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INTERPOLATION BY RATIONAL FUNCTIONS

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Dedicated to the 65th birthday of Professor Imre Kátai

Abstract. The main contribution of the paper is to construct discrete biorthogonal systems to a given system of rational functions. This can be used to construct interpolation processes on nodes (1.2) of the unite circle. Using a discrete analogue of the Cauchy integral formula the biorthogonal systems and the interpolation operators can be given in a useful explicit form. In special cases a lower and upper estimation is given for the norm of the interpolation operators.

1. Introduction

Denote by $\mathbb{C}$ the set of complex numbers and let $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ be the open unite disc. The disc algebra, i.e. the set of functions continuous on $\overline{\mathbb{D}} := \{z \in \mathbb{C} : |z| \le 1\}$ and analytic in $\mathbb{D}$ will be denoted by $\mathcal{A}$ (see [2], [3]). In this paper we interpolate by rational functions belonging to the m-dimensional subspace $\mathcal{R}_m \subset \mathcal{A}$, generated by the collection

$$(1.1) \qquad \phi_{k\ell}(z) := \frac{z^{\ell-1}}{(1-\overline{a}_k z)^{\ell}} \qquad (z \in \mathbb{C},\ 1 \le \ell \le m_k,\ k = 1, 2, \cdots, n).$$

Here $a_k \in \mathbb{D}$, $m_k \in \mathbb{N}^* := \{1, 2, \cdots\}$ $(k = 1, 2, \cdots, n)$ are fixed numbers and $m_1 + m_2 + \cdots + m_n = m$. We note that the function $\phi_{k\ell}$ has a pole in

$a_k^* := 1/\overline{a}_k \notin \overline{\mathbb{D}}$ of multiplicity ℓ. In the case $a_k \neq 0 \quad (k = 1, 2, \cdots, n) \quad \mathcal{R}_m$ is the same as the set of rational functions generated by the collection

$$\varphi_{k\ell}(z) := \frac{1}{(1 - \overline{a}_k z)^\ell} \qquad (z \in \mathbb{C}, \quad 1 \leq \ell \leq m_k, \ k = 1, 2, \cdots, n).$$

If $n = 1$ and $a_1 = 0$ then the set $\mathcal{R}_m$ coincides with the set of polynomials $\mathcal{P}_{m-1}$ of degree $m - 1$.

We shall consider interpolation processes on the set of nodes

$$\mathbb{T}_N := \left\{ e^{2\pi i \ell / N} : \ell = 0, 1, \cdots, N - 1 \right\} \subset \mathbb{T} := \{z \in \mathbb{C} : |z| = 1\} \tag{1.2}$$

in the unit circle $\mathbb{T}$. We construct a collection $\Phi_{k\ell}$ $(1 \leq \ell \leq m_k, \ k = 1, 2, \cdots, n)$ of polynomials, biorthogonal to (1.1) with respect to the following scalar product on $\mathbb{T}_N$:

$$[F, G] := [F, G]_N := \frac{1}{N} \sum_{z \in \mathbb{T}_N} F(z)\overline{G(z)}. \tag{1.3}$$

The polynomials $\Phi_{k\ell}$ are weighted fundamental polynomials of Hermite interpolation with respect the nodes a_k $(k = 1, 2, \cdots, n)$.

Denote

$$\epsilon_n(z) := z^n \quad (z \in \mathbb{T}, \ n \in \mathbb{Z}) \tag{1.4}$$

the trigonometric system (see [7]). The restriction of the functions ϵ_k $(k = 0, 1, \cdots, N - 1)$ to $\mathbb{T}_N$, i.e. the discrete trigonometric system is orthonormal with respect to the scalar product (1.3), i.e.

$$[\epsilon_k, \epsilon_\ell]_N = \delta_{k\ell} \quad (0 \leq k, \ell < N). \tag{1.5}$$

This implies that for any two polynomials

$$F(z) := \sum_{k=0}^{N-1} c_k z^k, \quad G(z) := \sum_{k=0}^{N-1} b_k z^k \quad (z \in \mathbb{C})$$

we have

$$[F, G]_N = \sum_{k=0}^{N-1} c_k \overline{b}_k. \tag{1.6}$$

In Section 2 we prove a discrete analogue of the Cauchy integral formula. This will be used in Section 3 and 4 to construct systems biorthogonal to the rational system introduced in (1.1). In Section 3 we investigate the Lagrange interpolation. In the special case $a_k := \rho e^{2\pi ik/N}$ $(k = 0, 1, \cdots, N-1)$ we give an explicit formula for the interpolation operator and we estimate the norm of this operator.

