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$\mathbf{H}^p$ MULTIPLIERS ON THE DYADIC FIELD

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Dedicated to Professor I. Kátai on the occasion of his 65th birthday

Abstract. In this paper we consider a classical multiplier condition, the Hörmander-Mihlin condition, originally introduced for the trigonometric case. It implies that the multiplier operator is bounded on $\mathbf{L^p}$, $1 < p < \infty$. Here we study the corresponding problem with respect to the Walsh transform and the noncompact dyadic Hardy spaces $\mathbf{H^p}[\mathbf{0}, \infty)$, $0 < p < 1$. We also show that our result is sharp. We note that a similar program was carried out for the trigonometric case and the classical Hardy spaces, and for the Walsh system and the dyadic Hardy spaces on $[0, 1]$ in our previous papers [1] and [2].

1. Introduction

Set $\mathbb{R}^+ = [0, \infty)$. The binary expansion of $x \in \mathbb{R}^+$ is $x = \sum\limits_{j=-\infty}^{\infty} x_j 2^{-j-1}$, where $x_j = 0$ or 1. In case of dyadic rationals, i.e. when there are two expansions of this form, we take the one that terminates in 0's. Then the Walsh functions are defined as

$$w_x(y) = (-1)^{\sum\limits_{k=-\infty}^{\infty} x_k y_{-k-1}} \qquad (x, y \in \mathbb{R}^+). \tag{1.1}$$

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We note that if $x = 2^k$ $(k \in \mathbb{Z})$ then $w_x(y) = w_{2^k}(y) = (-1)^{y_{-k-1}}$. Consequently, w_{2^k} is equal to the k-th Rademacher function.

Let the Walsh-Dirichlet kernels be denoted by D_t :

$$D_t(y) = \int_0^t w_x(y)\,dx \qquad (t, y \in \mathbb{R}^+).$$

It is known (see [5] or [12]) that

$$D_{2^n}(y) = \begin{cases} 2^n & 0 \le y < 2^{-n}, \\ 0 & 2^{-n} \le y < \infty \qquad (n \in \mathbb{Z}). \end{cases} \tag{1.2}$$

It is known that the Walsh system can be considered as the dual group of a locally compact Vilenkin group, the dyadic group. Taibleson ([13]) has developed a distribution theory for local fields. Following his concept of distributions we will consider the dyadic Hardy spaces $\mathbf{H}^p(\mathbb{R}^+)$ $(0 < p < 1)$ as subspaces of the space of dyadic distributions. More precisely, $\mathbf{H}^p(\mathbb{R}^+)$ will be defined by means of atomic decomposition of distributions. To this order let the intervals of the form $[k2^{-n}, (k+1)2^{-n})$ $(k \in \mathbb{N}, n \in \mathbb{Z})$ be called dyadic intervals. The Lebesgue measure of a measurable set A will be denoted by $|A|$. Then a function $\mathbf{a} : \mathbb{R}^+ \mapsto \mathbb{R}$ is a p-atom if there exists a dyadic interval I such that

$$\text{i) } \operatorname{supp} \mathbf{a} \subset I, \qquad \text{ii) } \|\mathbf{a}\|_{\mathbf{L}^\infty(\mathbb{R}^+)} \le |I|^{-1/p}, \qquad \text{iii) } \int_I \mathbf{a} = 0.$$

We say that a dyadic distribution f belongs to $\mathbf{H}^p(\mathbb{R}^+)$ $(0 < p < 1)$ if there exist α_k real numbers with $\sum_{k=0}^{\infty} |\alpha_k|^p < \infty$ and $\mathbf{a}_k$ p-atoms such that

$$f = \sum_{k=0}^{\infty} \alpha_k \mathbf{a}_k. \tag{1.3}$$

The decomposition is understood in the sense of distributions. The $\mathbf{H}^p(\mathbb{R}^+)$ norm is defined by

$$\|f\|_{\mathbf{H}^p(\mathbb{R}^+)} = \inf \left(\sum_{k=0}^{\infty} |\alpha_k|^p \right)^{1/p}$$

with taking the infimum over all decompositions of the form (1.3).

