

# WEIGHTED (0,2)–INTERPOLATION WITH INTERPOLATORY BOUNDARY CONDITIONS

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*Dedicated to Professor Imre Kátai,  
on the occasion of his 65th birthday*

**Abstract.** The weighted (0,2)-interpolation is studied in a unified way with two additional interpolatory conditions. The question is how to choose the nodal points and the weight function  $w$  so that the problem is regular. We formulate sufficient conditions on the nodal points and on the weight function. In the regular cases we find simple explicit forms of the interpolational polynomial. Special cases are presented when the nodes are the zeros of the classical orthogonal polynomials.

## 1. Introduction

P. Turán initiated the study of (0,2)-interpolation in order to get an approximate solution to the differential equation

$$y'' + f \cdot y = 0.$$

The first results were published by J. Surányi and P. Turán [12] in 1955. In 1961 J. Balázs [2] introduced a generalization of this problem, the *weighted (0,2)-interpolation problem*: Let the system of nodes

$$(1) \quad -\infty \leq a < x_{n,n} < x_{n-1,n} < \dots < x_{1,n} < b \leq \infty$$

be given in the finite (or infinite) open (or closed) interval  $(a, b)$  and let  $w \in C^2(a, b)$  be a weight function. Find a polynomial  $R_n$  of minimal degree satisfying the conditions

$$(2) \quad R_n(x_{k,n}) = y_{k,n}; \quad (wR_n)''(x_{k,n}) = y_{k,n}'' \quad (k = 1, \dots, n; n \in N),$$

where  $y_{k,n}, y_{k,n}''$  are arbitrary given real numbers.

The questions are how to choose the nodal points and the weight function  $w$ , so that the problem is regular (it has a unique solution) and in the regular case to find simple explicit form of  $R_n$  in order to prove convergence.

J. Balázs [2] investigated the above problem on the interval  $[-1, 1]$ , when the nodes are the roots of the ultraspherical polynomial  $P_n^{(\alpha)}$  ( $\alpha > -1$ ), and the weight function is  $w(x) = (1 - x^2)^{(\alpha+1)/2}$ . He showed, that in this case there does not exist a polynomial of degree  $\leq 2n - 1$  satisfying the requirements (2). He proved, that if  $n$  is even, then under the condition

$$(3) \quad R_n(0) = \sum_{k=1}^n y_{k,n} l_{k,n}^2(0)$$

there exists a unique polynomial of degree  $\leq 2n$  which satisfies (2) (if  $n$  is odd, then the uniqueness fails). (Here  $l_{k,n}(x)$  represent the Lagrange-fundamental polynomials corresponding to the nodal points  $x_{k,n}$ .) He gave the explicit form of this polynomial and proved convergence theorem.

Several authors investigated the weighted (0,2)-interpolation with the additional Balázs-type condition (3) on the roots of the classical orthogonal polynomials (I. Joó [5], I. Joó and L. Szili [6], J. Prasad [7], [8], [9], [10], L. Szili [14]). Then L. Szili [15] treated the weighted (0,2)-interpolational problem with Balázs-type condition in a unified way on the roots of all classical orthogonal polynomials with respect to the existence, uniqueness and representation (explicit formulae).

In special cases J. Bajpai [1], S. Enedunya [4], and J. Balázs [3] substituted the additional condition (3) with interpolatory type conditions. For more results on (0,2) interpolation we refer to the survey paper of L. Szili [16].

In this paper we study the weighted (0,2)-interpolation problem in a unified way with different interpolatory conditions. In these cases we determine sufficient conditions on the nodes and the weight function, for the problem to be regular. In the corollaries we give examples, when the nodes are the zeros of the classical orthogonal polynomials.

**The problem:** On the finite or infinite interval  $[a, b]$  let  $x_{i,n}$ ,  $i = 0, \dots, n$  be distinct points (the *nodal points of interpolation*), and let  $w \in C^2(a, b)$  be a

*weight function* on  $(a, b)$ . Find a polynomial  $Q_n$  of minimal degree satisfying the weighted (0,2)-interpolational conditions

$$Q_n(x_{i,n}) = y_{i,n}, \quad (wQ_n)''(x_{i,n}) = y_{i,n}'' \quad (i = 1, \dots, n-1),$$

with the additional interpolatory conditions

$$Q_n(x_{0,n}) = y_{0,n}, \quad Q_n(x_{n,n}) = y_{n,n}, \quad (\text{boundary - type})$$

or

$$Q_n(x_{0,n}) = y_{0,n}, \quad Q'_n(x_{0,n}) = y'_{0,n}, \quad (\text{initial - type})$$

or

$$Q'_n(x_{0,n}) = y'_{0,n}, \quad Q'_n(x_{n,n}) = y'_{n,n},$$

where  $y_{i,n}$ ,  $y'_{0,n}$ ,  $y'_{n,n}$  and  $y_{i,n}''$  are arbitrary real numbers.

In what follows, let  $n$  be a fixed positive integer and for the sake of simplicity we will use  $x_i$  instead of the double indexed  $x_{i,n}$ . Let  $[a, b]$  be a finite or infinite interval, and let

$$(4) \quad x_0, x_1, \dots, x_{n-1}, x_n \in [a, b]$$

be distinct nodes. Let  $p_{n-1}$  be a polynomial of degree  $n-1$ , for which

$$(5) \quad p_{n-1}(x_i) = 0 \quad (i = 1, \dots, n-1),$$

and let

$$(6) \quad l_j(x) = \frac{p_{n-1}(x)}{p'_{n-1}(x_j)(x - x_j)} \quad (j = 1, \dots, n-1)$$

be the fundamental polynomials of Lagrange interpolation corresponding to the nodal points  $x_1, \dots, x_{n-1}$ . Hence

$$(7) \quad l_j(x_i) = \delta_{i,j} \quad (i, j = 1, \dots, n-1),$$

and

$$(8) \quad l'_j(x_j) = \frac{p''_{n-1}(x_j)}{2p'_{n-1}(x_j)} \quad (j = 1, \dots, n-1).$$

