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ON MULTIPLICATIVE FUNCTIONS SATISFYING CONGRUENCE PROPERTIES II.

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Dedicated to Professor Imre Kátai on the ocassion of his 65th birthday

1. Introduction

The function $f : \mathbb{N} \to \mathbb{Y}$ is called multiplicative ($f \in \mathcal{M}$) if the condition

$$(*) \qquad f(nm) = f(n)f(m)$$

is satisfied for all pairs $n, m \in \mathbb{N}$, $(n, m) = 1$. The f is completely multiplicative ($f \in \mathcal{M}^*$), if $(*)$ holds for all pairs $n, m \in \mathbb{N}$. The function $f(n) = n^\alpha$ ($\alpha \in \mathbb{N}_0$) is multiplicative and has many nice properties. For example:

$$(**) \qquad (n+m)^\alpha \equiv m^\alpha \pmod{n} \ (\forall n, m \in \mathbb{N}).$$

As it was noticed by M.V. Subbarao (1966), namely we have

Theorem A. (M.V. Subbarao, 1966 [5]) *If $f \in \mathcal{M}$ and*

$$f(n+m) \equiv f(m) \pmod{n} \ (\forall n, m \in \mathbb{N}),$$

then $f(n) = n^\alpha$ ($\alpha \in \mathbb{N}_0$).

Let $M, N \subset \mathbb{N}$, and for $f : \mathbb{N} \to \mathbb{Y}$ assume

$$(***) \qquad f(n+m) \equiv f(m) \pmod{n} \ (\forall n \in N, \ \forall m \in M).$$

First let us remind a few variants of Theorem A. In them all f satisfy the condition $(***)$.

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Theorem B. (A. Iványi, 1972 [2]) *If* $f \in \mathcal{M}^*,\ N = \mathbb{N},\ M = \{m\}$ *and* $f(m) \neq 0$, *then* $f(n) = n^\alpha\ (\alpha \in \mathbb{N}_0)$.

The latter result was improved, namely we have

Theorem C. (B.M. Phong and J. Fehér, 1985 [4]) *If* $f \in \mathcal{M},\ N = \mathbb{N},\ M = \{m\}$ *and* $f(m) \neq 0$ *then* $f(n) = n^\alpha\ (\alpha \in \mathbb{N})$.

Theorem D. (I. Joó and B.M. Phong, 1992 [3]) *If* $f \in \mathcal{M},\ N = \{n \mid n \in \mathbb{N},\ A \mid n\}\ M = \{B\},\ (A, B) = 1$ *and* $f(B) \neq 0$, *then there are a real valued Dirichlet character* $\chi \pmod A$ *and* $\alpha \in \mathbb{N}_0$, *such that* $f(n) = \chi(n)n^\alpha\ (\forall n \in \mathbb{N},\ (n, A) = 1)$.

Theorem E. (J. Fehér, 1994, [1]) *If* $f \in \mathcal{M},\ N = \{n^2 \mid n \in \mathbb{N}\},\ M = \{1\}$, *then* $f(2) = 2^\beta$ *and* $f(q^k) = q^{K\alpha(q)}$ *for all primes of the form* $q = 4k + 1$.

Notice that the function f occuring in Theorme E satisfies also the following condition:

$$ab \in H \Rightarrow f(ab) = f(a)f(b),$$

where

$$H := \left\{ 2^\varepsilon \prod_i q_i^{h_i} \mid \varepsilon = 0, 1;\ q_i \in \mathcal{P},\ q_i \equiv 1 \pmod 4 \right\}.$$

In this paper we prove the following theorem.

Theorem. *Let* $f : \mathbb{N} \to \mathbb{Z}$ *be a multiplicative function. Assume that for all primes* p *and* $n \in \mathbb{N}$

$$f(n^2 + p) \equiv f(p) \pmod n. \tag{1}$$

Then: if there is a prime p_0 *such that* $f(p_0) \neq 0$, *then*

$$|f(q^k)| = q^{\alpha(q^k)}$$

for all q *primes and* $k \in \mathbb{N}$.

2. Lemmas

The proof of Theorem is based on the four lemmas as follows.