Let $\phi : \mathbb{R}^+ \to \mathbb{R}$, then the Walsh multiplier operator T_ϕ is said to be bounded on $\mathbf{H}^p(\mathbb{R}^+)$ $(0 < p < 1)$ if for every $f \in \mathbf{H}^p(\mathbb{R}^+)$ there exists a $T_\phi \in \mathbf{H}^p(\mathbb{R}^+)$ such that

$$\widehat{T_\phi f}(x) = \phi_k \hat{f}(x) \qquad (0 \le x < \infty),$$

where $\hat{f}$ stands for the Walsh-Fourier transform. Throughout the paper C will denote an absolute positive constant not necessarily the same in different occurrences.

2. Results

In our first theorem we consider a Hörmander-Mihlin ([7], [9]) type condition. We prove that it is sufficient to give boundedness on certain $\mathbf{H}^p(\mathbb{R}^+)$ spaces.

Theorem 2.1. *Let $1 < r \le 2$ and $\dfrac{r}{2r-1} < p < 1$. Suppose that $\varphi \in$ $\in L^\infty(\mathbb{R}^+)$ is differentiable and the inclusion $\varphi' \in L^r_{\mathrm{loc}}(\mathbb{R}^+)$ holds. If*

$$\left(\int_{2^j}^{2^{j+1}} |\varphi'(t)|^r \, dt \right)^{1/r} \le C 2^{-j(1-1/r)} \qquad (j \in \mathbb{Z}) \tag{2.1}$$

then T_φ is bounded on $\mathbf{H}^p(\mathbb{R}^+)$.

In our next theorem we show that *Theorem 2.1* is sharp in the sense that the condition on p can not be relaxed by replacing the right side by any number smaller than $r/(2r-1)$.

Theorem 2.2. *Let $1 \le r \le 2$. If $p < r/(2r-1)$ then there exists a differentiable $\varphi \in L^\infty(\mathbb{R}^+)$ that satisfies (2.1), but T_ϕ is not bounded from $H^p(\mathbb{R}^+)$ to $L^p(\mathbb{R}^+)$.*

For previous results on multipliers on the dyadic Hardy spaces, and Hardy spaces over locally compact Vilenkin groups we refer the reader to the papers [1], [3], [4] and [11].

3. Proofs

For the proof of *Theorem 2.1* we need the following lemma which is a Sidon type inequality. The trigonometric version of it was proved by Móricz [10].

Lemma 3.1. *Let $n, N \in \mathbb{Z}$, and $1 < q \leq 2$. Then for any $\gamma \in L^1_{\mathrm{loc}}(\mathbb{R}^+)$ we have*

$$\int_{2^N}^{\infty} \left| \int_0^{2^n} \gamma(t) D_t(x)\, dt \right| dx \leq C_q 2^{-N(1-1/q)} \left(\int_0^{2^n} |\gamma(t)|^q\, dt \right)^{1/q}. \tag{3.1}$$

Proof. Without loss of generality we may assume $n > N$. Let us start with the following decomposition formula ([6]) for the Dirichlet kernels

$$D_t(x) = w_t(x) \sum_{j=-\infty}^{\infty} t_j w_{2^{-j-1}}(x) D_{2^{-j-1}}(x) \qquad (t, x \in \mathbb{R}^+).$$

Before using this in the left side of (3.1) note that the integration with respect to x is over the interval $[2^N, \infty)$. By (1.2) we have that $D_{2^{-j-1}}(x) = 0$ holds for any $x \geq 2^N$ if $j \leq N-1$. Hence

$$\int_{2^N}^{\infty} \left| \int_0^{2^n} \gamma(t) D_t(x)\, dt \right| dx = \int_{2^N}^{\infty} \left| \sum_{j=N}^{\infty} w_{2^{-j-1}}(x) D_{2^{-j-1}}(x) \int_0^{2^n} t_j \gamma(t) w_t(x)\, dt \right| dx\,.$$