System identification based upon the partial fraction representation of the transfer function is recognized as a classical approach in systems science [1], [4], [5]. This type of biorthogonal expansion can be used to find the poles of rational functions [6].

2. Discrete Cauchy formula

For any function $F \in \mathcal{A}$ then Cauchy formula

$$\frac{1}{2\pi i}\int\limits_{\mathbb{T}} \frac{F(\zeta)}{(\zeta-a)^{n+1}}\,d\zeta = \frac{F^{(n)}(a)}{n!} \qquad (a \in \mathbb{D},\ n \in \mathbb{N}) \tag{2.1}$$

holds.

Replacing $\mathbb{T}$ by the discrete group $\mathbb{T}_N$ defined in (1.2) and the integral by the sum

$$\frac{1}{2\pi i}\int\limits_{\mathbb{T}_N} F(\zeta)\,d\zeta := \frac{1}{N}\sum_{\zeta\in\mathbb{T}_N} F(\zeta)\zeta \tag{2.2}$$

we get a similar formula for polynomials. For a function $F \in \mathcal{A}$ obviously

$$\lim_{N\to\infty}\frac{1}{2\pi i}\int\limits_{\mathbb{T}_N} F(\zeta)\,d\zeta = \frac{1}{2\pi i}\int\limits_{\mathbb{T}} F(\zeta)\,d\zeta.$$

In this paper we shall use the following discrete analogue of the Cauchy integral formula.

Theorem 1. *Let $n \in \mathbb{N}, N \in \mathbb{N}^*$ be fixed numbers and denote $P \in \mathcal{P}_{N+n-1}$ a polynomial. i) Then for any $a \in \mathbb{D}$ we have*

$$\frac{1}{2\pi i}\int\limits_{\mathbb{T}_N} \frac{P(\zeta)}{(\zeta-a)^{n+1}}\,d\zeta = \frac{1}{n!}\frac{d^n}{dz^n}\frac{P(z)}{1-z^N}\bigg|_{z=a}. \tag{2.3}$$

ii) Furthermore if $a_0, a_1, \cdots, a_n$ are distinct points in $\mathbb{D}$, then

$$\frac{1}{2\pi i}\int_{\mathbb{T}_N} \frac{P(\zeta)}{(\zeta-a_0)\cdots(\zeta-a_n)}\,d\zeta = \sum_{j=0}^{n} \frac{1}{\Omega_j(a_j)}\cdot\frac{P(a_j)}{1-a_j^N}, \tag{2.4}$$

where

$$\Omega_j(z) := \prod_{\ell=0,\ell\neq j}^{n}(z-a_\ell) \quad (z\in\mathbb{C},\ j=1,2,\cdots,n).$$

Proof. First we prove (2.3) for $n=0$. To this end write $P\in\mathcal{P}_{N-1}$ in the form

$$P(z)=\sum_{j=0}^{N-1} c_j z^j \quad (z\in\mathbb{C}).$$

Observe that for $\zeta\in\mathbb{T}_N$ we have $\zeta^N=1$ and consequently

$$\frac{\zeta}{\zeta-a}=\frac{1}{1-a\bar{\zeta}}=\frac{1}{1-a^N}\frac{1-(a\bar{\zeta})^N}{1-a\bar{\zeta}}=\frac{1}{1-a^N}\sum_{j=0}^{N-1}a^j\bar{\zeta}^j.$$

Applying (1.6) for

$$F(\zeta)=P(\zeta),\quad \overline{G(\zeta)}=\frac{\zeta}{\zeta-a}=\frac{1}{1-a^N}\sum_{j=0}^{N-1}a^j\bar{\zeta}^j \quad (\zeta\in\mathbb{T}_N)$$

we get

$$\frac{1}{2\pi i}\int_{\mathbb{T}_N}\frac{P(\zeta)}{\zeta-a}\,d\zeta=\frac{1}{N}\sum_{\zeta\in\mathbb{T}_N}P(\zeta)\frac{\zeta}{\zeta-a}=$$

$$=[F,G]_N=\frac{1}{1-a^N}\sum_{j=0}^{N-1}c_j a^j=\frac{P(a)}{1-a^N}$$

and for $n=0$ (2.3) is proved.