Lemma 1. *Let* $A, B, C \in \mathbb{N},\ (A, B) = 1$. *Then the diophantine equation*

$$Ax - By = 1$$

has got the solution (x, y) *such that* $(x, c) = 1$.

Proof. Let (x_0, y_0) be a solution, $c = \prod_{i=1}^{s} p_i^{\alpha_i} \prod_{j=1}^{r} q_j^{\beta_j}$ be the primepower-decomposition of c, where $p_i \mid x_0$, and $q_j \nmid x_0$. Then the pair

$$x = x_0 + B(p_1 \dots p_s + 1)(q_1 \dots q_r),$$
$$y = y_0 + A(p_1 \dots p_s + 1)(q_1 \dots q_r)$$

is a solution satisfying the condition $(C, X) = 1$.

Lemma 2. *Let* p_0, ρ_0 *be two (not equal) odd primes such that* $\left(\frac{-p_0}{\rho_0}\right) = -1$. *Then there are infinitely many odd primes* q *such that* $\left(\frac{-p_0}{q}\right) = \left(\frac{-q}{\rho_0}\right) = 1$.

Proof. Let $q = 4Mp_0 + 1$ $(M \in \mathbb{N})$. Then the condition $\left(\frac{-p_0}{q}\right) = 1$ (where $(\cdot)$ is the Jacobi symbol) is fulfilled for all M. The diophantine equation

$$4Mp_0 + 1 = -1 + \rho_0 L$$

has a solution and its solutions are: $M = M_0 + \rho_0 N$, $L = L_0 + 4p_0 N$. Using we get

$$q = 4p_0\rho_0 N + 4p_0 M_0 + 1 = 4\rho_0 p_0 N + \rho_0 L_0 - 1 \quad (N \in \mathbb{N}),$$

and this shows that $\left(\frac{-q}{\rho_0}\right) = 1$. The condition $(4\rho_0 p_0,\ 4p_0 M_0 + 1) = 1$ implies that among q-s there are infinitely many primes.

Lemma 3. *Let* $2 < q$ *be a prime such that* $q \nmid A$ *and* $p \neq q$ *a prime such that* $\left(\frac{-p}{q}\right) = 1$. *Then for all* $\alpha \in \mathbb{N}$ *there exist* $x, u \in \mathbb{N}$ *such that*

$$q^\alpha up = x^2 A^2 + p, \quad (q, u) = (p, u) = 1.$$

Proof. Let T and v be positive integers such that

$$q^\alpha v = A^2 \cdot T + 1. \tag{2}$$

The relation (2) shows that $\left(\frac{T}{q}\right) = \left(\frac{-1}{q}\right) \Rightarrow \left(\frac{Tp}{q}\right) = \left(\frac{-p}{q}\right) = 1$, hence there is $x_0 \in \mathbb{N}$ such that

$$x_0^2 \equiv Tp \pmod{q^{\alpha+1}}. \tag{3}$$

The numbers $x = x_0 + kq^{\alpha+1}$ are also solutions of (3), hence we can choose the k so that $p|x$. So we can assume that in (3) $p|x_0$. By the Lemma 1, we can choose v satisfying (2) and also $(v, pq) = 1$. The relation (3) shows that, denoting

$$L := \frac{x_0^2 - Tp}{q^\alpha}, \quad u^* := vp + LA^2,$$

we get $q|L,\ q\not|v$ and so $q\not|u^*$. The relation (2) implies

$$q^\alpha vp = TpA^2 + p. \tag{4}$$

From this we see that

$$q^\alpha vp = q^\alpha(u^* - LA^2) = q^\alpha\left(\frac{x_0^2 - Tp}{q^\alpha}A^2\right) = q^\alpha u^* - x_0^2A^2 + TpA^2,$$

and (4) also implies that $q^\alpha u^* = x_0^2A^2 + p$. Here $p \mid x_0$ implies $p\|X_0^2A_2 + p$ and this in turn implies $u^* = up,\ p\not|u$.

