite limit value $\lim_{x\to\infty} L(x) = c_0$, then for all real numbers $x > 0$ the following statement is true

$$\lim_{n\to\infty} P\left(\left|\frac{\hat\alpha_{n,s}}{\alpha} - 1 + \frac{\log c_0}{\log n}\right| > \frac{x}{\log n}\right) = G_s(e^{-sx}) + 1 - G_s(e^{sx}).$$

The case $0 < c_0 < \infty$ coincides with the so called Zipf of Pareto type distribution functions, for which $L(x) = c_0 + o(1)$, i.e. the asymptotic relation

$$1 - F(x) = (c_0 + o(1))x^{-\alpha}, \quad x \to \infty$$

holds. In this case $h(n) = c_0 + o(1)$.

Remark. For all slowly varying at infinity function l the following convergence is true (see Seneta [12])

$$\lim_{n\to\infty} \frac{\log l(n)}{\log n} = 0,$$

which is important from the point of view of Theorem 2.

Corollary 2. Since $\delta_n = \sup_{|\varepsilon|\le 1/2} \delta_n(\varepsilon)$ is a slowly varying function (see later formula (21), then $\log\delta_n/\log n \to 0$, $n \to \infty$ and $\varepsilon_n \to 0$, $n \to \infty$. Thus, from the Theorem 3 it follows that the relation $E|\hat\alpha_{n,s} - \alpha|^p \to 0$, $n \to \infty$ is true for all $p > 0$, therefore the sequence of estimates $\hat\alpha_{n,s}$ is asymptotically unbiased and strictly consistent.

Proof of Theorem 1. The assertion of Theorem 1 is equivalent to the convergence of $t_{n,s} \to s\alpha$, $n \to \infty$ with probability 1, which is valid (see Petrov [10] p. 215., Lemma 6) for every $\varepsilon > 0$ if and only if the following convergence is true

$$P\left(\bigcup_{m=n}^{\infty} \{|t_{m,s} - s\alpha| > \varepsilon\alpha\}\right) \to 0, \quad n \to \infty.$$

It is evident that we can suppose that $0 < \varepsilon < 1$.

The event $\{|t_{m,s} - s\alpha| > \varepsilon\alpha\}$ can be expressed with the disjoint events in the following way:

$$\{|t_{m,s} - s\alpha| > \varepsilon\alpha\} =$$

$$= \{t_{m,s} = 0\} \cup \{0 < t_{m,s} < (s-\varepsilon)\alpha\} \cup \{t_{m,s} > (s+\varepsilon)\alpha\} =$$
$$= \{t_{m,s} = 0\} \cup \{Z_m((s-\varepsilon)\alpha > m^s)\} \cup \{Z_m(s+\varepsilon)\alpha < m^s,\ t_{m,s} > 0\} =$$
$$= \{Z_m((s-\varepsilon)\alpha > m^s)\} \cup \{Z_m(s+\varepsilon)\alpha < m^s\}.$$

From this it can be seen that

$$P\left(\bigcup_{m=n}^{\infty} \{|t_{m,s} - s\alpha| > \varepsilon\alpha\}\right) = P\left(\bigcup_{m=n}^{\infty} \{Z_m((s-\varepsilon)\alpha > m^s)\}\right) + $$
$$+ P\left(\bigcup_{m=n}^{\infty} \{Z_m((s+\varepsilon)\alpha < m^s)\}\right).$$

At first we prove that for every $0 < \varepsilon < 1$ the following convergence is true

$$\frac{Z_n((s-\varepsilon)\alpha)}{n^s} \to 0, \quad n \to \infty \text{ with probability } 1, \tag{4}$$

from which by the above mentioned lemma it follows that

$$P\left(\bigcup_{m=n}^{\infty} \left\{\frac{Z_m((s-\varepsilon)\alpha)}{m^s} > 1\right\}\right) \to 0, \quad n \to \infty.$$