To show (2.4) write $P\in\mathcal{P}_{n+N-1}$ in the form

$$P(z)=Q(z)(z-a_0)\cdots(z-a_n)+R(z), \tag{2.5}$$

where $R\in\mathcal{P}_n, Q\in\mathcal{P}_{N-2}$. Applying Lagrange interpolation formula to R we get

$$R(z)=\sum_{j=0}^{n}R(a_j)\frac{\Omega_j(z)}{\Omega_j(a_j)} \quad (z\in\mathbb{C}).$$

By (2.5) $P(a_j) = R(a_j)$ $(j = 0, 1, \cdots, n)$ and consequently

$$\frac{P(z)}{(z-a_0)\cdots(z-a_n)} = Q(z) + \frac{R(z)}{(z-a_0)\cdots(z-a_n)} =$$
$$= Q(z) + \sum_{j=0}^{n} \frac{R(a_j)}{(z-a_j)\Omega_j(a_j)} = Q(z) + \sum_{j=0}^{n} \frac{P(a_j)}{(z-a_j)\Omega_j(a_j)}.$$

Since $Q \in \mathcal{P}_{N-2}$, the orthogonality of the discrete trigonometric system implies

$$\int_{\mathbb{T}_N} Q(\zeta)\, d\zeta = 0,$$

and applying (2.3) in the case $n = 0$ for the constant polynom we get (2.4).

Observe that the right hand side in (2.4) can be expressed by the divided differences of the function

$$H(z) := \frac{P(z)}{1 - z^N} \quad (z \in \mathbb{D}).$$

Namely (2.4) is equivalent to

$$\text{(2.6)} \qquad \frac{1}{2\pi i} \int_{\mathbb{T}_N} \frac{P(\zeta)}{(\zeta-a_0)\cdots(\zeta-a_n)}\, d\zeta = H(a_n, \cdots, a_1, a_0).$$

(Compare e.g. [3], p. 247.)

Since for any $H \in \mathcal{A}$

$$H(a_n, \cdots, a_1, a_0) \to \frac{H^{(n)}(a)}{n!} \quad \text{as} \quad a_j \to a \quad (j = 0, 1, \cdots, n),$$

for $n \geq 1$ (2.3) follows from (2.6).

Taking the limit in (2.3) as $N \to \infty$ we obtain the continuous variant of the formula.

3. Lagrange interpolation, biorthogonal expansion

In this section using the scalar product $[\cdot,\cdot]_N$ introduced by (1.3) we shall construct discrete biorthogonal systems depending on the vector parameter

$$\mathfrak{a} = (a_1, a_2, \cdots, a_n) \in \mathbb{D}^n, \tag{3.1}$$

where $a_i \neq a_j$, if $i \neq j$. Namely for $z \in \mathbb{C}$ and $k = 1, 2, \ldots, n$ set

$$\phi_k(z) := \frac{1}{1 - \overline{a}_k z}. \tag{3.2}$$

First we show

Theorem 2. *Let $N \geq n$. Then the system*

$$\begin{aligned} \Phi_k^N(z) &:= \Phi_k^N(z, \mathfrak{a}) := \frac{1 - a_k^N}{\omega_k(a_k)} \omega_k(z), \\ \omega_k(z) &:= \omega_k(z, \mathfrak{a}) := \prod_{j=1, j\neq k}^{n} (z - a_j) \qquad (k = 1, 2, \ldots, n) \end{aligned} \tag{3.3}$$

is biorthogonal to the system (3.2) with respect to $[\cdot,\cdot]_N$, i.e.

$$[\Phi_k^N, \phi_\ell]_N = \delta_{k\ell} \quad (1 \leq k, \ell \leq n), \tag{3.4}$$

where $\delta_{k\ell}$ is the Kronecker symbol. Especially if $n = N$ and

$$a_k := \rho e^{2\pi i k/N} \quad (0 < \rho < 1, k = 1, \ldots, N) \tag{3.5}$$

then using the notation $\Phi_k^N(z, \rho) := \Phi_k^N(z, \mathfrak{a})$ we get

$$\Phi_k^N(z, \rho) = \frac{(1 - \rho^N)(z^N - \rho^N)}{N \rho^N} \frac{a_k}{z - a_k} \quad (z \neq a_k, k = 1, \ldots, N). \tag{3.6}$$

