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ON THE CLASS OF GENERALIZING DIFFERENTIAL OPERATORS IN CLIFFORD ALGEBRA

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Dedicated to Professor Imre Kátai on his 65th birthday

Abstract. Let $\mathcal{A}$ be a universal Clifford algebra induced by an m-dimensional real linear space. It is well-know that the differential operator $\mu = \sum\limits_{k=0}^{m} \frac{\partial}{\partial x_k} e_k$ satisfies the relations $\mu.\overline{\mu} = \overline{\mu}.\mu = \Delta_{m+1}$, where $\overline{\mu}$ is the conjugate operator of μ and $\Delta_{m+1} = \sum\limits_{k=0}^{m} \frac{\partial^2}{\partial x_k^2}$ (see [1]). Let $m \equiv 2 \ (mod\ 4)$ and $L(e_0, e_{A_1}, ..., e_{A_{m+1}})$ be the invertible subspace in $\mathcal{A}$ (see [3]). In this paper we give the some conditions for the generalizing differential $D^* = \sum\limits_{k=0}^{m+1} \alpha_k \frac{\partial}{\partial x_k}$, where $\alpha_k \in L(e_0, e_{A_1}, ..., e_{A_{m+1}})$ such that any solution of a differential equation $D^* u = 0$ is always a solution of Laplace's equation $\Delta_{m+2} u = 0$, where $\Delta_{m+2} = \sum\limits_{k=0}^{m+1} \frac{\partial^2}{\partial x_k^2}$.

1. Preliminaries

Consider the 2^m-dimensional real space $\mathcal{A}$ with basis

$$E = \{e_0, e_1, ..., e_m, e_{12}, ..., e_{12...m}\}.$$

The product of two elements $e_A, e_B \in E$ is given by

$$e_A.e_B = (-1)^{\sharp(A\cap B)}(-1)^{P(A,B)}e_{A\Delta B}; \quad A, B \subset \{1, 2, ..., m\},$$

where

$$\begin{cases} P(A,B) & = \sum\limits_{j\in B} P(A,j), \\ P(A,j) & = \sharp\{i \in A : i > j\}, \\ A\Delta B & = (A\backslash B) \cup (B\backslash A), \end{cases}$$

and $\sharp A$ denotes the number of elements of A.

Each element $a = \sum\limits_A a_A e_A \in \mathcal{A} \quad (a_A \in I\!R)$ is called a Clifford number. The product of two Clifford numbers $a = \sum\limits_A a_A e_A; \quad b = \sum\limits_B b_B e_B$ is defined by the formula

$$ab = \sum_A \sum_B a_A b_B e_A e_B.$$

It is easy to check that in such way $\mathcal{A}$ is turned into an associative non-commutative algebra over $I\!R$. It is called the Clifford algebra.

The involution for basic vector $e_{k_1k_2\ldots k_t} \in E$; $k_1, k_2, \ldots, k_t \in \{1, 2, \ldots, m\}$ is given by $\overline{e}_{k_1\ldots k_t} = (-1)^{\frac{t(t+1)}{2}} e_{k_1k_2\ldots k_t}$.

For any $a = \sum\limits_A a_A e_A \in \mathcal{A}$, we write $\overline{a} = \sum\limits_A a_A \overline{e}_A$ and $|a| = \left(\sum\limits_A a_A^2\right)^{\frac{1}{2}}$.

2. Generalizing differential operators

Definition 1 (see [3]). i) An element $a \in \mathcal{A}$ is said to be invertible if there exists an element $a^{-1} \in \mathcal{A}$ such that $a.a^{-1} = a^{-1}.a = e_0; \quad a^{-1}$ is said to be the inverse of a.

ii) A subspace $X \subset \mathcal{A}$ is said to be invertible if every non-zero element in X is invertible in $\mathcal{A}$.

iii) $L(u_1, u_2, \ldots, u_n) = lin\{u_1, u_2, \ldots, u_n\}, \quad u_i \in \mathcal{A} \quad (i = 1, 2, \ldots, n)$.

Let $m \equiv 2 \ (mod\ \ 4)$ and $L(e_0, e_{A_1}, ..., e_{A_{m+1}})$ be the invertible subspace in $\mathcal{A}$ (see [3]).

