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DYNAMICAL SYSTEMS ORIGINATED IN THE OSTROWSKI ALPHA–EXPANSION

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Dedicated to Professor Imre Kátai
on the occasion of his 65th birthday

Abstract. The Ostrowski system of numeration has for scale of numeration the denominators of the convergents in the continued fraction expansion of a given real number. The aim of this paper is to study the topological and ergodic properties of various dynamical systems arising from this scale, especially flows associated with multiplicative sequences for which a complete description is given.

1. Introduction

1.1. The Ostrowski α-expansion

Let α be an irrational number in the unit interval $[0, 1)$ and $[0; a_1, a_2, \cdots]$ its continued fraction expansion. Recall that the convergents of α are the rational numbers p_n/q_n $(n \geq 0)$ given by the classical relations $p_{-1} = 1$, $p_0 = 0$, $p_n = a_n p_{n-1} + p_{n-2}$ (hence $p_1 = 1$) and $q_{-1} = 0$, $q_0 = 1$ $q_n = a_n q_{n-1} + q_{n-2}$ (hence $q_1 = a_1$), such that

$$[0; a_1, \cdots, a_n] := \frac{p_n}{q_n} = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots + \cfrac{1}{a_n}}}.$$

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Then any positive integer m can be expanded as

$$m = e_0(m)q_0 + \cdots + e_r(m)q_r\ , \quad e_r(m) \neq 0, \tag{1}$$

where each $e_j(m)$ is an integer (called j-th *Ostrowski α-digit*) of m, such that for all j, $0 \le j < r$, the inequalities

$$e_0(m)q_0 + \cdots + e_j(m)q_j < q_{j+1} \tag{2}$$

hold. These inequalities ensure the uniqueness of the α-digits and (1) defines the so-called *Ostrowski α-expansion* of m. The Ostrowski α-digits are extended for $j > r$ by putting $e_j(m) := 0$. This expansion has been introduced by A. Ostrowski in [44].

The system of inequalities (2), which are now satisfied for all $j \ge 0$, is equivalent to

$$\begin{cases} e_0(m) \le a_1 - 1; \\ \forall\, j \ge 1,\ e_j(m) \le a_{j+1}; \\ \forall\, j \ge 1,\ (e_j(m) = a_{j+1} \Rightarrow e_{j-1}(m) = 0\,). \end{cases} \tag{3}$$

In the case $\alpha = \dfrac{-1+\sqrt{5}}{2}$, the sequence q_n is nothing but the classical Fibonacci sequence $F_0 = 1$, $F_1 = 1$, $F_{n+2} = F_{n+1} + F_n$ $(n \ge 0)$. Inequality (3) for $j = 0$ implies $e_0(m) = 0$, so that the first digit can be omitted and then the corresponding α-expansion is the so-called *Zeckendorf expansion* [53]. More generally, $\alpha < 1/2$ gives $a_1 \ge 2$ and yields a system of numeration $(q_n)_{n\ge 0}$ in the usual sense (see [25]). If $\alpha > 1/2$, then $a_1 = 1$, thus $e_0(m) = 0$ for any m. Then $(q_n)_{n\ge 1}$ is a system of numeration, derived from $\alpha' = [0; a_2+1, a_3, a_4, \ldots] \in$ $\in (0, 1/2)$. Notice that $\alpha' = 1 - \alpha$. The cancellation of this useless 0 at the beginning of the expansion can be interpreted from Theorem 2 (see Section 3.3) by the identification between the isomorphic rotations R_α and $R_{1-\alpha}$. Without loss of generality, we will hence suppose that $a_1 \ge 2$.

This paper is devoted to the study of various dynamical systems which derive from the Ostrowski α-expansion. Although it is staying out of our present interests, we mention that the Ostrowski α-expansion furnishes a powerful tool to study the discrepancy of the sequence $(n\alpha \pmod 1)_n$ and to give combinatorial properties of sturmian sequences (see for instance [5, 18, 19, 20, 21, 24, 35, 36, 47, 49, 52]). In fact, we shall especially pay attention to isomorphisms between them.

1.2. Almost topological isomorphisms

There are many different notions of isomorphism of dynamical systems (topological, metric, spectral, of measured algebras, almost topological, a.s.o.). It turns out that the relevant notion in the context is this of almost topological isomorphism introduced by Denker and Keane (see [17]).