Proof. By (2.3) and (3.2) for $1 \leq k, \ell \leq n$ and for $\zeta \in \mathbb{T}_N$ we get

$$\begin{aligned} [\Phi_k^N, \phi_\ell] &= \frac{1}{N} \sum_{\zeta \in \mathbb{T}_N} \Phi_k^N(\zeta) \frac{1}{1 - a_\ell \overline{\zeta}} = \frac{1}{N} \sum_{\zeta \in \mathbb{T}_N} \Phi_k^N(\zeta) \frac{\zeta}{\zeta - a_\ell} = \\ &= \frac{1}{2\pi i} \int_{\mathbb{T}_N} \frac{\Phi_k^N(\zeta)}{\zeta - a_\ell} \, d\zeta = \frac{\Phi_k^N(a_\ell)}{1 - a_\ell^N} = \frac{1 - a_k^N}{1 - a_\ell^N} \frac{\omega_k(a_\ell)}{\omega_k(a_k)} = \delta_{k\ell} \end{aligned}$$

and (3.4) is proved.

The numbers a_k $(k = 1, \ldots, N)$ in (3.5) are the roots of the equations $z^N - \rho^N = 0$ and consequently

$$z^N - \rho^N = \prod_{j=1}^{N} (z - a_j) \quad (z \in \mathbb{C}).$$

Hence we get

$$\omega_k(z) = \frac{\prod_{j=1}^{N} (z - a_j)}{z - a_k} = \frac{z^N - \rho^N}{z - a_k} \quad (z \in \mathbb{C}, z \neq a_k).$$

Taking the limit as $z \to a_k$ we get

$$\omega_k(a_k) = \lim_{z \to a_k} \frac{z^N - \rho^N}{z - a_k} = N a_k^{N-1} = \frac{N\rho^N}{a_k}.$$

Thus in the special case (3.6) follows from (3.3).

The biorthogonal expansion of the function $f : \mathbb{T} \to \mathbb{C}$ with respect to the system ϕ_k $(k = 1, \cdots, n)$ is defined by

$$(I_{n,\mathfrak{a}}^N f)(z) := \sum_{k=1}^{n} [f, \Phi_k^N(\cdot, \mathfrak{a})]_N \, \phi_k(z) \quad (z \in \mathbb{C}). \tag{3.7}$$

Obviously $I_{n,\mathfrak{a}}^N f \in \mathcal{A}$ and in the case $n = N$ the function $I_{N,\mathfrak{a}}^N f$ interpolates f in the points of $\mathbb{T}_N$:

$$(I_{N,\mathfrak{a}}^N f)(z) = f(z) \quad (z \in \mathbb{T}_N). \tag{3.8}$$

Indeed the existence of biorthogonal system implies that the system $(\phi_k, 1 \leq k \leq N)$ is linearly independent on $\mathbb{T}_N$ and consequently every function $f : \mathbb{T}_N \to \mathbb{C}$ can be written in the form

$$f(z) = \sum_{k=1}^{N} c_k \phi_k(z) \quad (z \in \mathbb{T}_N),$$

where $c_k \in \mathbb{C}$. Hence by biorthogonality we get $c_k = [f, \Phi_k]$ $(1 \leq k \leq N)$, and consequently on the set $\mathbb{T}_N$ (3.8) is satisfied.

Introducing the kernel function

$$K_n^N(z,w;\mathfrak{a}) := \sum_{k=0}^{n-1} \phi_k(z)\overline{\Phi_k^N(w,\mathfrak{a})} \quad (z,w\in\mathbb{C}) \tag{3.9}$$

the operator $I_{n,\mathfrak{a}}^N$ can be written in the form

(3.10)

$$(I_{n,\mathfrak{a}}^N f)(z) = \frac{1}{N}\sum_{k=0}^{n-1}\sum_{\zeta\in\mathbb{T}_N} f(\zeta)\overline{\Phi_k^N(\zeta,\mathfrak{a})}\phi_k(z) = \frac{1}{N}\sum_{\zeta\in\mathbb{T}_N} f(\zeta)K_n^N(z,\zeta;\mathfrak{a}) \ \ (z\in\mathbb{C}).$$

It is easy to see that the norm of the operator

$$I_{n,\mathfrak{a}}^N : C(\mathbb{T}) \to \mathcal{A}$$

is

$$\|I_{n,\mathfrak{a}}^N\| = \max_{z\in\mathbb{T}} \frac{1}{N}\sum_{\zeta\in\mathbb{T}_N} \big|K_n^N(z,\zeta;\mathfrak{a})\big|. \tag{3.11}$$

If $n = N$ and a_k is defined by (3.5) then the operator (3.7) depends on N and ρ and will be denoted by $I_{N,\rho}$:

$$I_{N,\rho} f := \sum_{k=0}^{N-1} \big[f, \Phi_k^N(\cdot,\rho)\big]\phi_k \quad (a_k := \rho e^{2\pi i k/N}, k = 1,\ldots,N-1).$$

In this case the kernel K_N^N can be written in a useful closed form.