Definition 2. i) The operator $D^* = \sum\limits_{k=0}^{m+1} \alpha_k \frac{\partial}{\partial x_k}$, where

$$\alpha_k \in L(e_0, e_{A_1}, ..., e_{A_{m+1}}) \quad (k = 0, 1, ..., m+1)$$

is called a generalizing operator in $\mathcal{A}$.

ii) Let $\alpha_k = \sum\limits_{i=0}^{m+1} a_{ik} e_{A_i} \quad k = 0, 1, ..., m+1$, where $e_{A_0} = e_{A_0}$. The matrix $A = (a_{ij})_{m+2}$ is called the symbol of D^*.

iii) The operator $\overline{D^*} = \sum\limits_{k=0}^{m+1} \overline{\alpha}_k \frac{\partial}{\partial x_k}$ is called the conjugate operator of D.

iv) The matrix $\overline{A} = (a'_{ij})_{m+2}$ defined by

$$\begin{cases} a'_{0j} & = a_{0j}, \\ & \\ a'_{ij} & = -a_{ij} \end{cases} \quad \text{for} \quad i = 1, ..., m+1 \quad \text{and} \quad j = 0, ..., m+1$$

is called the conjugate of the matrix A.

Remark. From these notations we can write

$$D^* = (e_0 \ \ e_{A_1} \dots e_{A_{m+1}}) A \left(\frac{\partial}{\partial x_0} \ \ \frac{\partial}{\partial x_1} \cdots \frac{\partial}{\partial x_{m+1}} \right)^T,$$

where M^T denotes the transpose of the matrix M.

Lemma 1 (see [2]). *If $L(e_0, e_{A_1}, ..., e_{A_l})$, $\quad e_{A_i} \in E$, $\ e_{A_i} \neq e_{A_j}$ for all $i \neq j$, $i, j \in \{0, 1, \dots, l\}$, is invertible then*

$$\textit{either} \quad \#A_j = 4p_j + 1 \quad \textit{or} \quad \#A_j = 4p_j + 2 \quad (p_j \in I\!N, j = \overline{1, l}).$$

Proof. Suppose that there exits $e_{A_i} \in \{e_{A_1}, e_{A_2}, \dots, e_{A_l}\}$ such that

$$\sharp A_j = 4p_j \quad \text{or} \quad \sharp A_j = 4p_j + 3.$$

Hence

$$\big(e_0 + e_{A_i}\big).\big(e_0 - \overline{e}_{A_i}\big) = e_0 + e_{A_i} - \overline{e}_{A_i} - e_{A_i}.\overline{e}_{A_i} = 0.$$

So $e_0 + e_{A_i}$ is not invertible.

Lemma 2 (see [2]). *$L(e_0, e_{A_1}, ..., e_{A_l})$, $\quad e_{A_i} \in E, e_{A_i} \neq e_{A_j}$ for all $i \neq j$, is invertible if and only if*

$$e_{A_i}\overline{e}_{A_j} + e_{A_j}\overline{e}_{A_i} = 0 \quad \textit{for all} \quad i \neq j; \ \ i, j \in \{0, 1, ..., l\},$$

where $e_{A_0} = e_0$.

Proof. *Sufficiency.* Suppose that $e_{A_i}\overline{e}_{A_j} + e_{A_j}\overline{e}_{A_i} = 0 \quad \text{for} i \neq j, \quad i, j \in$ $\in \{1, 2, \dots l\}$. From $e_0\overline{e}_{A_j} + e_{A_j}\overline{e}_0 = 0$, we have $\overline{e}_{A_j} + e_{A_j} = 0, \quad j \in$ $\{1, 2, \dots, l\}$.