## 2. Preliminaries

### 2.1. The classical orthogonal polynomials

Let us consider the homogeneous differential equation

$$(9) \quad u'' + f \cdot u = 0.$$

**Lemma 1.** *If  $P_n^{(\alpha, \beta)}$  ( $\alpha, \beta > -1$ ) denotes the Jacobi polynomial of degree  $n$ , and*

$$w(x) = (1-x)^{\frac{\alpha+1}{2}}(1+x)^{\frac{\beta+1}{2}},$$

*then  $u = wP_n^{(\alpha, \beta)}$  satisfies the differential equation (9) with*

$$f(x) = \frac{1}{4} \frac{1-\alpha^2}{(1-x)^2} + \frac{1}{4} \frac{1-\beta^2}{(1+x)^2} + \frac{2n(n+\alpha+\beta+1) + (\alpha+1)(\beta+1)}{2(1-x^2)}.$$

**Proof.** Cf. (4.24.1) in [13].

**Lemma 2.** *If  $L_n^{(\alpha)}$  ( $\alpha > -1$ ) denotes the Laguerre polynomial of degree  $n$ , and*

$$w(x) = e^{-\frac{x}{2}} x^{\frac{\alpha+1}{2}},$$

*then  $u = wL_n^{(\alpha)}$  satisfies the differential equation (9) with*

$$f(x) = \frac{2n+\alpha+1}{2x} + \frac{1-\alpha^2}{4x^2} - \frac{1}{4}.$$

**Proof.** Cf. (5.1.2) in [13].

**Lemma 3.** *If  $H_n$  denotes the Hermite polynomial of degree  $n$ , and*

$$w(x) = e^{-\frac{x^2}{2}},$$

*then  $u = wH_n$  satisfies the differential equation (9) with*

$$f(x) = 2n+1-x^2.$$

**Proof.** Cf. (5.5.2) in [13].

## 2.2. The fundamental polynomials

Let us introduce the notations

$$r(x) = (x - x_0)^{\varepsilon_1} (x - x_n)^{\varepsilon_2},$$

$$q(x) = (x - x_0)^{\delta_1} (x - x_n)^{\delta_2},$$

where  $\varepsilon_i, \delta_i \in \{0, 1, 2, \dots\}$  and  $\varepsilon_i \geq \delta_i$  for  $i = 1, 2$ .

**Lemma 4.** *If on the system of nodes (4) the weight function  $w$  satisfies the conditions*

$$(10) \quad w(x_i) \neq 0, \quad (qwp_{n-1})''(x_i) = 0 \quad (i = 1, \dots, n-1),$$

then for  $k = 1, \dots, n-1$  the polynomials

$$(11) \quad A_k(x) = \frac{r(x)}{r(x_k)} l_k^2(x) + \frac{q(x)p_{n-1}(x)}{r(x_k)p'_{n-1}(x_k)} \times \\ \times \left\{ c_k + \int_{x_0}^x \left[ \frac{l'_k(x_k)l_k(t) - l'_k(t)}{t - x_k} \cdot \frac{r(t)}{q(t)} + a_k l_k(t) + b_k p_{n-1}(t) \right] dt \right\}$$

satisfy the weighted (0,2)-interpolational conditions

$$(12) \quad A_k(x_i) = \delta_{i,k}, \quad (wA_k)''(x_i) = 0 \quad (i = 1, \dots, n-1),$$

where

$$(13) \quad a_k = -\frac{(rw)''(x_k)}{2(qw)(x_k)} - 2l'_k(x_k) \left( \frac{r}{q} \right)'(x_k),$$

and  $b_k, c_k$  are arbitrary constants. Furthermore, for  $k = 1, \dots, n-1$  the polynomials

$$(14) \quad B_k(x) = \frac{q(x)p_{n-1}(x)}{2q(x_k)w(x_k)p'_{n-1}(x_k)} \left\{ \tilde{a}_k + \int_{x_0}^x [l_k(t) + \tilde{b}_k p_{n-1}(t)] dt \right\}$$

satisfy the weighted (0,2)-interpolational conditions

$$(15) \quad B_k(x_i) = 0, \quad (wB_k)''(x_i) = \delta_{i,k} \quad (i = 1, \dots, n-1),$$

where  $\tilde{a}_k$  and  $\tilde{b}_k$  are arbitrary constants.

**Proof.** We are looking for  $A_k$  ( $k = 1, \dots, n-1$ ) in the form of

$$(16) \quad A_k(x) = \frac{1}{r(x_k)} \left\{ r(x) l_k^2(x) + q(x) p_{n-1}(x) g_k(x) \right\},$$

where  $g_k$  is a polynomial. From (7) it is obvious, that  $A_k(x_i) = \delta_{i,k}$  for  $i, k = 1, \dots, n-1$ . For  $i \neq k$ , the condition  $(wA_k)''(x_i) = 0$  is equivalent to the equation

$$2l_k'^2(x_i)w(x_i)r(x_i) + 2w(x_i)q(x_i)p_{n-1}'(x_i)g_k'(x_i) = 0,$$

and because of  $w(x_i) \neq 0$ ,  $q(x_i) \neq 0$ , and  $p_{n-1}'(x_i) \neq 0$ , we have

$$g_k'(x_i) = -\frac{l_k'^2(x_i)r(x_i)}{p_{n-1}'(x_i)q(x_i)} = \frac{-1}{p_{n-1}'(x_k)} \cdot \frac{l_k'(x_i)}{x_i - x_k} \cdot \frac{r(x_i)}{q(x_i)}.$$