Theorem 3. *Let $a_k := \rho e^{2\pi i k/N}$ $(0\le k<N)$. Then the kernel K_N^N is of the form*

$$\begin{aligned} K_N^N(z,w;\mathfrak{a}) &= \frac{(1-\rho^N)w}{1-(\rho z)^N}\ \frac{z^N-w^N}{z-w} \qquad (w\in\mathbb{T}_N, z\in\mathbb{C}, z\ne w),\\ K_N^N(w,w;\mathfrak{a}) &= N \quad (w\in\mathbb{T}_N). \end{aligned} \tag{3.12}$$

Furthermore there exist constants C_1, C_2 independent on N and ρ such that

$$C_1\frac{1-\rho^N}{1+\rho^N}\log N \le \|I_{N,\rho}\| \le 1 + C_2\frac{1-\rho^N}{1+\rho^N}\log N \quad (N\in\mathbb{N}^*). \tag{3.13}$$

Proof. If $w \in \mathbb{T}_N, z \in \mathbb{C}$ and $z \neq w$ then by (3.6) we get

$$K_N^N(z,w;\mathfrak{a}) =$$

$$= \frac{(1-\rho^N)^2}{N\rho^N}\sum_{k=0}^{N-1}\frac{1}{1-\overline{a}_k z}\frac{\overline{a}_k}{\overline{w}-\overline{a}_k} = \frac{(1-\rho^N)^2 w}{N\rho^N}\sum_{\zeta\in\mathbb{T}_N}\frac{\rho\zeta}{(\zeta-\rho z)(\zeta-\rho w)} =$$

$$= \frac{(1-\rho^N)^2 w}{\rho^N}\frac{1}{2\pi i}\int_{\mathbb{T}_N}\frac{\rho\, d\zeta}{(\zeta-\rho z)(\zeta-\rho w)} =$$

$$= \frac{(1-\rho^N)^2 w}{\rho^N}\frac{1}{z-w}\frac{1}{2\pi i}\int_{\mathbb{T}_N}\left(\frac{1}{\zeta-\rho z}-\frac{1}{\zeta-\rho w}\right)d\zeta.$$

Applying (2.3) we get

$$K_N^N(z,w;\mathfrak{a}) =$$

$$= \frac{(1-\rho^N)^2 w}{\rho^N}\frac{1}{z-w}\left(\frac{1}{1-(\rho z)^N}-\frac{1}{1-(\rho w)^N}\right) = \frac{w(1-\rho^N)}{1-(z\rho)^N}\frac{z^N-w^N}{z-w}$$

and the first part of (3.12) is proved. Taking the limit as $z \to w$, we get the second part of (3.12).

To prove (3.13) for any $\zeta = e^{2\pi i k/N} \in \mathbb{T}_N$ and $z = e^{2\pi i t} \in \mathbb{T}$ set

$$F_N(t) := \frac{1}{N}\sum_{\zeta\in\mathbb{T}_N}\frac{|z^N-\zeta^N|}{|z-\zeta|},\quad G_N(t) := \frac{|\sin N\pi t|(1-\rho^N)}{|1-\rho^N e^{2\pi i N t}|}\quad (t\in\mathbb{R}).$$

The function F_N is periodic with period $1/N$. Thus we can assume that $|t| \le \frac{1}{2N}$ and $F_N(t)$ can be written in the form

$$F_N(t) = \frac{|\sin N\pi t|}{N|\sin \pi t|}+\frac{|\sin N\pi t|}{N}\sum_{k=1}^{N-1}\frac{1}{\sin\pi(\frac{k}{N}-t)} = \frac{|\sin N\pi t|}{N|\sin \pi t|}+|\sin N\pi t|\, L_N(t),$$

where

$$L_N(t) := \frac{1}{N}\sum_{k=1}^{N-1}\frac{1}{\sin\pi(\frac{k}{n}-t)}\qquad \left(|t| \le \frac{1}{2N}\right).$$

Then by (3.12)

$$G_N(t)L_N(t) \le \frac{1}{N}\sum_{\zeta\in\mathbb{T}_N}\left|K_N^N(z,\zeta)\right| \le G_N(t)L_N(t)+\frac{(1-\rho^N)}{|1-\rho^N e^{2\pi i N t}|}.$$

It is easy to see that

$$\max_{t\in\mathbb{R}} G_N(t) = \frac{1-\rho^N}{1+\rho^N}, \quad \min_{t\in\mathbb{R}} |1-\rho^N e^{2\pi i N t}| = 1-\rho^N \tag{3.14}$$

and

$$\frac{1}{\pi}(\log N - 1) \le L_N(t) \le \log N + 2 \quad (t\in\mathbb{R}, N\ge 2) \tag{3.15}$$

and (3.13) follows from (3.14) and (3.15).