Take $a = a_0e_0 + \sum\limits_{i=1}^{l} a_ie_{A_i} \in L(e_0, e_{A_1}, e_{A_2}, \dots, e_{A_l})$, $(a \neq 0)$. Write

$$a^{-1} = \frac{1}{|a|^2}\left(a_0e_0 + \sum_{i=1}^{l} a_i\overline{e}_{A_i}\right).$$

Then

$$\begin{aligned} aa^{-1} &= \frac{1}{|a|^2}\left(a_0e_0 + \sum_{i=1}^{l} a_ie_{A_i}\right)\left(a_0e_0 + \sum_{j=1}^{l} a_j\overline{e}_{A_j}\right) = \\ &= \frac{1}{|a|^2}\left[a_0^2e_0 + a_0\left(\sum_{i=1}^{l} a_ie_{A_i} + \sum_{j=1}^{l} a_j\overline{e}_{A_j}\right) + \right. \\ &\quad \left. + \sum_{i=1}^{l} a_i^2e_{A_i}\overline{e}_{A_i} + \sum_{i<j} a_ia_j(e_{A_i}\overline{e}_{A_j} + e_{A_j}\overline{e}_{A_i})\right] = \\ &= \frac{1}{|a|^2}\left(\sum_{i=0}^{l} a_i^2\right)e_0 = e_0. \end{aligned}$$

Similarly, one can check the equality $a^{-1}a = e_0$.

Neccessity. Suppose that $L(e_0, e_{A_1}, e_{A_2}, \dots, e_{A_l})$ is invertible. By Lemma 1 we have

$$\sharp A_j = 4p_j + 1 \quad \text{or} \quad \sharp A_j = 4p_j + 2, \;\; p_j \in I\!N, j \in \{1, 2, \dots, l\}.$$

Hence

$$e_0\overline{e}_{A_j} + e_{A_j}\overline{e}_0 = \overline{e}_{A_j} + e_{A_j} = 0 \quad \text{for} \quad j \in \{1, 2, \dots, l\}.$$

Suppose that there exists $e_{A_i}, e_{A_j} \in \{e_{A_1}, e_{A_2}, \dots, e_{A_l}\}$ such that

$$e_{A_i}\overline{e}_{A_j} + e_{A_j}\overline{e}_{A_i} \neq 0.$$

By Lemma 1 we have $-e_{A_i}e_{A_j} - e_{A_j}e_{A_i} \neq 0$. It is easy to see that

$$\text{either} \quad e_{A_\mu}e_{A_\nu} = e_{A_\nu}e_{A_\mu} \quad \text{or} \quad e_{A_\mu}e_{A_\nu} = -e_{A_\nu}e_{A_\mu}, \quad \forall \; e_{A_\mu}, e_{A_\nu} \in E.$$

Hence $e_{A_i}e_{A_j} = e_{A_j}e_{A_i}$. Write $a = e_{A_i} + e_{A_j}$ and $b = \overline{e}_{A_i} - \overline{e}_{A_j}$. Then we get

$$ab = \left(e_{A_i}+e_{A_j}\right)\left(\overline{e}_{A_i}-\overline{e}_{A_j}\right) = e_0+e_{A_j}\overline{e}_{A_i}-e_{A_i}\overline{e}_{A_j}-e_0 = -e_{A_j}e_{A_i}+e_{A_i}e_{A_j} = 0.$$

So a is not invertible.

Lemma 3. *The generalized differential operator* $D = \sum\limits_{k=0}^{m+1} \dfrac{\partial}{\partial x_k} e_{A_k}$ *with* $e_{A_0} = e_0$ *satisfies* $D.\overline{D} = \overline{D}.D = \Delta_{m+2}$, *where* $\overline{D} = \sum\limits_{k=0}^{m+1} \dfrac{\partial}{\partial x_k}\overline{e}_{A_k}$ *is the conjugate operator of* D *and* $\Delta_{m+2} = \sum\limits_{k=0}^{m+1} \dfrac{\partial^2}{\partial x_k^2}$.

Proof. By Lemma 1 and Lemma 2 we get

$$D.\overline{D} = \sum_{k=0}^{m+1} \frac{\partial}{\partial x_k} e_{A_k}.\sum_{l=0}^{m+1} \frac{\partial}{\partial x_l}\overline{e}_{A_l} =$$

$$= \sum_{k=0}^{m+1} \frac{\partial^2}{\partial x_k^2} e_{A_k}.\overline{e}_{A_k} + \sum_{k\neq l} \frac{\partial^2}{\partial x_k \partial x_l}\left(e_{A_k}\overline{e}_{A_l} + e_{A_l}\overline{e}_{A_k}\right) = \Delta_{m+2}.$$

Similarly, one can check that $\overline{D}.D = \Delta_{m+2}$.