Since

$$Z_m((s-\varepsilon)\alpha) = \sum_{j=1}^{m} X_j^{(s-\varepsilon)\alpha}$$

is a sum of i.i.d. r.v., and since for the sequence $a_k = k^s,\ k = 1, 2, \ldots$

$$\sum_{k=n}^{\infty} \frac{1}{a_k} = O(n/a_n)\ (= o(1/n)), \quad n \to \infty$$

then (see Petrov [10] p. 226., Theorem 16) the convergence (4) is satisfied, if the following condition holds:

$$\sum_{n=1}^{\infty} P(X^{(s-\varepsilon)\alpha} \geq n^s) < \infty. \tag{5}$$

From (1) it immediately follows that

$$P(X^{(s-\varepsilon)\alpha} \geq n^s) = P\left(X \geq n^{s/[(s-\varepsilon)\alpha]}\right) = n^{-1-\varepsilon/(s-\varepsilon)} L(n^{[s/(s-\varepsilon)\alpha]}),$$

where the function $L(n^{s/(s-\varepsilon)\alpha})$ is slowly varying and the exponent $-1-\varepsilon/(s-\varepsilon)$ of n in the right hand side is less than -1, therefore (see Seneta [12], 1.5.§.) the series (5) is convergent for all $0 < \varepsilon < 1$.

Now we prove the convergence

$$I_n = P\left(\bigcup_{m=n}^{\infty} \{Z_m((s+\varepsilon)\alpha) < m^s\}\right) \to 0, \quad n \to \infty.$$

With the use of the well-known inequality $\log(1+x) \leq x$, $|x| < 1$ we have

$$I_n \leq \sum_{m=n}^{\infty} P(Z_m((s+\varepsilon)\alpha) < m^s) = \sum_{m=n}^{\infty} P\left(\sum_{j=1}^{m} X_j^{(s+\varepsilon)\alpha} < m^s\right) \leq$$

$$\leq \sum_{m=n}^{\infty} P\left(\bigcap_{j=1}^{m} \{X_j < m^{s/[(s+\varepsilon)\alpha]}\}\right) = \sum_{m=n}^{\infty} \left[P\left(X < m^{s/[(s+\varepsilon)\alpha]}\right)\right]^m =$$

$$= \sum_{m=n}^{\infty} \left[1 - P\left(X \geq m^{s/[(s+\varepsilon)\alpha]}\right)\right]^m =$$

$$= \sum_{m=n}^{\infty} \exp\left\{m \log\left(1 - P\left(X \geq m^{s/[(s+\varepsilon)\alpha]}\right)\right)\right\} \leq$$

$$\leq \sum_{m=n}^{\infty} \exp\left\{-mP\left(X \geq m^{s/[(s+\varepsilon)\alpha]}\right)\right\}.$$

By (1) we get

$$mP\left(X \geq m^{s/[(s+\varepsilon)\alpha]}\right) =$$

$$= m m^{-s/(s+\varepsilon)} L\left(m^{s/[(s+\varepsilon)\alpha]}\right) = m^{\varepsilon/(s+\varepsilon)} L\left(m^{s/[(s+\varepsilon)\alpha]}\right),$$

therefore

$$\sum_{m=1}^{\infty} \exp\left\{-m^{\varepsilon/(s+\varepsilon)} L\left(m^{s/[(s+\varepsilon)\alpha]}\right)\right\} < \infty,$$

from which it follows the convergence

$$I_n \le \sum_{m=n}^{\infty} \exp\left\{-m^{\varepsilon/(s+\varepsilon)} L\left(m^{s/((s+\varepsilon)\alpha)}\right)\right\} \to 0, \quad n \to \infty.$$