To show (3.15) set $N = 2N' + r$ $(r = 0,1)$ and take the decomposition

$$L_N(t) =$$

$$= \frac{1}{N}\sum_{k=1}^{N-1} \frac{1}{\sin\pi(\frac{k}{N}-t)} = \frac{1}{N}\sum_{k=1}^{N'} \frac{1}{\sin\pi(\frac{k}{N}-t)} + \frac{1}{N}\sum_{k=N'+1}^{N-1} \frac{1}{\sin\pi(\frac{k}{N}-t)} =$$

$$= \frac{1}{N}\sum_{k=1}^{N'} \frac{1}{\sin\pi(\frac{k}{N}-t)} + \frac{1}{N}\sum_{k=1}^{N'+r-1} \frac{1}{\sin\pi(\frac{N-k}{N}-t)} =$$

$$= \frac{1}{N}\sum_{k=1}^{N'} \frac{1}{\sin\pi(\frac{k}{N}-t)} + \frac{1}{N}\sum_{k=1}^{N'+r-1} \frac{1}{\sin\pi(\frac{k}{N}+t)}.$$

Hence we get

$$\frac{1}{N}\sum_{k=1}^{N'-1} \frac{1}{\sin\dfrac{(2k+1)\pi}{2N}} < L_N(t) < \frac{2}{N}\sum_{k=1}^{N'} \frac{1}{\sin\dfrac{(2k-1)\pi}{2N}}.$$

Applying the inequality

$$\frac{2}{\pi}x \le \sin x \le x \quad \left(0 \le x \le \frac{\pi}{2}\right)$$

we have

$$\frac{1}{\pi}\sum_{k=1}^{N'-1} \frac{1}{k+1/2} \le L_N(t) \le \left(1 + \sum_{k=1}^{N'-1} \frac{1}{k+1/2}\right)$$

and consequently (3.15) holds. From (3.15) we get that (3.13) is satisfied for $C_1 = 1/10, C_2 = 2$ and $N \ge 4$.

Applying (3.13) for a sequence ρ_N $(n \ge 4)$ we get

Corollary. *If for the sequence* ρ_N $(n \geq 4)$

$$\left(1 - \frac{1}{\log N}\right)^{1/N} \leq \rho_N < 1 \quad (N \geq 4)$$

is satisfied, then the interpolation operators I_{N,ρ_N} $(N \geq 4)$ *are uniformly bounded*

$$\sup_{N \geq 4} \|I_{N,\rho_N}\| \leq 3.$$

4. Biorthogonal systems in the general case

Generalizing the construction of Section 3 we fix the complex vector

$$\mathfrak{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{D}^n,$$

where $a_i \neq a_j$, if $i \neq j$ and consider the rational functions

$$\phi_{k\ell}(z) := \phi_{(k,\ell)}(z) := \frac{z^\ell}{(1 - \overline{a}_k z)^{\ell+1}} \tag{4.1}$$

$$(z \in \mathbb{C},\ k = 1, 2, \ldots, n,\ \ \ell = 0, 1, \ldots, m_k - 1),$$

where $m_1, m_2, \cdots, m_n \in \mathbb{N}^*$ are given numbers. Obviously the numbers $a_k^* := 1/\overline{a}_k$ are the poles of $\phi_{k\ell}$ with the multiplicity $\ell + 1$. Set

$$\begin{aligned} &\mathfrak{m} = (m_1, m_2, \ldots, m_n), \quad m := m_1 + \cdots + m_n, \\ &\mathcal{J}_{\mathfrak{m}} := \{(i, j) : j \in \mathbb{N}, 0 \leq j < m_i,\ \ i = 1, 2, \cdots, n\}. \end{aligned} \tag{4.2}$$