This inspires us to define  $g_k'$  as

$$g_k'(x) = \frac{1}{p_{n-1}'(x_k)} \left\{ \frac{l_k'(x_k)l_k(x) - l_k'(x)}{x - x_k} \cdot \frac{r(x)}{q(x)} + a_k l_k(x) + b_k p_{n-1}(x) \right\}.$$

It is clear, that  $g_k'$  is a polynomial, and

$$\begin{aligned} g_k(x) &= \\ &= \frac{1}{p_{n-1}'(x_k)} \left\{ c_k + \int_{x_0}^x \left[ \frac{l_k'(x_k)l_k(t) - l_k'(t)}{t - x_k} \cdot \frac{r(t)}{q(t)} + a_k l_k(t) + b_k p_{n-1}(t) \right] dt \right\}, \end{aligned}$$

furthermore

$$(17) \quad g_k'(x_k) = \frac{1}{p_{n-1}'(x_k)} \left\{ \frac{r(x_k)}{q(x_k)} [l_k'^2(x_k) - l_k''(x_k)] + a_k \right\}.$$

The coefficient  $a_k$  we determine from the condition  $(wA_k)''(x_k) = 0$ , which is equivalent to

$$(18) \quad \begin{aligned} (wr)''(x_k) + 4r(x_k)l_k'(x_k)[w'(x_k) + w(x_k)l_k'(x_k)] + \\ + 4w(x_k)r'(x_k)l_k'(x_k) + 2w(x_k)q(x_k)a_k = 0, \end{aligned}$$

where we substituted (17). On using  $p_{n-1}(x_k) = 0$ , from the condition (10)

$$(wp_{n-1})''(x_k) = -\frac{2q'(x_k)w(x_k)p_{n-1}'(x_k)}{q(x_k)},$$

and hence, by (8)

$$\begin{aligned} w'(x_k) + w(x_k)l'_k(x_k) &= \frac{1}{2p'_{n-1}(x_k)} \left[ 2p'_{n-1}(x_k)w'(x_k) + w(x_k)p''_{n-1}(x_k) \right] = \\ &= \frac{(wp_{n-1})''(x_k)}{2p'_{n-1}(x_k)} = -\frac{q'(x_k)}{q(x_k)}w(x_k). \end{aligned}$$

Thus the equation (18) can be written in the form

$$(wr)''(x_k) + 4l'_k(x_k)w(x_k) \left[ -r(x_k)\frac{q'(x_k)}{q(x_k)} + r'(x_k) \right] + 2(wq)(x_k)a_k = 0,$$

and we obtain (13) for  $a_k$  ( $k = 1, \dots, n-1$ ).

Finally, applying (10) it is easy to verify that

$$(wB_k)''(x_i) = \frac{1}{2q(x_k)w(x_k)p'_{n-1}(x_k)} \times 2q(x_i)w(x_i)p'_{n-1}(x_i)l_k(x_i) = \delta_{i,k},$$

and  $B_k(x_i) = 0$  for  $i, k = 1, \dots, n-1$ .

In the next section we will determine the constants  $b_k$ ,  $c_k$ ,  $\tilde{a}_k$  and  $\tilde{b}_k$ , such that, the polynomials  $A_k$  and  $B_k$  are of minimal degree and fulfil the additional interpolational conditions for different choices of  $q$ .

### 3. Results

**Theorem 1.** For  $n \geq 2$  let  $\{x_i\}_{i=0}^n$  be a set of distinct nodes in  $[a, b]$ , and  $p_{n-1}(x) = c(x - x_1) \dots (x - x_{n-1})$ . Let  $w \in C^2(a, b)$  be a weight function. If

$$(19) \quad \int_{x_0}^{x_n} p_{n-1}(t)dt \neq 0,$$

and

$$(20) \quad w(x_i) \neq 0, \quad (wp_{n-1})''(x_i) = 0 \quad (i = 1, \dots, n-1),$$

then there exists a unique polynomial  $Q_n$  of degree at most  $2n - 1$ , which fulfils weighted  $(0, 2)$ -interpolational conditions at  $x_1, \dots, x_{n-1}$  with boundary-type conditions at  $x_0$  and  $x_n$ , that is

$$(21) \quad \begin{aligned} Q_n(x_i) &= y_i & (i = 0, 1, \dots, n-1, n), \\ (wQ_n)''(x_i) &= y_i'' & (i = 1, \dots, n-1), \end{aligned}$$

where  $y_i, y_i''$  are arbitrary real numbers.

**Proof.** Applying Lemma 4 with  $r(x) = (x - x_0)(x - x_n)$  and  $q(x) = 1$ , and from the condition  $A_k(x_0) = 0$  we obtain for  $k = 1, \dots, n-1$

$$(22) \quad \begin{aligned} A_k(x) &= \frac{(x - x_0)(x - x_n)}{(x_k - x_0)(x_k - x_n)} l_k^2(x) + \frac{p_{n-1}(x)}{(x_k - x_0)(x_k - x_n) p'_{n-1}(x_k)} \times \\ &\times \int_{x_0}^x \left[ \frac{l'_k(x_k) l_k(t) - l'_k(t)}{t - x_k} (t - x_0)(t - x_n) + a_k l_k(t) dt + b_k p_{n-1}(t) \right] dt, \end{aligned}$$

where

$$(23) \quad a_k = - \frac{((x - x_0)(x - x_n)w)''(x_k)}{2w(x_k)} - 2l'_k(x_k)(2x_k - x_0 - x_n).$$

From the condition  $A_k(x_n) = 0$  we get

$$(24) \quad b_k = \frac{-1}{\int_{x_0}^{x_n} p_{n-1}(t) dt} \left\{ \int_{x_0}^{x_n} \frac{l'_k(x_k) l_k(t) - l'_k(t)}{t - x_k} (t - x_0)(t - x_n) dt + a_k \int_{x_0}^{x_n} l_k(t) dt \right\}.$$