Lemma 4. *If the matrix* A *is a symbol of the generalizing differential operator* D^* *then* $\overline{A}$ *is the symbol of* $\overline{D^*}$.

Proof. By Lemma 1 we have $\sharp A_j = 4p_j+1$ or $\# A_j = 4p_j+2$ ($p_j \in I\!N; j = = \overline{1, m+1}$). Hence $\overline{e}_{A_i} = -e_{A_i} \quad i = \overline{1, m+1}$.

Suppose that $\alpha_k = \sum\limits_{i=0}^{m+1} a_{ik}e_{A_i} \quad (k = 0, 1, ..., m+1)$, then

$$\overline{\alpha}_k = \sum_{i=0}^{m+1} a_{ik}\overline{e}_{A_i} = a_{0k}\overline{e}_0 + \sum_{i=1}^{m+1} a_{ik}\overline{e}_{A_i} = a_{0k}e_0 - \sum_{i=1}^{m+1} a_{ik}e_{A_i}.$$

Since

$$\overline{D^*} = \sum_{k=0}^{m+1} \overline{\alpha}_k \frac{\partial}{\partial x_k} = (e_0, e_{A_1}, ..., e_{A_{m+1}})\overline{A}\left(\frac{\partial}{\partial x_0}, \frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_{m+1}}\right)^T.$$

The Lemma is proved.

Definition 3. i) Let Ω be a certain open domain in $I\!R^{m+2}$. A function $f \in C^1(\Omega; \mathcal{A})$ is said to be left monogenic in Ω if and only if $D^* f = 0$ in Ω.

ii) The set of left monogenic functions in Ω is denoted by $H(\Omega; \mathcal{A})$, and the set of orthogonal matrices of order $m+2$ is denoted by $O(m+2)$.

Lemma 5. *Let A be the symbol of the generalizing differential operator $D^* = \sum\limits_{k=0}^{m+1} \alpha_k \frac{\partial}{\partial x_k}$, where $\alpha_k \in L(e_0, e_{A_1}, ..., e_{A_{m+1}})$. Then $A = \lambda O$, where $\lambda > 0$ and $O \in O(m+2)$ if and only if*

$$\begin{cases} \alpha_i \overline{\alpha}_j + \alpha_j \overline{\alpha}_i & = 0 \\ \alpha_i & = \lambda \end{cases} \qquad i, j = 0, 1, \ldots, m+1;\ i \neq j.$$

Proof. By Lemma 1 we have $\# A_j = 4p_j + 1$ or $\# A_j = 4p_j + 2 \quad (p_j \in I\!N,$ $j = \overline{1, m+1})$. Hence it is easy to check that $e_{A_k} \overline{e}_{A_k} = 1$ and $\alpha_k \overline{\alpha}_k = |\alpha_k|^2$ for $k = \overline{0, m+1}$.

Let $B = A^T.A$. Then $b_{ij} = \sum\limits_{k=0}^{m+1} a_{ki} a_{kj} \quad i, j = 0, ..., m+1$. By Lemma 2 we get

$$\alpha_i \overline{\alpha}_j + \alpha_j \overline{\alpha_i} = \left(\sum_{l=0}^{m+1} a_{li} e_{A_l} \right) \left(\sum_{k=0}^{m+1} a_{kj} \overline{e}_{A_k} \right) + \left(\sum_{k=0}^{m+1} a_{kj} e_{A_k} \right) \left(\sum_{l=0}^{m+1} a_{li} \overline{e}_{A_l} \right) =$$

$$= \sum_{l=0}^{m+1} \sum_{k=0}^{m+1} a_{li} a_{kj} (e_{A_l} \overline{e}_{A_k} + e_{A_k} \overline{e}_{A_l}) =$$

$$= 2 \sum_{k=0}^{m+1} a_{ki} a_{kj} (e_{A_k} \overline{e}_{A_k}) = 2 \sum_{k=0}^{m+1} a_{ki} a_{kj} = 2 b_{ij}.$$

Thus

$$A = \lambda O, \text{ where } O \in O(m+2) \iff \begin{cases} b_{ij} & =0 \text{ for } i \neq j \\ b_{ii} & = \lambda^2 \end{cases} \qquad i, j = 0, ..., m+1$$

$$\iff \begin{cases} \alpha_i \overline{\alpha}_j + \alpha_j \overline{\alpha}_i & = 0 \text{ for } i \neq j \\ 2 \alpha_i \overline{\alpha}_i & = 2 |\alpha_i|^2 = 2 b_{ij} = 2 \lambda^2 \end{cases} \qquad i, j = 0, 1, ..., m+1.$$

Lemma 5 is proved.