Proof of Theorem 2. From the definitions of $\hat{\alpha}_{n,s}$ and $t_{n,s}$ it follows that for arbitrarily chosen sequence $x_n > 0$ and for sufficiently large n we have

$$\begin{aligned}
&P\left(\left|\frac{\hat{\alpha}_{n,s}}{\alpha} - 1 + \frac{\log h(n)}{\log n}\right| > \frac{x}{\log n}\right) = \\
&= P\left(\left|t_{n,s} - s\alpha + \frac{\log h(n)}{\log n} s\alpha\right| > \frac{xs\alpha}{\log n}\right) = \\
(6)\qquad &= P\left(t_{n,s} - s\alpha + \frac{\log h(n)}{\log n} s\alpha < -\frac{xs\alpha}{\log n}\right) + \\
&+ P\left(t_{n,s} - s\alpha + \frac{\log h(n)}{\log n} s\alpha > \frac{xs\alpha}{\log n}\right) = \\
&= P\left(t_{n,s} < u_{n,s,\alpha}\right) + P\left(t_{n,s} > \nu_{n,s,\alpha}\right) = \\
&= P\left(Z_n(u_{n,s,\alpha}) > n^s\right) + P\left(Z_n(\nu_{n,s,\alpha}) < n^s\right),
\end{aligned}$$

where

$$u_{n,s,\alpha} = s\alpha - \frac{\log h(n)}{\log n} s\alpha - \frac{xs\alpha}{\log n} \quad \text{and} \quad \nu_{n,s,\alpha} = s\alpha - \frac{\log h(n)}{\log n} s\alpha + \frac{xs\alpha}{\log n}.$$

First of all we note that in case of any sequence of real numbers

$$\varepsilon_n \to 0, \quad n \to \infty \quad (s + \varepsilon_n > 0)$$

we can get (see Szeidl [14])

$$(7)\qquad \overline{Z}_n((s+\varepsilon_n)\alpha) = \frac{Z_n((s+\varepsilon_n)\alpha)}{D_n^{(s+\varepsilon_n)\alpha}} \xrightarrow{d} \zeta_s, \quad n \to \infty,$$

where the sign $\xrightarrow{d}$ denotes the convergence in the distribution. It is clear that the following relations are true

$$\begin{aligned}
(8)\qquad P(Z_n(u_{n,s,\alpha}) > n^s) &= P\left(\frac{Z_n((u_{n,s,\alpha}))}{D_n^{u_{n,s,\alpha}}} > n^s D_n^{-u_{n,s,\alpha}}\right) = \\
&= P(\overline{Z}_n(u_{n,s,\alpha}) > \exp\{s \log n - u_{n,s,\alpha} \log D_n\}),
\end{aligned}$$

$$
(9) \quad \begin{aligned} P(Z_n(\nu_{n,s,\alpha}) < n^s) &= P\left(\frac{Z_n(\nu_{n,s,\alpha})}{D_n^{\nu_{n,s,\alpha}}} < n^s D_n^{-\nu_{n,s,\alpha}}\right) = \\ &= P(\overline{Z}_n(\nu_{n,s,\alpha}) < \exp\{s \log n - \nu_{n,s,\alpha} \log D_n\}). \end{aligned}
$$

Since from the well-known result concerning slowly varying functions (see Seneta [12], §.1.5.) for arbitrary slowly varying at infinity function $l(x)$ and for any contant $\beta > 0$,

$$
(10) \quad \lim_{n\to\infty} \frac{(\log l(n))^\beta}{\log n} = 0,
$$

and moreover

$$
(11) \quad \begin{aligned} s \log n - u_{n,s,\alpha} \log D_n &= \\ &= s \log n + s\alpha \left(1 - \frac{\log h(n)}{\log n} - \frac{x}{\log n}\right) \frac{1}{\alpha} (\log n + \log h(n)) = \\ &= s\left(x + x\frac{\log h(n)}{\log n} + \frac{\log^2 h(n)}{\log n}\right) \to sx, \quad n \to \infty, \end{aligned}
$$

and

$$
(12) \quad \begin{aligned} s \log n - \nu_{n,s,\alpha} \log D_n &= \\ &= s \log n + s\alpha \left(1 - \frac{\log h(n)}{\log n} + \frac{x}{\log n}\right) \frac{1}{\alpha} (\log n + \log h(n)) = \\ &= s\left(-x - x\frac{\log h(n)}{\log n} + \frac{\log^2 h(n)}{\log n}\right) \to -sx, \quad n \to \infty, \end{aligned}
$$

therefore from the relations (6)-(12) we can get immediately the assertion (2) of Theorem 2.