After changing the order of integration and summation we obtain

$$\int_{2^N}^{\infty} \left| \int_0^{2^n} \gamma(t) D_t(x)\, dt \right| dx \leq \sum_{j=N}^{\infty} \int_{2^N}^{\infty} \left| w_{2^{-j-1}}(x) D_{2^{-j-1}}(x) \int_0^{2^n} t_j \gamma(t) w_t(x)\, dt \right| dx\,.$$

We proceed by introducing the notation $g_j(x) = \operatorname{sgn} \int_0^{2^n} t_j \gamma(t) w_t(x)\, dt$, and rewriting $D_{2^{-j-1}}$ as $2^{-(j+1)} \chi_{[0,2^{j+1}]}$, where $\chi_{[0,2^{j+1}]}$ is the characteristic function of $[0, 2^{j+1}]$. Then, after performing a change in the order of integration, our estimate takes the form

$$\int_{2^N}^{\infty} \left| \int_0^{2^n} \gamma(t) D_t(x) dt \right| dx \leq \sum_{j=N}^{\infty} 2^{-(j+1)} \int_0^{2^n} t_j \gamma(t) \int_{2^N}^{\infty} \chi_{[0,2^{j+1}]}(x) g_j(x) w_t(x) dx dt.$$

The inner integral will be considered as the Walsh-Fourier transform, in notation $(\widehat{g_j\chi_{[0,2^{j+1}]}})(t)$, of $g_j\chi_{[0,2^{j+1}]}$ at t. By using Hölder's inequality for the outer integral and then the Hausdorff-Young inequality for the Walsh-Fourier transform we obtain

$$\int_{2^N}^{\infty}\left|\int_0^{2^n}\gamma(t)D_t(x)\,dt\right|dx \le \sum_{j=N}^{\infty}2^{-(j+1)}\|\chi_{[0,2^n]}\gamma\|_{L^q(\mathbb{R}^+)}\|(\widehat{g_j\chi_{[0,2^{j+1}]}})\|_{L^p(\mathbb{R}^+)} \le$$

$$\le C_q\left(\int_0^{2^n}|\gamma(t)|^q\,dt\right)^{1/q}\sum_{j=-N}^{\infty}2^{-(j+1)}\|\chi_{[0,2^{j+1}]}g_j\|_{L^q(\mathbb{R}^+)},$$

where $1/p+1/q=1$.

By the definition of g_j we have $\|\chi_{[0,2^{j+1}]}g_j\|_{L^q(\mathbb{R}^+)}\le 2^{(j+1)/q}$. Therefore

$$\sum_{j=-N}^{\infty}2^{-(j+1)}\|\chi_{[0,2^{j+1}]}g_j\|_{L^q(\mathbb{R}^+)} \le \sum_{j=N}^{\infty}2^{-(j+1)(1-1/q)} \le$$
$$\le C_q 2^{-N(1-1/q)}$$

which is the desired estimate.

Proof of Theorem 2.1. We will show that (2.1) implies that φ satisfies the following condition:

$$\text{(3.2)}\quad \sum_{n=-\infty}^{\infty}2^{n(p-1)}\left(\int_{2^{-(n+1)}}^{2^{-n}}\left|\int_{2^{j-1}}^{2^j}\varphi(t)w_t(x)\,dt\right|dx\right)^p \le C2^{j(p-1)} \qquad (j\in\mathbb{Z}).$$

It was proved by Kitada [8] that (3.2) is sufficient for T_ϕ be bounded on $\mathbf{H}^p(\mathbb{R}^+)$, $0<p<1$. Let us split the sum in (3.2) at $n=j$ and consider the case $n\ge j$ first

$$I_2=\sum_{n=j}^{\infty}2^{n(p-1)}\left(\int_{2^{-(n+1)}}^{2^{-n}}\left|\int_{2^{j-1}}^{2^j}\varphi(t)w_t(x)\,dt\right|dx\right)^p.$$