Theorem. *Let $D^* = \sum\limits_{k=0}^{m+1} \alpha_k \frac{\partial}{\partial x_k}$, where $\alpha_k \in L(e_{A_0}, ..., e_{A_{m+1}})$ and A be the symbol of D^*. Then $D^* \overline{D}^* = \overline{D}^* D^* = \lambda^2 \Delta_{m+2}$ if and only if $A = \lambda O$, where $O \in O(m+2)$ and $\Delta_{m+2} = \sum\limits_{k=0}^{m+1} \frac{\partial^2}{\partial x_k^2}$, $\lambda > 0$.*

Proof. By Lemma 5 we have

$$D^*\overline{D}^* = \left(\sum_{i=0}^{m+1} \alpha_i \frac{\partial}{\partial x_i}\right)\left(\sum_{j=0}^{m+1} \overline{\alpha}_j \frac{\partial}{\partial x_j}\right) = \sum_{i,j=0}^{m+1} \alpha_i \overline{\alpha}_j = \lambda^2 \Delta_{m+2} \iff$$

$$\iff \begin{cases} \alpha_i\overline{\alpha}_j + \alpha_j\overline{\alpha}_i = 0 \\ \alpha_i|^2 = \lambda^2 \end{cases} \quad \text{for} \quad i,j = 0,1,...,m+1 \quad \text{and} \quad i \neq j \iff$$

$$\iff A = \lambda O.$$

Corollary 1. *Let D^* be a generalizing differential operator in $\mathcal{A}, f \in$ $\in \mathcal{H}(\Omega;\mathcal{A})$. If $A = \lambda O$, where $\lambda > 0, O \in O(m+1), A$ is the symbol of D^*, then f is the solution of Laplace's equation $\Delta_{m+2} f = 0$.*

Proof. From $A = \lambda O$ and $D^* f = 0$ we get $\overline{D}^* D^* f = \lambda^2 \Delta_{m+2} f = 0$. So $\Delta_{m+2} f = 0$.

Corollary 2. *If the generalizing operator $D^* = \sum\limits_{k=0}^{m+1} \alpha_k \frac{\partial}{\partial x_k}$ satisfies the relation $D^*\overline{D}^* = \overline{D}^* D^* = \lambda^2 \Delta_{m+2}$, where $\Delta_{m+2} = \sum\limits_{k=0}^{m+1} \frac{\partial^2}{\partial x_k^2}$, then exits the linear transformation $y_i = \sum\limits_{j=0}^{m+1} p_{ij} x_j$ such that $D^* = \sum\limits_{k=0}^{m+1} \frac{\partial}{\partial y_k} e_{A_k}$.*

Proof. Let A be the symbol of D^*. By Theorem we have $A = \lambda O$, where $\lambda > 0$ and $O \in O(m+2)$. If we choose $P = (A^T)^{-1} = (\lambda O^T)^{-1} = \dfrac{1}{\lambda} O = \dfrac{1}{\lambda^2 A}$, then we get

$$\begin{aligned} D^* &= \Big(e_0 \;\; e_{A_1} \dots e_{A_{m+1}}\Big) A \left(\frac{\partial}{\partial x_0} \;\; \frac{\partial}{\partial x_1} \cdots \frac{\partial}{\partial x_{m+1}}\right)^T = \\ &= \Big(e_0 \;\; e_{A_1} \dots e_{A_{m+1}}\Big) A P^T \left(\frac{\partial}{\partial y_0} \;\; \frac{\partial}{\partial y_1} \cdots \frac{\partial}{\partial y_{m+1}}\right)^T = \\ &= \Big(e_0 \;\; e_{A_1} \dots e_{A_{m+1}}\Big) \left(\frac{\partial}{\partial y_0} \;\; \frac{\partial}{\partial y_1} \cdots \frac{\partial}{\partial y_{m+1}}\right)^T . \end{aligned}$$

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