Now we are looking for  $A_0$  in the form of

$$A_0(x) = p_{n-1}(x)g_0(x),$$

where  $g_0$  is a polynomial of degree at most  $n$ . It is obvious that  $A_0(x_i) = 0$  ( $i = 1, \dots, n-1$ ). For the weighted second derivative at  $x_i$  ( $i = 1, \dots, n-1$ ) we have

$$(wA_0)''(x_i) = 2w(x_i)p'_{n-1}(x_i)g'_0(x_i) = 0,$$

hence

$$g'_0(x_i) = 0 \quad i = 1, \dots, n-1,$$

that is

$$g'_0(x) = a_0 p_{n-1}(x),$$



and it follows

$$g_0(x) = a_0 \int_{x_n}^x p_{n-1}(t) dt + c_0.$$

From the condition  $A_0(x_n) = 0$  we have  $c_0 = 0$ , and from  $A_0(x_0) = 1$

$$a_0 = \frac{1}{p_{n-1}(x_0) \int_{x_n}^{x_0} p_{n-1}(t) dt},$$

and hence

$$(25) \quad A_0(x) = \frac{p_{n-1}(x)}{p_{n-1}(x_0) \int_{x_0}^{x_n} p_{n-1}(t) dt} \int_x^{x_n} p_{n-1}(t) dt.$$

In a similar way we construct

$$(26) \quad A_n(x) = \frac{p_{n-1}(x)}{p_{n-1}(x_n) \int_{x_0}^{x_n} p_{n-1}(t) dt} \int_{x_0}^x p_{n-1}(t) dt.$$

It is obvious, that the polynomials  $A_k$  ( $k = 0, 1, \dots, n$ ) are of degree at most  $2n - 1$ , and  $A_k(x_i) = \delta_{i,k}$  for  $i = 0, 1, \dots, n$ , and  $(wA_k)''(x_i) = 0$  for  $i = 1, \dots, n - 1$ .

Now applying (14), from the condition  $B_k(x_0) = 0$  we get  $\tilde{a}_k = 0$ , and

$$(27) \quad B_k(x) = \frac{p_{n-1}(x)}{2w(x_k)p'_{n-1}(x_k)} \int_{x_0}^x [l_k(t) + \tilde{b}_k p_{n-1}(t)] dt,$$

where

$$(28) \quad \tilde{b}_k = -\frac{\int_{x_0}^{x_n} l_k(t) dt}{\int_{x_0}^{x_n} p_{n-1}(t) dt}$$

is determined by the condition  $B_k(x_n) = 0$ . Hence  $B_k$  is a polynomial of degree  $\leq 2n - 1$ , furthermore  $B_k(x_i) = 0$  ( $i = 0, \dots, n$ ), and  $(wB_k)''(x_i) = \delta_{i,k}$  ( $i = 1, \dots, n - 1$ ).

As the polynomials  $A_k$  ( $k = 0, 1, \dots, n$ ) and  $B_k$  ( $k = 1, \dots, n-1$ ) defined by (22)-(28) are the basis polynomials of the interpolational problem (21), the polynomial

$$(29) \quad Q_n(x) = \sum_{k=0}^n y_k A_k(x) + \sum_{k=1}^{n-1} y_k'' B_k(x)$$

is of degree at most  $2n-1$  and fulfils the equations (21).

For the proof of the uniqueness we study the homogeneous problem: Find a polynomial  $\bar{R}_n$  of degree at most  $2n-1$  such that  $\bar{R}_n(x_i) = 0$  ( $i = 0, 1, \dots, n$ ), and  $(w\bar{R}_n)''(x_i) = 0$  ( $i = 1, \dots, n-1$ ). From these conditions it is obvious, that

$$\bar{R}_n(x) = (x - x_0)(x - x_n)p_{n-1}(x)\bar{g}_{n-2}(x),$$

where  $\bar{g}_{n-2}$  is a polynomial of degree at most  $n-2$ . As for  $i = 1, \dots, n-1$

$$(w\bar{R}_n)''(x_i) = 2w(x_i)p'_{n-1}(x_i)[(x - x_0)(x - x_n)\bar{g}_{n-2}]'(x_i) = 0,$$

and  $w(x_i) \neq 0$ ,  $p'_{n-1}(x_i) \neq 0$ , therefore with a constant  $\bar{c}$

$$(x - x_0)(x - x_n)\bar{g}_{n-2}(x) = \bar{c} \int_{x_0}^x p_{n-1}(t)dt.$$

Substituting  $x = x_n$  we get  $\bar{c} \int_{x_0}^{x_n} p_{n-1}(t)dt = 0$ , and hence  $\bar{c} = 0$ , that is  $R_n(x) \equiv 0$ , which completes the proof.

**Corollary 1.** *Let the set of nodes be*

$$-1 = x_n < x_{n-1} < \dots < x_1 < x_0 = 1$$

where  $\{x_i\}_{i=1}^{n-1}$  are the roots of the Jacobi polynomial  $P_{n-1}^{(\alpha, \beta)}$  of degree  $n-1$  ( $\alpha, \beta > -1; n \geq 2$ ), and let the weight function be

$$w(x) = (1-x)^{\frac{\alpha+1}{2}}(1+x)^{\frac{\beta+1}{2}}.$$

If

$$\int_{-1}^1 P_{n-1}^{(\alpha, \beta)}(t)dt \neq 0,$$

then there exists a unique polynomial  $Q_n$  of degree at most  $2n - 1$ , which fulfils weighted (0,2)-interpolational conditions at the zeros of  $P_{n-1}^{(\alpha,\beta)}$  with boundary-type conditions at  $x_0 = 1$  and  $x_n = -1$ .

