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REGULARITY OF SOLUTIONS OF A FUNCTIONAL EQUATION ON LIE GROUPS

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Dedicated to Professor Imre Kátai on the occasion of his 65th birthday

Abstract. It is proved that if G is a connected Lie group, X is a $\mathcal{C}^\infty$ manifold, and $f : G \to X$ is measurable or has the Baire property and f satisfies the functional equation

$$f(x) = h\left(x, y, f\left(\left(x^{\varepsilon_1} y^{k_1}\right)^{\delta_1}\right), \dots, f\left(\left(x^{\varepsilon_n} y^{k_n}\right)^{\delta_n}\right)\right), \quad x, y \in G,$$

where $h : G^2 \times X^n \to X$ is $\mathcal{C}^\infty$ and $\varepsilon_i \in \{0,1\}$, $\delta_i \in \{-1,1\}$, $k_i \in \mathbb{N}$, $k_i > 0$ for $i = 1, 2, \dots, n$, then f is $\mathcal{C}^\infty$.

1. Introduction

In connection with his fifth problem Hilbert [5] suggested that although the method of reduction to differential equations makes it possible to solve functional equations in an elegant way, the inherent differentiability assumptions are typically unnatural (see Aczél [2]). Such shortcomings can be overcome by appealing to regularity theorems.

In this short note two theorems will be proven (Theorem 1 and Theorem 4). The first makes possible to deduce continuity of measurable solutions for a general type of functional equations on certain locally compact groups,

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especially on Lie groups. The second theorem makes possible to deduce $\mathcal{C}^\infty$ of solutions directly from measurability, but for a more special type of equations, and only works on connected Lie groups. We follow the terminology of Bourbaki about topology, hence compact and locally compact spaces are supposed to be Hausdorff. About measure theory we follow the terminology of Federer [5].

2. Results

First we will prove the following “measurability implies continuity” type theorem:

Theorem 1. *Suppose that Z, Z_i $(i = 1, 2, \ldots, n)$ are completely regular spaces, Z_0 is a σ-compact completely regular space, G is a locally compact topological group and*

$$(1)\qquad f(x) = h\Big(x, y, f_0(y), f_1\big(g_{l,1}(x)y^{k_1}g_{r,1}(x)\big), \ldots, f_n\big(g_{l,n}(x)y^{k_n}g_{r,n}(x)\big)\Big),$$

$$x, y \in G,$$

where $h : G^2 \times Z_0 \times Z_1 \times \cdots \times Z_n \to Z$ is continuous and $g_{l,i} : G \to G$, $g_{r,i} : G \to G$ are continuous, $k_i \in \mathbb{N}$ and $k_i > 0$ for $i = 1, 2, \ldots, n$. Suppose, that $f_0 : G \to Z_0$ is an arbitrary function and $f_i : G \to Z_i$ is Lusin measurable for $1 \le i \le n$, moreover

(2) G has a compact subset K with positive left Haar measure such that for any compact subset $C \subset K$ with positive left Haar measure and for any exponent k_i in (1) the left Haar measure of the set $\{x^{k_i} : x \in C\}$ is positive, too.

Then f is continuous.

The theorem generalizes a result of Járai [4] about D'Alembert's functional equation

$$f(xy) + f(xy^{-1}) = 2f(x)f(y).$$

The proof is based to the following general theorem (see Járai [7], Theorem 8.2 and Járai [5]):

Theorem 2. *Let Z, Z_i $(i = 1, 2, \cdots, n)$ be completely regular spaces, Z_0 a σ-compact completely regular space, Y and X_i $(i = 1, 2, \cdots, n)$ locally compact spaces, X an arbitrary topological space. Suppose that $D \subset X \times Y$. Let $f : X \to Z$, $f_0 : Y \to Z_0$, $g_i : D \to X_i$ $(i = 1, 2, \cdots, n)$, $f_i : X_i \to Z_i$ $(i = 1, 2, \cdots, n)$, and $h : D \times Z_0 \times Z_1 \times \ldots \times Z_n \to Z$ be functions, ν a Radon* *measure on Y and μ_i a Radon measure on X_i $(i = 1, 2, \cdots, n)$. Suppose that $x_0 \in X$ and the following conditions are satisfied:*