If $x<2^{-n}$ then $x_k=0$ for every $k<n$. Similarly, $t<2^j$ means $t_k=0$ for every $k<-j$. Since $j\le n$ we have by definition (1.1) that $w_t(x)=1$. Therefore,

$$I_2=\sum_{n=j}^{\infty}2^{n(p-1)}\left(2^{-(n+1)}\left|\int_{2^{j-1}}^{2^j}\varphi(t)\,dt\right|\right)^p.$$

Making use of the fact that φ is bounded, we obtain

$$I_2 \leq \sum_{n=j}^{\infty} 2^{n(p-1)} \left(2^{-(n+1)} 2^j C\right)^p \leq C 2^{j(p-1)},$$

which is corresponds to (3.2).

Let us take the $n < j$ part:

$$I_1 = \sum_{n=-\infty}^{j-1} 2^{n(p-1)} \left(\int_{2^{-(n+1)}}^{2^{-n}} \left| \int_{2^{j-1}}^{2^j} \varphi(t) w_t(x)\, dt \right| dx \right)^p .$$

We start with using integration by parts for the integral with respect to t

$$\int_{2^{j-1}}^{2^j} \varphi(t) w_t(x)\, dt = \varphi(t) D_t(x) \Big]_{2^{j-1}}^{2^j} - \int_{2^{j-1}}^{2^j} \varphi'(t) D_t(x)\, dt\,.$$

Hence

$$\left| \int_{2^{j-1}}^{2^j} \varphi(t) w_t(x)\, dt \right| \leq |\varphi(2^j)| D_{2^j}(x) + |\varphi(2^{j-1})| D_{2^{j-1}}(x) + \left| \int_{2^{j-1}}^{2^j} \varphi'(t) D_t(x)\, dt \right| .$$

Then we have

$$\begin{aligned} I_1 \leq & \sum_{n=-\infty}^{j-1} 2^{n(p-1)} \left(\int_{2^{-(n+1)}}^{2^{-n}} |\varphi(2^{j-1})| D_{2^{j-1}}(x) + |\varphi(2^j)| D_{2^j}(x)\, dx \right)^p + \\ & + \sum_{n=-\infty}^{j-1} 2^{n(p-1)} \left(\int_{2^{-(n+1)}}^{2^{-n}} \left| \int_{2^{j-1}}^{2^j} \varphi'(t) D_t(x)\, dt \right| dx \right)^p = I_{11} + I_{12}. \end{aligned}$$

If $n < j-1$ then $[2^{-(n+1)}, 2^{-n}] \subset [2^{-j+1}, 1]$. Recall that $D_{2^j} = 2^j \chi_{[0,2^{-j})}$, and $D_{2^{j-1}} = 2^{j-1} \chi_{[0,2^{-j+1})}$. This means that the sum in I_{11} reduces to a single term

$$I_{11} = 2^{(j-1)(p-1)} \left(2^{-j} |\varphi(2^{j-1})| 2^{j-1}\right)^p.$$

Again, it follows from the boundedness of φ that $I_{11} \leq C 2^{j(p-1)}$.

Applying *Lemma 3.1* to the integral in I_{12} we obtain

$$\int\limits_{2^{-(n+1)}}^{2^{-n}} \left| \int\limits_{2^{j-1}}^{2^j} \varphi'(t) D_t(x)\, dt \right| dx \leq C 2^{(n+1)(1-1/r)} \left(\int\limits_{2^{j-1}}^{2^j} |\varphi'(t)|^r \, dt \right)^{1/r} .$$

Hence we have by (2.1)

$$I_{22} \leq C \sum_{n=-\infty}^{j-1} 2^{n(p-1)} \left(2^{(n+1)(1-1/r)} 2^{j(1/r-1)} \right)^p =$$

$$= C 2^{jp(1/r-1)} \sum_{n=-\infty}^{j-1} 2^{n(2p-1-p/r)} .$$

It follows from the assumption $p > \dfrac{r}{2r-1}$ that $2p - 1 - \dfrac{p}{r} > 0$. Consequently,

$$I_{12} \leq C 2^{jp(1/r-1)} 2^{j(2p-1-p/r)} = C 2^{j(p-1)}.$$

Combining the estimates for I_1, and I_2 we obtain the claimed estimate.