One can prove (in a similar way) the following

Lemma 4. *Let $2\not|A$ and $\alpha \in \mathbb{N}$. Then there are infinitely many primes $p > 2$ such that*

$$2^\alpha up = x^2A^2 + p, \quad (u, 2p) = 1. \tag{5}$$

3. Proof of the theorem

Assume that f fulfills the conditions of the theorem. First we show that $f(p) \neq 0$ for all primes p.

Let p, q be primes such that $p \neq q,\ p \neq p_0,\ q \neq p_0,\ q^k\|f(p_0)$ and assume

$$pu\,p_0 = x^2q^{2(k+1)} + p_0,\ (u, pp_0) = 1.$$

Then

$$f(p)f(u)f(p_0) \equiv f(p_0) \pmod{q^{k+1}},$$

which implies

$$f(p)f(u) \equiv 1 \pmod{q},$$

showing that $f(p) \neq 0$.

By the Lemmas 2, 3, 4 we see that

(α) $p_0 = 2 \Rightarrow f(11) = 0$.

$p \neq p_0$ and $p \neq 11$ and $\left(\frac{-11}{p}\right) = 1 \Rightarrow f(p) \neq 0$,

$p \neq p_0$ and $p \neq 11$ and $\left(\frac{-11}{p}\right) = -1 \Rightarrow \exists q$ prime, for which $\left(\frac{-11}{q}\right) = \left(\frac{-q}{p}\right) = 1$, and so $f(11) \neq 0 \Rightarrow f(q) \neq 0 \Rightarrow f(p) \neq 0$.

(β) $2 < p_0$ and $2 < p \neq p_0$ and $\left(\frac{-p_0}{p}\right) = 1 \Rightarrow f(p) = 0$,

$2 < p_0$ and $2 < p \neq p_0$ and $\left(\frac{-p_0}{p}\right) = -1 \Rightarrow \exists q > 0$ prime, for which $\left(\frac{-p_0}{q}\right) = \left(\frac{-q}{p}\right) = 1$, and so

$f(p_0) \neq 0 \Rightarrow f(q) \neq 0 \Rightarrow f(p) \neq 0$.

Finally, let q^α be a given power of the prime q, and a prime ρ, such that $\rho \neq q$. Then there are $u, x \in \mathbb{N}$ and a prime $p(\neq q)$, such that

$$q^\alpha up = x^2\rho^{2k} + p, \quad (u, pq) = 1.$$

From this we see that

$$f(q^\alpha)f(u)f(p) \equiv f(p) \pmod{p^k}. \tag{6}$$

For $f(p) \neq 0$ $\exists s \in \mathbb{N}_0$, $\rho^s \| f(p)$. Assuming that $k > s$ the relation (6) shows that

$$f(q^\alpha)f(u) \equiv 1 \pmod{\rho},$$

consequently $\rho \nmid f(q^\alpha)$.

4. Remarks

(a) It seems that the function f satisfying the conditions of the Theorem as well as the congruence (1) are power functions. It seems to us that to prowe the independence of $\alpha(p^k)$ upon k and p is not easy task.

(b) If for some prime p_0, $f(p_0) = 0$, then obviously $f(p) = \{\circ\}$. In this case there is a solution f of (1) such that $f \in \mathcal{M}\backslash\mathcal{M}^*$. An example of such function:

$$f(1) = 1,\ f(9) = 2 \quad \text{and} \quad f(n) = 0 \text{ if } n \neq 1, 9.$$

References

[1] **Fehér J.,** On integer valued multiplicative and additive functions, *Annales Univ. Sci. Budapest. Sect. Comp.,* **14** (1994), 39-45.

[2] **Iványi A.,** On multiplicative functions with congruence property, *Annales Univ. Sci. Budapest. Sect. Math.,* **15** (1972), 133-137.

[3] **Joó I. and Phong B.M.,** Arithmetical functions with congruence properties, *Annales Univ. Sci. Budapest. Sect. Math.,* **35** (1992), 151-155.

[4] **Phong B.M. and Fehér J.,** Note on multiplicative functions satisfying a congruence property, *Annales Univ. Sci. Budapest. Sect. Math.,* **33** (1990), 261-265.

[5] **Subbarao M.V.,** Arithmetic functions satisfying congruences property, *Canad. Math. Bull.,* **9** (1966), 143-146.

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