Let us consider the sequence of r.v.s $Z_n(\alpha s)$, $n = 1, 2, \ldots$, $s_0 \le s \le s_1$, where $1 < s_0 < s_1 < \infty$. By the definition

$$
Z_n(\alpha s) = X_1^{\alpha s} + \ldots + X_n^{\alpha s}, \quad n = 1, 2, \ldots .
$$

Here the distribution function of the i.i.d. r.v.s $X_j^{\alpha s}$ is the following

$$
F_{\alpha s}(x) = P(X_1^{\alpha s} < x) = F(x^{1/\alpha s}), \quad x > 0.
$$

Let us denote

$$
\overline{F}_{\alpha s}(x) = \overline{F}(x^{1/\alpha s}) = 1 - x^{-1/s} L(x^{1/\alpha s}), \quad x > 0.
$$

In our case for a fixed constant s, $s_0 \le s \le s_1$ the Corollary 2.1. of Borovkov [1] states that there exists a function $\varphi_{\alpha s}(t) \downarrow 0$, $t \downarrow 0$ such that

$$\sup_{x, x \ge t} \frac{P(Z_n(\alpha s) > x)}{n\overline{F}_{\alpha s}(x)} \le 1 + \varphi_{\alpha s}(1/t).$$

For the proof of Theorem 3 we need to verify that this asymptotic relation holds uniformly in s, $s_0 \le s \le s_1$.

Lemma 1. *If the condition (1) holds, then there exists a function* $\varphi(t) \downarrow 0$, $t \downarrow 0$ *such that*

$$\sup_{x : x \ge t} \frac{P(Z_n(\alpha s) > x)}{n\overline{F}_{\alpha s}(x)} \le 1 + \varphi(1/t), \quad s_0 \le s \le s_1.$$

Proof of Lemma 1. Using the proofs of Theorem 2.1. and Corollary 2.1. of Borovkov [1] it is enough to prove that there exists a constant c and a function $\overline{\varphi}(t) \downarrow 0$, $t \downarrow 0$ such that the following assertions are true ($2/(\mu s) < y$, $\mu \downarrow 0$, $\lambda = \mu y \to \infty$)

$$G(\mu) = \int_0^{2/(\mu s)} e^{\mu t} dF_{\alpha s}(t) \le 1 + c\overline{F}_{\alpha s}(1/\mu), \tag{13}$$

$$H(\mu, y) = \int_{1/(\mu s)}^{y} e^{\mu t} dF_{\alpha s}(t) \le e^{\mu y}\overline{F}_{\alpha s}(y)(1 + \overline{\varphi}(1/\lambda)). \tag{14}$$

Firstly we prove (13). Let us denote

$$M_s = 2/(\mu s), \quad g(x) = \int_1^x t^{\alpha(s_0 - 1) - 1} L(t) dt.$$

We note that by the known property of regularly varying functions (see Seneta [12]) the following asymptotic relation satisfies

$$g(x) \sim \frac{1}{\alpha(s_0 - 1)} x^{\alpha(s_0 - 1)} L(x), \quad x \to \infty.$$