**Proof.** Now  $p_{n-1}(x) = P_{n-1}^{(\alpha,\beta)}(x)$ . By Lemma 1 the condition (20) is satisfied.

**Remark.** Corollary 1 was stated and proved by L. Szili [15] in 1993.

In the special case  $\alpha = \beta = -1/2$ , when the inner nodes are the zeros of the Tchebyscheff polynomials of first kind, the explicit form of the interpolational polynomial was given by S. Eneudanya [4] in 1985.

In 1994 P. Bajpai [1] studied the special case  $\alpha = \beta = 1/2$ , when the inner nodes are the zeros of the Tchebyscheff polynomials of second kind. He also proved convergence theorem.

In 1969 J. Prasad and A. Verma [11] studied the special case  $\alpha = \beta$ , they also proved convergence theorem.

**Theorem 2.** For  $n \geq 2$  let  $\{x_i\}_{i=0}^n$  be a set of distinct nodes in  $[a, b]$ , and  $p_{n-1}(x) = c(x - x_1) \dots (x - x_{n-1})$ . Let  $w \in C^2(a, b)$  be a weight function. If

$$(30) \quad w(x_i) \neq 0, \quad ((x - x_n)wp_{n-1})''(x_i) = 0 \quad (i = 1, \dots, n-1),$$

then there exists a unique polynomial  $Q_n$  of degree at most  $2n - 1$ , which fulfils weighted (0,2)-interpolational conditions at  $x_1, \dots, x_{n-1}$  with boundary-type conditions at  $x_0$  and  $x_n$ .

**Proof.** We apply Lemma 4 with  $r(x) = (x - x_0)(x - x_n)$  and  $q(x) = (x - x_n)$ . In order to get the minimal degree  $2n - 1$  for  $A_k$ , let  $b_k = 0$ , and  $c_k = 0$  due to the condition  $A_k(x_0) = 0$ . Hence we obtain for  $k = 1, \dots, n-1$

$$(31) \quad A_k(x) = \frac{(x - x_0)(x - x_n)}{(x_k - x_0)(x_k - x_n)} l_k^2(x) + \frac{(x - x_n)p_{n-1}(x)}{(x_k - x_0)(x_k - x_n)p'_{n-1}(x_k)} \times \\ \times \int_{x_0}^x \left[ \frac{l'_k(x_k)l_k(t) - l'_k(t)}{t - x_k} (t - x_0) + a_k l_k(t) \right] dt,$$

where

$$(32) \quad a_k = -\frac{((x - x_0)(x - x_n)w)''(x_k)}{2(x_k - x_n)w(x_k)} - 2l'_k(x_k).$$

Furthermore let

$$(33) \quad A_0(x) = \frac{(x - x_n)p_{n-1}(x)}{(x_0 - x_n)p_{n-1}(x_0)},$$

and

$$(34) \quad A_n(x) = \frac{p_{n-1}^2(x)}{p_{n-1}^2(x_n)} - \frac{(x - x_n)p_{n-1}(x)}{p_{n-1}^2(x_n)} \times \\ \times \left\{ \frac{p_{n-1}(x_0)}{x_0 - x_n} + \frac{1}{p_{n-1}(x_n)} \int_{x_0}^x \frac{p_{n-1}(x_n)p'_{n-1}(t) - p'_{n-1}(x_n)p_{n-1}(t)}{t - x_n} dt \right\}.$$

It is obvious that the polynomials  $A_k$  ( $k = 0, 1, \dots, n$ ) are of degree at most  $2n - 1$ , and  $A_k(x_i) = \delta_{i,k}$  for  $i = 0, 1, \dots, n$ , and  $(wA_k)''(x_i) = 0$  for  $i = 1, \dots, n - 1$ .

Now applying (14), for  $k = 1, \dots, n - 1$  we obtain

$$(35) \quad B_k(x) = \frac{(x - x_n)p_{n-1}(x)}{2w(x_k)(x_k - x_n)p'_{n-1}(x_k)} \int_{x_0}^x l_k(t) dt,$$

which is a polynomial of degree  $\leq 2n - 1$ , and also  $B_k(x_i) = 0$  ( $i = 0, \dots, n$ ), and  $(wB_k)''(x_i) = \delta_{i,k}$  ( $i = 1, \dots, n - 1$ ).

As the polynomials  $A_k$  ( $k = 0, 1, \dots, n$ ) and  $B_k$  ( $k = 1, \dots, n - 1$ ) defined by (31) - (35) are the basis polynomials of the interpolational problem (21), the polynomial

$$(36) \quad Q_n(x) = \sum_{k=0}^n y_k A_k(x) + \sum_{k=1}^{n-1} y_k'' B_k(x)$$

is of degree at most  $2n - 1$ , and fulfils the equations (21). The uniqueness can be proved in a similar way as in Theorem 1.

**Corollary 2.** *If the nodes are*

$$-1 = x_n < x_{n-1} < \dots < x_1 < x_0 = 1$$

where  $\{x_i\}_{i=1}^{n-1}$  are the roots of the Jacobi polynomial  $P_{n-1}^{(\alpha, \beta)}$  of degree  $n - 1$  ( $\alpha, \beta > -1; n \geq 2$ ), and

$$w(x) = (1 - x)^{\frac{\alpha+1}{2}} (1 + x)^{\frac{\beta-1}{2}}$$

is the weight function, then there exists a unique polynomial  $Q_n$  of degree at most  $2n - 1$ , which fulfils weighted  $(0, 2)$ -interpolational conditions at the zeros of  $P_{n-1}^{(\alpha, \beta)}$  with boundary-type conditions at  $x_0 = 1$  and  $x_n = -1$ .

**Proof.** Now  $p_{n-1}(x) = P_{n-1}^{(\alpha, \beta)}(x)$ . By Lemma 1 the function  $(1+x)wp_{n-1}$  fulfils the conditions (30).