(1) for each $(x, y) \in D$

$$f(x) = h\Big(x, y, f_0(y), f_1\big(g_1(x, y)\big), \ldots, f_n\big(g_n(x, y)\big)\Big);$$

(2) h is continuous;

(3) f_i is Lusin μ_i measurable on the subset A_i of X_i $(i = 1, 2, \cdots, n)$;

(4) g_i is continuous on D $(i = 1, 2, \cdots, n)$;

(5) there exist sets V and K such that V is open, K is compact, $V \times K \subset D$, $x_0 \in V$, $\nu(K) > 0$, and $K \subset \bigcap\limits_{i=1}^{n} g_{i,x_0}^{-1}(A_i)$;

(6) for each $\varepsilon > 0$ there exists a $\delta > 0$ such that $B \subset K$ and $\nu(B) \geq \varepsilon$ implies $\mu_i\big(g_{i,x}(B)\big) \geq \delta$ whenever $1 \leq i \leq n$ and $x \in V$.

Then f is continuous on a neighborhood of x_0.

Proof of Theorem 1. Let λ denote a left Haar measure on G. Let x_0 denote an arbitrary element of G. We prove, that f is continuous at x_0. Let X be a compact set containing a neighborhood of x_0, let K be the set from (3), $D = X \times K$, and let $Y = X_i = G$ and $g_i(x, y) = g_{l,i}(x) y^{k_i} g_{r,i}(x)$ whenever $(x, y) \in D$. We will use Theorem 2 with sets $A_i = G$. The only non-trivial fact is that the last condition is satisfied by g_i. Since if $B \subset K$ then the left translation does not change the left Haar measure of $\{y^{k_i} : y \in B\}$ and the right translation only changes this measure with a positive constant depending on $g_{r,i}(x)$ but bounded by positive constants from below and above for all $x \in X$, it is enough to prove that for each $\varepsilon > 0$ there exists a $\delta > 0$ such that $B \subset K$ and $\lambda(B) \geq \varepsilon$ implies $\lambda\{y^{k_i} : y \in B\} \geq \delta$ for $i = 1, 2, \ldots, n$. Suppose, that this is not the case. Then there exists a $1 \leq i \leq n$, $\varepsilon_0 > 0$ and for every natural number j an open subset U_j of G for which $\lambda(U_j) < 1/2^j$ and

$$\lambda\{y : y \in C,\ y^k \in U_j\} \geq \varepsilon_0,$$

where $k = k_i$. Since the mapping $y \mapsto y^k$ is continuous, the sets $\{y : y^k \in U_j\}$ are open, and so the sets $\{y : y^k \in U_j\} \cap K$ are measurable. Let

$$A_j = \{y^k : y \in C\} \cap \left(\bigcup_{i=j}^{\infty} U_i\right).$$

Then $A_1 \supset A_2 \supset \ldots$, $\lambda(A_j) < 1/2^{j-1}$, and the sets $\{y : y^k \in A_j\}$ are measurable with finite measures which are not less than ε_0. If

$$B = \left\{ y : y^k \in \bigcap_{j=1}^{\infty} A_j \right\},$$

then $\lambda(B) \geq \varepsilon_0 > 0$, but

$$\lambda\{y^k : y \in B\} = \lambda\left(\bigcap_{j=1}^{\infty} A_j\right) = 0.$$

If C is a compact subset of B with positive left Haar measure we have a contradiction with (2). Hence the proof is complete.

Remark 3. Condition (2) of Theorem 1 is fulfilled in many important locally compact groups, but not in all of them. For example, if $G = \{-1, 1\}$ with multiplication and discrete topology, $\mathfrak{n}$ is a cardinal number and for some i we have $k_i = 2$, then $G^{\mathfrak{n}}$ satisfies (2) if and only if $\mathfrak{n}$ is finite.

If G_1 and G_2 satisfy (2) with subsets K_1 and K_2 then $G = G_1 \times G_2$ satisfies (2) with $K = K_1 \times K_2$. This easily follows from Fubini's theorem.