From this relation it follows that

$$g(x) \le \frac{2}{\alpha(s_0-1)} x^{\alpha(s_0-1)} L(x), \quad x \ge x_1,$$

if x_1 is large enough. It is easy to see that the following inequalities hold

$$G(\mu) =$$

$$= 1 - e^{2/s}\overline{F}_{\alpha s}(M_s) + \mu \int_0^{M_s} e^{\mu t}\overline{F}_{\alpha s}(t)dt \le 1 + \mu e^{\mu} + \mu e^{2/s_0} \int_1^{M_s} t^{-1/s} L(t^{1/\alpha s})dt =$$

$$= 1 + \mu e^{\mu} + \mu\alpha s e^{2/s_0} \int_1^{M_s^{1/\alpha s}} u^{-\alpha} u^{\alpha s-1} L(u)du =$$

$$= 1 + \mu e^{\mu} + \mu\alpha s e^{2/s_0} \int_1^{M_s^{1/\alpha s}} u^{\alpha(s-s_0)} u^{\alpha(s_0-1)-1} L(u)du \le$$

$$\le 1 + \mu e^{\mu} + \mu\alpha s e^{2/s_0} M_s^{1-s_0/s} g(M_s^{1/\alpha s}) \le$$

$$\le 1 + \mu e^{\mu} + \mu\alpha s e^{2/s_0} M_s^{1-s_0/s} \frac{2}{\alpha(s_0-1)} M_s^{(s_0-1)/s} L(M_s^{1/\alpha s}) \le$$

$$\le 1 + \mu e^{\mu} + \alpha e^{2/s_0} \frac{4}{\alpha(s_0-1)} \overline{F}_{\alpha s}(M_s) \le 1 + c\overline{F}_{\alpha s}(1/\mu).$$

Proof of (14).

$$H(\mu, y) = \int_{M_s}^{y} e^{\mu t} dF_{\alpha s}(t) \le e^{2/s}\overline{F}_{\alpha s}(M_s) + \mu \int_{M_s}^{y} e^{\mu t}\overline{F}_{\alpha s}(t)dt =$$

$$= e^{2/s}\overline{F}_{\alpha s}(M_s) + e^{\mu y} \int_0^{(y-M_s)\mu} e^{-u}\overline{F}_{\alpha s}(y - u/\mu)du.$$

With simple calculation we have

$$\int_0^{(y-M_s)\mu} e^{-u}\overline{F}_{\alpha s}(y-u/\mu)du \le \overline{F}_{\alpha,s}(y) \int_0^{\lambda-2/s} e^{-u} \frac{\overline{F}_{\alpha s}(y-u/\mu)}{\overline{F}_{\alpha s}(y)} du =$$

$$= \overline{F}_{\alpha,s}(y) \int_0^{\lambda-2/s} e^{-u} \left(\frac{\lambda}{\lambda - s}\right)^{1/s} \frac{L([(\lambda-u)/\mu]^{1/\alpha s})}{L(y^{1/\alpha s})} du.$$

By the Karamata theorem the slowly varying function L can be represented as follows

$$L(x) = \exp\left\{ \vartheta_0(x) + \int_1^x \frac{\vartheta(x)}{x} dx \right\}, \quad x > 0, \tag{15}$$

where $\vartheta_0(x)$ and $\vartheta(x)$ are bounded measurable functions, $\vartheta(x) \equiv 0$, $0 < x < 1$, $\vartheta(x)$ is continuous for $1 \le x < \infty$ and $\vartheta_0(x) \to \vartheta_0$, $\vartheta(x) \to 0$ as $x \to \infty$. Using this representation of the function L we get

$$\frac{L([(\lambda-u)/\mu]^{1/\alpha s})}{L(y^{1/\alpha s})} \le \exp\left\{ \Theta(u,y,\mu) + \int_{[(\lambda-u)/\mu]^{1/\alpha s}}^{y^{1/\alpha s}} \frac{|\vartheta(x)|}{x} dx \right\},$$

where the function

$$\Theta(u,y,\mu) = |\vartheta_0([(\lambda-u)/\mu]^{1/\alpha s}) - \vartheta_0(y^{1/\alpha s})|$$

is uniformly bounded and it uniformly converges to zero on every finite interval $0 \le u \le u_0$ as $\lambda = \mu y \to \infty$, $\mu \to 0$. Let us denote

$$\vartheta_1 = \max\{|\vartheta(x)| : x \ge 1\},$$

then

$$\int_{[(\lambda-u)/\mu]^{1/\alpha s}}^{y^{1/\alpha s}} \frac{|\vartheta(x)|}{x} dx \le \vartheta_1 \frac{1}{\alpha s_0} \log \frac{\lambda}{\lambda - u}.$$