**Theorem 3.** For  $n \geq 2$  let  $\{x_i\}_{i=0}^{n-1}$  be a set of distinct nodes in  $[a, b]$ , and  $p_{n-1}(x) = c(x-x_1)\dots(x-x_{n-1})$ . Let  $w \in C^2(a, b)$  be a weight function. If

$$(37) \quad w(x_i) \neq 0, \quad (wp_{n-1})''(x_i) = 0 \quad (i = 1, \dots, n-1),$$

then there exists a unique polynomial  $Q_n$  of degree at most  $2n-1$ , which fulfils weighted (0,2)-interpolational conditions at  $x_1, \dots, x_{n-1}$  with initial-type conditions at  $x_0$ , that is

$$(38) \quad \begin{aligned} Q_n(x_i) &= y_i & (i = 0, 1, \dots, n-1), \\ Q'_n(x_0) &= y'_0 \\ (wQ_n)''(x_i) &= y''_i & (i = 1, \dots, n-1), \end{aligned}$$

where  $y_i, y''_i, y'_0$  are arbitrary real numbers.

**Proof.** Applying Lemma 4 with  $r(x) = (x-x_0)^2$  and  $q(x) = 1$ , and using the condition  $A_k(x_0) = 0$ , we obtain for  $k = 1, \dots, n-1$

$$(39) \quad \begin{aligned} A_k(x) &= \frac{(x-x_0)^2}{(x_k-x_0)^2} l_k^2(x) + \frac{p_{n-1}(x)}{(x_k-x_0)^2 p'_{n-1}(x_k)} \times \\ &\times \int_{x_0}^x \left[ \frac{l'_k(x_k) l_k(t) - l'_k(t)}{t-x_k} (t-x_0)^2 + a_k l_k(t) + b_k p_{n-1}(t) \right] dt, \end{aligned}$$

where

$$(40) \quad a_k = -\frac{((x-x_0)^2 w)''(x_k)}{2w(x_k)} - 4l'_k(x_k)(x_k-x_0).$$

From the condition  $A'_k(x_0) = 0$  we get

$$(41) \quad b_k = -\frac{a_k l_k(x_0)}{p_{n-1}(x_0)}.$$

Furthermore, let

$$(42) \quad A_0(x) = \frac{p_{n-1}(x)}{p_{n-1}(x_0)} \left\{ 1 - \frac{p'_{n-1}(x_0)}{p_{n-1}^2(x_0)} \int_{x_0}^x p_{n-1}(t) dt \right\}.$$

It is obvious that the polynomials  $A_k$  ( $k = 0, 1, \dots, n-1$ ) are of degree at most  $2n-1$ , and  $A_k(x_i) = \delta_{i,k}$  for  $i = 0, 1, \dots, n-1$ ,  $A'_k(x_0) = 0$  and  $(wA_k)''(x_i) = 0$  for  $i = 1, \dots, n-1$ .

The polynomial

$$(43) \quad C_0(x) = \frac{p_{n-1}(x)}{p_{n-1}^2(x_0)} \int_{x_0}^x p_{n-1}(t) dt$$

is also of degree at most  $2n-1$ , and  $C_0(x_i) = 0$  for  $i = 0, 1, \dots, n-1$ ,  $C'_0(x_0) = 1$  and  $(wC_0)''(x_i) = 0$  for  $i = 1, \dots, n-1$ .

Now applying (14), we have

$$(44) \quad B_k(x) = \frac{p_{n-1}(x)}{2w(x_k)p'_{n-1}(x_k)} \int_{x_0}^x [l_k(t) + \tilde{b}_k p_{n-1}(t)] dt,$$

where

$$(45) \quad \tilde{b}_k = -\frac{l_k(x_0)}{p_{n-1}(x_0)}.$$

It is easy to verify, that  $B_k$  is a polynomial of degree  $\leq 2n-1$ , furthermore  $B_k(x_i) = 0$  ( $i = 0, \dots, n-1$ ),  $B'_k(x_0) = 0$ , and  $(wB_k)''(x_i) = \delta_{i,k}$  ( $i = 1, \dots, n-1$ ).

As the polynomials  $A_k$  ( $k = 0, 1, \dots, n-1$ ),  $B_k$  ( $k = 1, \dots, n-1$ ) and  $C_0$  defined by (39)-(45) are the basis polynomials of the interpolational problem (38), the polynomial

$$(46) \quad Q_n(x) = \sum_{k=0}^{n-1} y_k A_k(x) + \sum_{k=1}^{n-1} y''_k B_k(x) + y'_0 C_0(x)$$

is of degree at most  $2n-1$  and fulfils the equations (38). The uniqueness can be proved in a similar way as in Theorem 1.

**Remark.** Theorem 3 was stated and proved by J. Balázs [3] in 1998. In [3] the basis polynomials  $A_k$  are derived in a different form.

**Theorem 4.** For  $n \geq 2$  let  $\{x_i\}_{i=0}^{n-1}$  be a set of distinct nodes in  $[a, b]$ , and  $p_{n-1}(x) = c(x-x_1)\dots(x-x_{n-1})$ . Let  $w \in C^2(a, b)$  be a weight function. If

$$(47) \quad w(x_i) \neq 0, \quad ((x-x_0)wp_{n-1})''(x_i) = 0 \quad (i = 1, \dots, n-1),$$

then there exists a unique polynomial  $Q_n$  of degree at most  $2n - 1$ , which fulfils weighted (0,2)-interpolational conditions at  $x_1, \dots, x_{n-1}$  with initial-type conditions at  $x_0$ .