Proof of Theorem 2.2. Set

$$\sigma(t) = \begin{cases} \dfrac{1}{2}(1 - \cos 2\pi t) & \text{if} \quad 0 \leq t \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Define $\varphi \in L^\infty(\mathbb{R}^+)$ as follows

$$\varphi(t) = \sum_{k=0}^{\infty} 2^{-k(1-1/r)} \tau_{2^k} \sigma(t) \qquad (t \in \mathbb{R}^+),$$

where $\tau_x \sigma(t) = \sigma(t - x)$, $x \in \mathbb{R}$. Then $\varphi \in L^\infty(\mathbb{R}^+)$, $\operatorname{supp} \varphi = \bigcup\limits_{k=0}^{\infty} [2^k, 2^k + 1]$, and φ is differentiable. Moreover

$$\left(\int\limits_{2^k}^{2^{k+1}} |\varphi'(t)|^r \, dt \right)^{1/r} = \left(\int\limits_{2^k}^{2^k+1} 2^{-k(1-1/r)} |2\pi \sin 2\pi t|^r \, dt \right)^{1/r} < 2\pi 2^{-k(1-1/r)}.$$

Consequently, φ satisfies condition (2.1).

We will define the function $f \in H^p(\mathbb{R}^+)$ by means of the p-atoms

$$a_k = 2^{k(1/p-1)}(D_{2^{k+1}} - D_{2^k}) \quad (k \in \mathbb{N}).$$

Let us choose the coefficients λ_k as

$$\lambda_k = 2^{-k(1/p+1/r-2)} \qquad (k \in \mathbb{N}).$$

Then it follows from the condition $p < r/(2r-1)$ that $1/p + 1/r - 2 > 0$. Thus $\sum\limits_{k=0}^{\infty} |\lambda_k|^p < \infty$, i.e. $f = \sum\limits_{k=0}^{\infty} \lambda_k a_k \in H^p(\mathbb{R}^+)$.

The action of the multiplier φ on f can be calculated as follows

$$T_\varphi f(x) = \sum_{k=0}^{\infty} \lambda_k 2^{k(1/p-1)} 2^{-k(1-1/r)} \int\limits_{2^k}^{2^{k+1}} \tau_{2^k}\sigma(t) w_t(x)\, dt \qquad (x \in \mathbb{R}^+).$$

We will show that $\chi_{[0,1)} T_\varphi \notin L^p[0,1)$. To this order let us calculate

$$\int\limits_{2^k}^{2^{k+1}} \tau_{2^k}\sigma(t) w_t(x)\, dt, \qquad 0 \le x < 1.$$

Since $w_t(x) = w_{[t]}(x)$ $(x \in [0,1),\ t \in \mathbb{R}^+)$ (see e.g. [12]) we have

$$\int\limits_{2^k}^{2^{k+1}} \tau_{2^k}\sigma(t) w_t(x)\, dt = w_{2^k}(x) \int\limits_0^1 \frac{1}{2}(1 - \cos 2\pi t)\, dt = w_{2^k}(x) \qquad (x \in [0,1))\,.$$

Consequently, $\chi_{[0,1)} T_\varphi$ takes the form of a Rademacher series. i.e.

$$T_\varphi(x) = \sum_{k=0}^{\infty} r_k(x) \qquad (x \in [0,1))\,.$$

By the Khintchin inequality, $\left\| \sum\limits_{k=0}^{\infty} c_k r_k \right\|_{L^p([0,1)} \approx \left(\sum\limits_{k=0}^{\infty} c_k^2 \right)^{1/2}$. In particular, $\int\limits_0^1 |T_\varphi(x)|^p\, dx = \infty$, i.e. $T_\varphi \notin L^p(\mathbb{R}^+)$.

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