From the relations above we get immediately

$$H(\mu, y) \le$$

$$\le e^{\mu y}\overline{F}_{\alpha,s}(y) \int_0^{\lambda-2/s} \exp\{-u - (1/s_0 + \vartheta_1/\alpha s_0)\log(1-u/\lambda) + \Theta(u,y,\mu)\} du.$$

Since $0 \le u \le \lambda - 2/s$, thus for all $k \ge 1$

$$\begin{aligned} -\log(1-u/\lambda) \le& \\ &\le u/\lambda + (u/\lambda)^2/2 + \ldots + (u/\lambda)^k/k + (1-u/\lambda)^{-1}(u/\lambda)^{k+1}/(k+1) \le \\ &\le (u/\lambda)(1 + 1/2 + \ldots + 1/k) + (1-u/\lambda)^{-1}(u/\lambda)/(k+1) \le \\ &\le (u/\lambda)\log(k+1) + us_1/2(k+1). \end{aligned}$$

Choosing $k = [\sqrt{\lambda}]$ we can get the following inequality for $\lambda = \mu y \to \infty$, $\mu \to 0$

$$H(\mu, y) \le e^{\mu y} \overline{F}_{\alpha,s}(y) \times$$
$$\times \int_0^\infty \exp\left\{-u + u(1/s_0 + \vartheta_1/\alpha s_0)(\lambda^{-1} \log([\sqrt{\lambda}] + 1) + s_1([\sqrt{\lambda}] + 1)^{-1})\right\} du =$$
$$= e^{\mu y} \overline{F}_{\alpha,s}(y)(1 + \overline{\varphi}(1/\lambda)),$$

where $\overline{\varphi}(1/\lambda) \to 0$, as $\lambda \to \infty$.

Proof of Theorem 3. First we note that if the inequality $\max\limits_{1 \le j \le n} X_j \ge A$ holds, then by the definition of the process Z_n we have $Z_n(u) \ge A^u$, $u > 0$. From this relation it follows immediately

$$t_{n,s} \le \min\{u : A^u = n^s\} =: \frac{s \log n}{\log A}.$$

Let ε, $0 < \varepsilon \le 1/2$ be an arbitrary constant, then

$$\begin{aligned} \alpha^{-p} E|\hat{\alpha}_{n,s} - \alpha|^p &= (s\alpha)^{-p} E|t_{n,s} - s\alpha|^p \le \\ &\le P(t_{n,s} < s\alpha(1 - \varepsilon)) + \alpha^{-p} \varepsilon^p P(|t_{n,s} - s\alpha| \le \varepsilon) + \\ &+ P(s\alpha(1 + \varepsilon) < t_{n,s} \le (s + 1)\alpha) + \\ &+ (s\alpha)^{-p} \left(\frac{s \log n}{A}\right)^p P(t_{n,s} > (s + 1)\alpha) = \\ &= I_1(\varepsilon) + I_2(\varepsilon) + I_3(\varepsilon) + I_4. \end{aligned} \tag{16}$$

Observe that from the definition of $t_{n,s}$ we have immediately

$$t_{n,s} < t \Leftrightarrow Z_n(t) > n^s,$$

and

$$t_{n,s} > t \Leftrightarrow Z_n(t) < n^s.$$

Using the asymptotic formula of Lemma 1 it is easy to verify that uniformly in ε, $0 \le \varepsilon \le 1/2$ the following relation holds ($\Psi_n \to 0$, $n \to \infty$)

$$I_1(\varepsilon) = P(Z_n(s\alpha(1 - \varepsilon)) > n^s) \le (1 + \psi_n) n \overline{F}_{\alpha, s(1-\varepsilon)}(n^s) =$$
$$= (1 + \psi_n) \exp\left\{-[\varepsilon/(1 - \varepsilon)] \log n + \log L(n^{1/\alpha(1-\varepsilon)})\right\} = I_{11}(\varepsilon).$$