Each Lie group satisfies this condition. Let G be an N-dimensional Lie group with unit element e. It is not hard to prove using some theorem on the left Haar measure of Lie groups (see Chevalley [2]) that there exist open subsets U, V, and a homeomorphism φ of U onto an open subset of $\mathbb{R}^N$ so that $e \in V \subset U$, $\varphi(e) = 0$, the mapping $x \mapsto \varphi\big(\varphi^{-1}(x)^{k_i}\big)$ is an analytic mapping of $\varphi(V)^{k_i}$ into $\varphi(U)$ for $i = 1, 2, \ldots, n$ and that for every compact subset C of V the left Haar measure of C and the Lebesgue measure of $\varphi(C)$ vanish together. Since the Jacobian of the mapping

$$x \mapsto \varphi\big(\varphi^{-1}(x)^k\big)$$

of $\varphi(V)$ into $\varphi(U)$ is equal to k^N at 0, we have that there exists an open neighborhood W of e in G so that $W^{k_i} \subset V$ for $i = 1, 2, \ldots, n$ and the Jacobian over $\varphi(W)$ is not less than 1. Hence, by the formula on transformation of integrals, for any compact subset C of W the Lebesgue measure of $\varphi\big(\{x^{k_i} : x \in$ $\in C\}\big)$ is not less than the Lebesgue measure of $\varphi(C)$. Hence G satisfies condition (2) with any compact subset K of W which has a positive left Haar measure.

In a rather general situation, we can even directly deduce that $f \in \mathcal{C}^\infty$. We can obtain the following

Theorem 4. *Suppose that G is a connected Lie group, X is a $\mathcal{C}^\infty$ manifold, and $f : G \to X$ satisfies the functional equation*

$$f(x) = h\Big(x, y, f\big((x^{\varepsilon_1} y^{k_1})^{\delta_1}\big), \dots, f\big((x^{\varepsilon_n} y^{k_n})^{\delta_n}\big)\Big), \quad x, y \in G,$$

where $h : G^2 \times X^n \to X$ is $\mathcal{C}^\infty$ and $\varepsilon_i \in \{0, 1\}$, $\delta_i \in \{-1, 1\}$, $k_i \in \mathbb{N}$, $k_i > 0$ for $i = 1, 2, \dots, n$. Then if f is measurable or has the Baire property, then f is $\mathcal{C}^\infty$.

The proof is based to the following general theorem (see Járai [7], Theorem 1.37 and Járai [6]):

Theorem 5. *Let X, Y, and Z be $\mathcal{C}^\infty$ manifolds, let D be an open subset of $X \times Y$ and let W be an open subset of $D \times Z^n$. Let $f : X \to Z$, $g_i : D \to X$ $(i = 1, 2, \dots, n)$, and $h : W \to Z$ be functions. Suppose that*

(1) if $(x, y) \in D$ then

$$\Big(x, y, f\big(g_1(x, y)\big), \dots, f\big(g_n(x, y)\big)\Big) \in W$$

and

$$f(x) = h\Big(x, y, f\big(g_1(x, y)\big), \dots, f\big(g_n(x, y)\big)\Big);$$

(2) h is $\mathcal{C}^\infty$;

(3) g_i is $\mathcal{C}^\infty$ and there exists a compact subset C of X such that for each $x_0 \in X$ there exists a y_0 for which $(x_0, y_0) \in D$, $g_i(x_0, y_0) \in C$ and $y \mapsto g_i(x_0, y)$ is a submersion at y_0 for $i = 1, 2, \dots, n$.

Then every f which is measurable or has the Baire property, is $\mathcal{C}^\infty$.

Proof of Theorem 4. Let $\mathfrak{g}$ denote the Lie algebra of G and let $K = \max_i k_i$. There exists an open neighborhood V_0 of $0 \in \mathfrak{g}$ such that the exponential mapping is a diffeomorphism of this neighborhood onto some open neighborhood U_0 of e in G (see Dieudonné [3], (19.8.5), (19.8.6)). Let us choose a convex symmetric neighborhood V of 0 in $\mathfrak{g}$ such that $K\overline{V} \subset V_0$. Then with the notation $U = \exp(V)$ we have $\overline{U} = \exp(\overline{V})$ and the mapping $y \mapsto y^k$ is a submersion in each point of U whenever $0 < k \le K$.