We show that there exists a collection of polynomials

$$\Phi_{k\ell} = \Phi^{\mathfrak{m}}_{(k,\ell)}(\cdot, \mathfrak{a}) \in \mathcal{P}_{m-1} \quad ((k, \ell) \in \mathcal{J}_{\mathfrak{m}})$$

such that the systems

$$(\phi_i, i \in \mathcal{J}_{\mathfrak{m}}), \quad (\Phi_i, i \in \mathcal{J}_{\mathfrak{m}})$$

are biorthogonal with respect the scalar product $[\cdot, \cdot]_N$, if $N \geq m$, i.e.

$$[\Phi_{k\ell}, \phi_{rs}]_N = \delta_{kr}\delta_{\ell s} \qquad ((k, \ell), (r, s) \in \mathcal{J}_{\mathfrak{m}}). \tag{4.3}$$

Moreover the polynomials $\Phi_{k\ell}$ can be written in the form

$$\Phi_{k\ell} = \omega_k P_{k\ell} \quad ((k,\ell) \in \mathcal{J}_{\mathfrak{m}}), \tag{4.4}$$

where
(4.5)

$$P_{k\ell} \in \mathcal{P}_{m_k-1}, \ \omega_k(z) := \omega_k(z,\mathfrak{a}) := \prod_{i=1, i\neq k}^{n} (z-a_i)^{m_i} \quad (k=1,2,\ldots,n, z\in\mathbb{C}).$$

The polynomials $P_{k\ell}$ can be expressed by the partial sums of the Taylor-series expansion

$$\begin{gathered} P_k(z) := P_k(z,\mathfrak{a}) := \frac{1-z^N}{\omega_k(z,\mathfrak{a})} = \sum_{j=0}^{\infty} p_{jk}(z-a_k)^j \\ (|z-a_k| < r_k, z\in\mathbb{D}, r_k := \min\{|a_j-a_k| : j=1,2,\cdots,n, j\neq k\}), \end{gathered} \tag{4.6}$$

namely

$$P_{k\ell}(z) = (z-a_k)^{\ell} \sum_{j=0}^{m_k-\ell-1} p_{jk}(z-a_k)^j \quad (z\in\mathbb{C}, (k,\ell)\in\mathcal{J}_{\mathfrak{m}}) \tag{4.7}$$

and by (4.6)

$$p_{jk} = \frac{P_k^{(j)}(a_k,\mathfrak{a})}{j!} := \frac{1}{j!}\,\frac{d^j}{dz^j} P_k(z,\mathfrak{a})\Big|_{z=a_k}.$$

We prove

Theorem 4. *Let $\mathfrak{a} = (a_1,\cdots,a_n) \in \mathbb{D}^n$, where $a_i \neq a_j$, if $1 \le i,j \le n$ and $i \neq j$ and fix the vector $\mathfrak{m} = (m_1,\cdots,m_n)$ with $m_j \in \mathbb{N}^*$ and the natural number $N \ge m := m_1 + \cdots + m_n$. Then there exists an unique system of polynomials $\Phi_{(k,\ell)} \in \mathcal{P}_{m-1}$ $((k,\ell)\in\mathcal{J}_{\mathfrak{m}})$ such that the systems $\phi_{(k,\ell)}$ and $\Phi_{(k,\ell)}$ $((k,\ell)\in\mathcal{J}_{\mathfrak{m}})$ are biorthogonal with respect the scalar product (1.3). Moreover the polynomials $\Phi_{(k,\ell)}$ can be written in the form (4.4) and the coefficients of $P_{k\ell}$ are defined by (4.6) and (4.7).*

Proof. By (2.3) for $\zeta \in \mathbb{T}_N$ we have

$$\begin{aligned} [\Phi_{k\ell}, \phi_{rs}]_N &= \frac{1}{N}\sum_{\zeta\in\mathbb{T}_N} \frac{\Phi_{k\ell}(\zeta)\bar{\zeta}^s}{(1-a_r\bar{\zeta})^{s+1}} = \frac{1}{N}\sum_{\zeta\in\mathbb{T}_N} \frac{\Phi_{k\ell}(\zeta)\zeta}{(\zeta-a_r)^{s+1}} = \\ &= \frac{1}{2\pi i}\int_{\mathbb{T}_N} \frac{\Phi_{k\ell}(\zeta)}{(\zeta-a_r)^{s+1}}\,d\zeta = \frac{1}{s!}\,\frac{d^s}{dz^s}\,\frac{\Phi_{k\ell}(z)}{1-z^N}\Big|_{z=a_r} \end{aligned}$$

and consequently the systems in question are biorthogonal if and only if

$$\frac{1}{s!}\,\frac{d^s}{dz^s}\,\frac{\Phi_{k\ell}(z)}{1-z^N}\bigg|_{z=a_r} = \delta_{kr}\delta_{\ell s} \tag{4.8}$$

for any couple $(k,\ell),(r,s)\in\mathcal{J}_{\mathfrak{m}}$.