We have with simple calculations

$$I_2(\varepsilon) \le \alpha^{-p}\varepsilon^p = I_{21}(\varepsilon), \tag{17}$$

and

$$\begin{aligned} I_3(\varepsilon) &\le P(s\alpha(1+\varepsilon) < t_{n,s}) = P(Z_n(s\alpha(1+\varepsilon)) < n^s) \le \\ &\le P\left(\max_{1\le j\le n} X_j < n^{1/\alpha(1+\varepsilon)}\right) = [P(X_1 < n^{1/\alpha(1+\varepsilon)})]^n = \\ &= [1 - P(X_1 > n^{1/\alpha(1+\varepsilon)})]^n = \exp\left\{n\log[1 - P(X_1 > n^{1/\alpha(1+\varepsilon)})]\right\} \le \\ &\le \exp\left\{-nP(X_1 > n^{1/\alpha(1-\varepsilon)})\right\} = \exp\left\{-nn^{-1/(1+\varepsilon)}L(n^{1/\alpha(1+\varepsilon)})\right\} = \\ &= \exp\left\{-n^{\varepsilon/(1+\varepsilon)}L(n^{1/\alpha(1+\varepsilon)})\right\} = I_{31}(\varepsilon). \end{aligned}$$

Since

$$\begin{aligned} P(t_{n,s} > (s+1)\alpha) &= P(Z_n((s+1)\alpha) < n^s) \le \\ \le P\left(\max_{1\le j\le n} X_j \le n^{s/(s+1)\alpha}\right) &= [1 - P(X_1 > n^{s/(s+1)\alpha})]^n = \\ = \exp\left\{-nn^{-s/(s+1)}L(n^{s/(s+1)\alpha})\right\} &= \exp\left\{-n^{1/(s+1)}L(n^{s/(s+1)\alpha})\right\}, \end{aligned}$$

therefore

$$I_4 \le I_{41} = (s\alpha)^{-p}\left(\frac{s\log n}{A}\right)^p \exp\left(-n^{1/(s+1)}L(n^{s/(s+1)\alpha})\right) = o(1/\log n). \tag{18}$$

We have the following estimations for sufficiently large n

(19)

$$I_{11}(\varepsilon_n) = (1+\psi_n)\exp\left\{-\varepsilon_n/(1-\varepsilon_n)\log n + L(n^{1/\alpha(1-\varepsilon_n)})\right\} \le (1+\psi_n)\frac{1}{\log n},$$

$$\begin{aligned} I_{31}(\varepsilon_n) &= \\ &= \exp\left\{-\exp[\varepsilon_n\log n - \varepsilon_n^2/(1+\varepsilon_n)\log n + \log L(n^{1/\alpha(1+\varepsilon_n)})]\right\} \le \\ &\le \exp\left\{-\exp[\log\log n - \varepsilon_n^2/(1+\varepsilon_n)\log n]\right\} = \exp\left\{-\log n\exp[-\varepsilon_n^2\log n]\right\}. \end{aligned}$$

Using the representation (15) it is easy to get the following inequality

$$\delta_n = \sup_{|\varepsilon|\le 1/2} \delta_n(\varepsilon) \le \exp\left\{ \vartheta_0(n^{1/\alpha(1-\varepsilon)}) + \int_1^{n^{1/\alpha(1-\varepsilon)}} \frac{\vartheta(x)}{x} dx \right\} \le \tag{20}$$

$$\le \Delta_n = \exp\left\{ |\vartheta_*| + \int_1^{n^{2/\alpha}} \frac{|\vartheta(x)|}{x} dx \right\},$$

where

$$\vartheta_* = \sup_{x>0} |\vartheta_0(x)| < \infty.$$

Since the function Δ_n is slowly varying at the infinity, thus by the relation (10) we get

$$\varepsilon_n^2 \log n \le \frac{(\log\log n + \log \Delta_n)^2}{\log n} = o(1), \quad n \to \infty,$$

therefore

$$I_{31}(\varepsilon_n) \le \frac{1}{n^{1+o(1)}} = o(1/\log n), \quad n \to \infty. \tag{21}$$

Summarizing the relations (16)-(21) we have proved the Theorem 3.