**Proof.** We apply Lemma 4 with  $r(x) = (x - x_0)^2$  and  $q(x) = (x - x_0)$ . Following the steps of the proof of Theorem 3, we obtain for  $k = 1, \dots, n - 1$

$$(48) \quad A_k(x) = \frac{(x - x_0)^2}{(x_k - x_0)^2} l_k^2(x) + \frac{(x - x_0)p_{n-1}(x)}{(x_k - x_0)^2 p'_{n-1}(x_k)} \times \\ \times \int_{x_0}^x \left[ \frac{l'_k(x_k)l_k(t) - l'_k(t)}{t - x_k} (t - x_0) + a_k l_k(t) \right] dt,$$

where

$$(49) \quad a_k = -\frac{((x - x_0)^2 w)''(x_k)}{2(x_k - x_0)w(x_k)} - 2l'_k(x_k);$$

(50)

$$A_0(x) = \frac{p_{n-1}^2(x)}{p_{n-1}^2(x_0)} - \frac{(x - x_0)p_{n-1}(x)}{p_{n-1}^2(x_0)} \times \\ \times \left\{ 2p'_{n-1}(x_0) + \frac{1}{p_{n-1}(x_0)} \int_{x_0}^x \frac{p_{n-1}(x_0)p'_{n-1}(t) - p'_{n-1}(x_0)p_{n-1}(t)}{t - x_0} dt \right\},$$

$$(51) \quad C_0(x) = \frac{(x - x_0)p_{n-1}(x)}{p_{n-1}(x_0)},$$

and

$$(52) \quad B_k(x) = \frac{(x - x_0)p_{n-1}(x)}{2w(x_k)(x_k - x_0)p'_{n-1}(x_k)} \int_{x_0}^x l_k(t) dt.$$

As the polynomials  $A_k$  ( $k = 0, 1, \dots, n$ ),  $B_k$  ( $k = 1, \dots, n - 1$ ) and  $C_0$ , defined by (48)-(52) are the basis polynomials of the interpolational problem (38), the polynomial

$$(53) \quad Q_n(x) = \sum_{k=0}^{n-1} y_k A_k(x) + \sum_{k=1}^{n-1} y''_k B_k(x) + y'_0 C_0(x)$$

is of degree at most  $2n - 1$  and fulfils the equations (38). The uniqueness can be proved in a similar way as in Theorem 1.

**Corollary 3.** *If the nodes are*

$$0 = x_0 < x_1 < \dots < x_{n-1},$$

where  $\{x_i\}_{i=1}^{n-1}$  are the roots of the Laguerre polynomial  $L_{n-1}^{(\alpha)}$  of degree  $n - 1$  ( $\alpha > -1; n \geq 2$ ), and

$$w_1(x) = e^{-\frac{x}{2}} x^{\frac{\alpha+1}{2}}$$

or

$$w_2(x) = e^{-\frac{x}{2}} x^{\frac{\alpha-1}{2}}$$

are weight functions, then there exists a unique polynomial  $Q_n$  of degree at most  $2n - 1$ , which fulfils weighted  $(0, 2)$ -interpolational conditions at the zeros of  $L_{n-1}^{(\alpha)}$  with initial-type conditions at  $x_0 = 0$ .

**Proof.** Let  $p_{n-1}(x) = L_{n-1}^{(\alpha, \beta)}(x)$ . By Lemma 2 the conditions (37) and (47) are satisfied with the weight functions  $w_1$  and  $w_2$ , respectively.

**Theorem 5.** *For  $n \geq 2$  let  $\{x_i\}_{i=0}^n$  be a set of distinct nodes in  $[a, b]$ , and  $p_{n-1}(x) = c(x - x_1) \dots (x - x_{n-1})$ . Let  $w \in C^2(a, b)$  be a weight function. If*

$$(54) \quad w(x_i) \neq 0, \quad (wp_{n-1})''(x_i) = 0 \quad (i = 1, \dots, n-1),$$

and

$$(55) \quad p'_{n-1}(x_0)p'_{n-1}(x_n) \int_{x_0}^{x_n} p_{n-1}(t)dt + \\ + p'_{n-1}(x_0)p_{n-1}^2(x_n) - p'_{n-1}(x_n)p_{n-1}^2(x_0) \neq 0,$$

then there exists a unique polynomial  $Q_n$  of degree at most  $2n - 1$ , which fulfils weighted  $(0, 2)$ -interpolational conditions at  $x_1, \dots, x_{n-1}$  with additional interpolatory conditions at  $x_0$  and  $x_n$ , that is

$$(56) \quad \begin{aligned} Q_n(x_i) &= y_i, & (wQ_n)''(x_i) &= y_i'', & (i = 1, \dots, n-1), \\ Q'_n(x_0) &= y'_0, & Q'_n(x_n) &= y'_n, \end{aligned}$$

where  $y_i, y_i'', y'_0$  and  $y'_n$  are arbitrary real numbers.



**Proof.** Applying Lemma 4 with  $r(x) = (x - x_0)^2(x - x_n)$  and  $q(x) = 1$ , we obtain for  $k = 1, \dots, n - 1$

$$(57) \quad A_k(x) = \frac{(x - x_0)^2(x - x_n)}{(x_k - x_0)^2(x_k - x_n)} l_k^2(x) + \frac{p_{n-1}(x)}{(x_k - x_0)^2(x_k - x_n) p'_{n-1}(x_k)} \times \\ \times \left\{ c_k + \int_{x_0}^x \left[ \frac{l'_k(x_k) l_k(t) - l'_k(t)}{t - x_k} (t - x_0)^2 (t - x_n) + a_k l_k(t) + b_k p_{n-1}(t) \right] dt \right\},$$

where

$$(58) \quad a_k = -\frac{((x - x_0)^2(x - x_n)w)''(x_k)}{2w(x_k)} - 2l'_k(x_k)(x_k - x_0)(3x_k - 2x_n - x_0).$$