The solutions of equations (4.8) are connected with the weighted Hermite interpolation problem

$$(\rho_N\Phi)^{(j)}(a_i) = b_{ij} \quad ((i,j)\in\mathcal{J}_{\mathfrak{m}}),$$

where $\rho_N(z) := (1-z^N)^{-1}$ $(z\in\mathbb{C})$ is the weight function and b_{ij} are given numbers. Namely the polynomials $\Phi_{k\ell}$ $((k,\ell)\in\mathcal{J}_{\mathfrak{m}})$ can be expressed by the fundamental polynomials of this interpolation problem.

From (4.8) it follows that $\Phi_{k\ell}$ is of the form

$$\Phi_{k\ell}(z) = P_{k\ell}(z)\prod_{j=1,j\neq k}^{n}(z-a_j)^{m_j} = P_{k\ell}(z)\omega_k(z) \quad (z\in\mathbb{C})$$

and by (4.8)

$$(\rho_N\omega_kP_{k\ell})^{(j)}(a_k) = \delta_j^{\ell}j! \quad (\ell\le j<m_k). \tag{4.9}$$

This is equivalent to

$$\sum_{i=0}^{j}\binom{j}{i}(\rho_N\omega_k)^{(j-i)}(a_k)P_{k\ell}^{(i)}(a_k) = \delta_j^{\ell}j! \quad (\ell\le j<m_k).$$

Thus $P_{k\ell}^{(i)}(a_k)=0$, if $i<\ell$ and

$$\sum_{i=\ell}^{j}\frac{(\rho_N\omega_k)^{(j-i)}(a_k)}{(j-i)!}\,\frac{P_{k\ell}^{(i)}(a_k)}{i!} = \delta_j^{\ell} \quad (\ell\le j<m_k). \tag{4.10}$$

We consider the infinite system of linear equations with respect to $p_{k0},p_{k1},\dots,p_{ki},\dots$:

$$\sum_{i=0}^{j}\frac{(\rho_N\omega_k)^{(j-i)}(a_k)}{(j-i)!}p_{ki} = \delta_0^{j} \quad (j\in\mathbb{N}). \tag{4.11}$$

The coefficient of p_{kj} in j-th equation is $(\rho_N \omega_k)(a_k) \neq 0$, consequently this system has a unique solution. Comparing this with (4.10) and (4.11) we get

$$\frac{P_{k\ell}^{(i)}(a_k)}{i!} = p_{k(\ell-i)} \quad (i \geq \ell).$$

It is clear that the Taylor-coefficients of the function

$$P_k(z) := \frac{1-z^N}{\omega_k(z)} = \sum_{j=0}^{\infty} p_{kj}(z-a_k)^j \quad ((|z-a_k| < r_k))$$

satisfy (4.11) and Theorem 2 is proved.

To evaluate the numbers p_{kj} we introduce the function

$$S_k(z) := \frac{P_k'(z)}{P_k(z)} = \sum_{j=0}^{N-1} \frac{1}{z-\epsilon_N^j} - \sum_{j=1, j\neq k}^{n} \frac{1}{z-a_j} \quad (|z-a_k| < r_k),$$

where $\epsilon_N^j = \exp(2\pi i j/N)$. Hence by

$$P_k^{(\ell+1)}(a_k) = \sum_{j=0}^{\ell} \binom{\ell}{j} P_k^{(j)}(a_k) S_k^{(\ell-j)}(a_k) \quad (\ell \in \mathbb{N})$$

we get the following recursion:

(4.12) $$p_{k(\ell+1)} = \frac{1}{\ell+1} \sum_{j=0}^{\ell} p_{kj} s_{k(\ell-j)} \qquad (\ell \in \mathbb{N}),$$

where

(4.13) $$s_{ki} := \frac{S_k^{(i)}(a_k)}{i!} = \sum_{j=0}^{N-1} \frac{(-1)^i}{(a_k-\epsilon_N^j)^{i+1}} - \sum_{j=1, j\neq k}^{n} \frac{(-1)^i}{(a_k-a_j)^{i+1}} \quad (i \in \mathbb{N}).$$

On the basis (4.12) and (4.13) the coefficients p_{kj} can be computed.

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