By Dieudonné [3], (19.9.11) for any $m > 1$ we have

$$\exp(v_1)\exp(v_2)\cdots\exp(v_m) =$$

$$= \exp\Big(v_1 + v_2 + \cdots + v_m + \frac{1}{2}\sum_{i<j}[v_i, v_j] + O\Big(\big(\|v_1\| + \cdots + \|v_m\|\big)^3\Big)\Big) =$$

$$= \exp\Big(v_1 + v_2 + \cdots + v_m + O\Big(\big(\|v_1\| + \cdots + \|v_m\|\big)^2\Big)\Big)$$

for sufficiently small $\|v_j\|$, $j = 1, 2, \ldots, m$. Hence by further shrinking V if necessary we may suppose that all of the mappings

$$(v_1, v_2, \ldots, v_K) \mapsto \exp(v_1)\exp(v_2)\cdots\exp(v_K)\cdot \cdot \exp\left(-\frac{v_1 + \cdots + v_K}{K}\right)^j,$$

$j = 1, 2, \ldots, K$ map KV into some compact subset C' of $\exp(KV)$.

We want to apply Theorem 5. Let us fix a $k \geq 2K$ and let $X = U^k$, $Y = U$. First let us observe that since the left shifts and the inversion are diffeomorphisms, for any $x_0 \in X$ all the mappings $y \mapsto (x_0^{\varepsilon_i} y^{k_i})^{\delta_i}$ are submersions at any point $y \in Y$ and for any $i = 1, 2, \ldots, n$.

Let $C = \overline{U}^{k-K} C' \cup C'^{-1} \overline{U}^{K-k}$. Clearly C is a symmetric compact set. Let us choose an open neighborhood W of e for which $C'W \subset \exp(KV)$. Then

$$\overline{U}^{k-K} C'W \subset \overline{U}^{k-K} \exp(KV) \subset \overline{U}^{k-K}\overline{U}^{K} = \overline{U}^{k} \subset \overline{U^k}.$$

But since $\overline{U}^{k-K} C'W$ is open, we have $\overline{U}^{k-K} C'W \subset U^k$, and hence $C \subset X = U^k$.

Let $x_0 \in X$ and let us represent x_0 as $x_0 = x_1 x_2 \cdots x_k$, where $x_j \in U$ for $j = 1, 2, \ldots, k$. For each j let v_j be the unique element of V for which $\exp(v_j) = x_j$. Let us define

$$y_0 = \exp\left(-\frac{v_k + v_{k-1} + \cdots + v_{k-K+1}}{K}\right).$$

Clearly, $y_0 \in Y$ and if $\varepsilon_i = 0$, then

$$x_0^{\varepsilon_i} y_0^{k_i} = y_0^{k_i} \in U^{k_i} \subset \overline{U}^{k-K} \subset C.$$

If $\varepsilon_i = 1$, then by the choice of C' we have

$$x_0^{\varepsilon_i} y_0^{k_i} = x_0 y_0^{k_i} = x_1 x_2 \cdots x_{k-K}(x_{k-K+1} \cdots x_k y_0^{k_i}) \in \in \overline{U}^{k-K} C' \subset C.$$

Since $C^{-1} = C$ we obtain that $(x_0^{\varepsilon_i} y_0^{k_i})^{\delta_i} \in C$ for each $i = 1, 2, \dots, n$.

Let us take for each (x_0, y_0) an open neighborhood $W_0 \subset X \times Y$ of this pair which is mapped by all maps $(x, y) \mapsto (x^{\varepsilon_i} y^{k_i})^{\delta_i}$ into X and let D be the union of all these W_0's. With this setting, Theorem 5 can be applied and implies that f is infinitely many times differentiable on $X = U^k$. Since G is connected, $G = \bigcup_{k \geq 2K} U^k$, hence the theorem is proved.

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