References

[1] **Borovkov A.A.**, Estimates for the distribution of sums and maxima of random variables without Cramér condition, *Siberian Math. J.*, **41** (5) (2000), 811-848.

[2] **Csörgő M., Csörgő S., Horváth L. and Mason M.D.**, Sup-norm convergence of the empirical process indexed by functions and applications, *Probab. Math. Stat.*, **7** (1) (1986), 13-26.

[3] **Csörgő S., Deheuvels P. and Mason D.**, Kernel estimates of the tail index of a distribution, *Ann. Statist.*, **12** (3) (1986), 1050-1077.

[4] **Csörgő S. and Viharos L.**, Estimating tail index, *Asymptotic methods in probability theory and statistics*, ed. B. Sziszkowicz, Elsevier Science, 1998, 833-881.

[5] **De Haan L. and Resnick S.**, A simple asymptotic estimate for the index of a stable distribution, *J.R. Statist. Soc. Ser. B,* **42** (1980), 83-87.

[6] **Gnedenko B.V. and Kolmogorov A.N.**, *Limit distribution for sums of independent random variables*, Addison-Wesley, Cambridge, 1954.

[7] **Hall P.**, On some simple estimates of an exponent of regular variation, *J.R. Statist. Soc. Ser. B*, **44** (1) (1982), 37-42.

[8] **Hill B.M.**, A simple approach to inference about the tail of a distribution, *Ann. Statist.*, **3** (1975), 1163-1174.

[9] **Igragimov I.A. and Linnik Yu.V.**, *Independent and stationary sequences of random variables*, Wolters-Noordhoff Publishing, Groningen, 1971.

[10] **Петров В.В.**, *Предельные теоремы для сумм независимых случайных величин*, Наука, Москва 1987. (Petrov V.V., Limit theorems for sums of independent random variables, Nauka, Moscow, 1987.)

[11] **Resnick S. and Stărică C.**, Smoothing the Hill estimator, *Ann. Appl. Probab.*, **27** (1997), 271-293.

[12] **Seneta E.**, *Regularly varying functions*, Lecture Notes in Mathematics **508**, Springer, 1976.

[13] **Szeidl L.**, Limit distribution of random symmetric functions, *Theor. Probab. Appl.*, **33** (2) (1986), 590-603.

[14] **Szeidl L.**, On power processes defined by independent identically distributed random variables having regularly varying distribution function at the infinity, *Annales Univ. Sci. Budapest. Sect. Comp.*, **14** (1994), 189-217.

[15] **Szeidl L.**, Some problems of investigations of extreme values of dynamic stress processes, *Studies in Vehicle Engineering and Transportation Science. A Festschrift in Honour of Professor Pál Michelberger on Occasion of his 70th Birthday*, eds. J. Bokor, E. Nándori and P. Várlaki, Hungarian Academy of Sciences and Budapest University of Technology and Economics, (ISBN 963 420 660 3), Budapest, 2000, 85-91.

[16] **Szeidl L. and Zolotarev V.M.**, *The theory of limit theorems for random polynomials and related fields*, Leaflets in Mathematics (ISSN 1416-0935), University of Pécs, 2003.

[17] **Viharos L.**, Tail index estimation based on linear combinations of intermediate order statistics, *Statistica Neelandica,* **51** (2) (1977), 164-177.

[18] **Zolotarev V.M.**, *One-dimensional stable distributions*, AMS, Providence, RI, 1986.

(Received December 21, 2004)

L. Szeidl
Institute of Mathematics and Informatics
University of Pécs
Ifjúság u. 6.
H-7624 Pécs, Hungary