The equations  $A'_k(x_0) = 0$  and  $A'_k(x_n) = 0$  are equivalent to the linear system

$$(59) \quad p'_{n-1}(x_0)c_k + p_{n-1}^2(x_0)b_k = -a_k p'_{n-1}(x_0)l_k(x_0), \\ p'_{n-1}(x_n)c_k + \left[ p'_{n-1}(x_n) \int_{x_0}^{x_n} p_{n-1}(t)dt + p_{n-1}^2(x_n) \right] b_k = \\ = -p'_{n-1}(x_n) \left\{ (x_n - x_0)^2 l_k^2(x_0) + \right. \\ \left. + \int_{x_0}^{x_n} \left[ \frac{l'_k(x_k) l_k(t) - l'_k(t)}{t - x_k} (t - x_0)^2 (t - x_n) + a_k l_k(t) \right] dt \right\} - \\ - a_k p_{n-1}(x_n) l_k(x_n),$$

which has unique solution for  $b_k$  and  $c_k$  if and only if its determinant is not 0, that is the condition (55) is fulfilled.

From (14) we have

$$(60) \quad B_k(x) = \frac{p_{n-1}(x)}{2w(x_k)p'_{n-1}(x_k)} \left\{ \tilde{a}_k + \int_{x_0}^x [l_k(t) + \tilde{b}_k p_{n-1}(t)] dt \right\},$$

where the constants  $\tilde{a}_k$  and  $\tilde{b}_k$  are determined from the equations

$$B'_k(x_0) = 0, \quad B'_k(x_n) = 0,$$

that is, from the linear system

(61)

$$\begin{aligned} p'_{n-1}(x_0)\tilde{a}_k + p_{n-1}^2(x_0)\tilde{b}_k &= -p'_{n-1}(x_0)l_k(x_0), \\ p'_{n-1}(x_n)\tilde{a}_k + [p'_{n-1}(x_n) \int_{x_0}^{x_n} p_{n-1}(t)dt + p_{n-1}^2(x_n)]\tilde{b}_k &= -p_{n-1}(x_n)l_k(x_n) - \\ &\quad - \int_{x_0}^{x_n} l_k(t)dt, \end{aligned}$$

which has unique solution for  $\tilde{a}_k$  and  $\tilde{b}_k$  if and only if the condition (55) is fulfilled.

Furthermore, the polynomial

$$(62) \quad C_0(x) = p_{n-1}(x) \left\{ c_0 \int_{x_0}^x p_{n-1}(t)dt + d_0 \right\}$$

fulfils the conditions

$$\begin{aligned} C_0(x_i) &= 0, & (wC_0)''(x_i) &= 0, & (i = 1, \dots, n-1), \\ C'_0(x_0) &= 1, & C'_0(x_n) &= 0, \end{aligned}$$

where the constants  $c_0$  and  $d_0$  are the unique solutions of the linear system

$$(63) \quad \begin{aligned} p'_{n-1}(x_0)d_0 + p_{n-1}^2(x_0)c_0 &= 1, \\ p'_{n-1}(x_n)d_0 + \left[ p'_{n-1}(x_n) \int_{x_0}^{x_n} p_{n-1}(t)dt + p_{n-1}^2(x_n) \right] c_0 &= 0, \end{aligned}$$

if and only if the condition (55) is fulfilled.

In a similar way we obtain, that the polynomial

$$(64) \quad C_n(x) = p_{n-1}(x) \left\{ c_n \int_{x_0}^x p_{n-1}(t)dt + d_n \right\}$$

fulfils the conditions

$$\begin{aligned} C_n(x_i) &= 0, & (wC_n)''(x_i) &= 0 & (i = 1, \dots, n-1), \\ C'_n(x_0) &= 0, & C'_n(x_n) &= 1, \end{aligned}$$

where the constants  $c_n$  and  $d_n$  are the unique solutions of the linear system

$$(65) \quad \begin{aligned} p'_{n-1}(x_0)d_n + p_{n-1}^2(x_0)c_n &= 0, \\ p'_{n-1}(x_n)d_n + \left[ p'_{n-1}(x_n) \int_{x_0}^{x_n} p_{n-1}(t)dt + p_{n-1}^2(x_n) \right] c_n &= 1, \end{aligned}$$

if and only if the condition (55) is fulfilled.

As the polynomials  $A_k$ ,  $B_k$  ( $k = 1, \dots, n-1$ ),  $C_0$  and  $C_n$  defined by (57)-(65) are the basis polynomials of the interpolational problem (56), the polynomial

$$(66) \quad Q_n(x) = \sum_{k=1}^{n-1} y_k A_k(x) + \sum_{k=1}^{n-1} y_k'' B_k(x) + y_0' C_0(x) + y_n' C_n(x)$$

is of degree at most  $2n-1$  and fulfils the equations (56). The uniqueness can be proved in a similar way as in Theorem 1.

**Corollary 4.** *On  $[-1, 1]$ , if the weight function is*

$$w(x) = (1 - x^2)^{\frac{\alpha+1}{2}},$$

*then for odd  $n$  there exists a unique polynomial  $Q_n$  of degree at most  $2n-1$ , which fulfils weighted (0,2)-interpolational conditions at the zeros of  $P_{n-1}^{(\alpha, \beta)}$  with additional interpolatory conditions for the first derivative at  $x_0 = 1$  and  $x_n = -1$ .*

**Proof.** Let  $p_{n-1}(x) = P_{n-1}^{(\alpha)}(x)$ . By Lemma 1 the function  $wp_{n-1}$  fulfils the conditions (54). For odd  $n$  the polynomial  $P_{n-1}^{(\alpha)}$  is even function, and using  $P_{n-1}^{(\alpha)}(1) = \binom{n-1+\alpha}{n-1}$  and  $P_{n-1}^{(\alpha)'}(x) = \frac{1}{2}(n+2\alpha)P_{n-2}^{(\alpha+1)}(x)$ , one can verify (55).

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