The context one has to begin with is purely topological. For $i = 1, 2$, let Ω_i be compact metrizable space, $\mathcal{B}(\Omega_i)$ its Borel $\sigma-$algebra and let $T_i : \Omega_i \to \Omega_i$ be a (Borel) measurable map. One says that there is an *almost topological isomorphism* Φ between $(\Omega_1, \mathcal{B}(\Omega_1), T_1)$ and $(\Omega_2, \mathcal{B}(\Omega_2), T_2)$ if there are subsets $Y_i \subset \Omega_i$ and a map $\Phi : Y_1 \to Y_2$ satisfying the following conditions:

- Y_i contains a dense G_δ subset of Ω_i.
- T_i induces a homeomorphism from Y_i onto Y_i for the induced topology.
- The map Φ is a homeomorphism such that the commutation relation

$$\Phi \circ T_2(y_2) = T_1 \circ \Phi(y_2)$$

  holds for any $y_2 \in Y_2$.

Usually, Φ is already defined on Ω_1 and we are concerned with its restriction on Y_1, still denoted by Φ. Furthermore, Φ is said to be an almost topological conjugacy (or isomorphism) from $(\Omega_1, \mathcal{B}(\Omega_1), T_1)$ to $(\Omega_2, \mathcal{B}(\Omega_2), T_2)$, and $(\Omega_i, \mathcal{B}(\Omega_i), T_i)$, or better $(\Omega_i, Y_i, \mathcal{B}(\Omega_i), T_i)$, is said to be an *almost topological dynamical system.*

In addition, let μ_i be a T_i-invariant Borel measure on Ω_i such that

$$\mu_i(\Omega_i \setminus Y_i) = 0.$$

If $\mu_1(B_1) := \mu_2(\Phi^{-1}(B_1))$ for any $B_1 \in \mathcal{B}(\Omega_1)$, then those almost topological dynamical systems are said *finitary isomorphic* under Φ and the corresponding measures $\mu_1 = \mu_2 \circ \Phi^{-1}$, μ_2. In the whole paper we will omit the mention of the $\sigma-$algebra $\mathcal{B}$ in the dynamical systems which will be always the Borel σ-algebra.

Notice that, if $(\Omega_2, \mathcal{B}(\Omega_2), T_2)$ is uniquely ergodic and if Y_2 has full measure w.r.t. the invariant measure, then there is a unique T_1-invariant measure μ_1 on Ω_1 such that $\mu_1(\Omega_1 \setminus Y_1) = 0$ and in fact $\mu_1 = \mu_2 \circ \Phi^{-1}$. In particular, if Y_1 is countable and contents no periodic orbit under T_1, then $(\Omega_1, \mathcal{B}(\Omega_1), T_1)$ is also uniquely ergodic.

1.3. Subshifts associated to sequences

The principal dynamical systems we will focus our attention on are flows originated in complex unimodular sequences. Let X be a compact metrizable space and let σ be the usual shift given by

$$\sigma(x_0, x_1, x_2, \ldots) := (x_1, x_2, \ldots)$$

on the compact topological product space

$$X^{\mathbf{N}} := \prod_{n=0}^{\infty} X_n,$$

where $X_n := X$ for all n. The natural projection $X^{\mathbf{N}} \to X_n$ will be denoted by π_n (so, the first projection is π_0). A sequence $x = (x_0, x_1, x_2, \ldots)$ in X is an element of $X^{\mathbf{N}}$ and will be also viewed as the infinite string $x_0 x_1 x_2 \cdots$ provided that no confusion from the context could be made with an infinite product in a topological multiplicative group. Given x in $X^{\mathbf{N}}$, the orbit of x under the shift is the set $\mathrm{Orb}(x) = \{\sigma^n x\,;\ n \in \mathbf{N}\}$ and we consider the orbit closure $\mathcal{O}_x := \overline{Orb(x)}$. The restriction of σ to $\mathcal{O}_x$ (still denoted by σ) defines the *flow* $\mathcal{F}(x) = (\mathcal{O}_x, \sigma)$. Notice that we do not require that σ is surjective on $\mathcal{O}_x$ and recall that the full shift $\mathcal{F}(X) := (X^{\mathbf{N}}, \sigma)$ is obtained as a flow $\mathcal{F}(x)$ for many x: if ν is a probability measure on X assigning positive measure to every non-empty open set, then almost all $x \in X^{\mathbf{N}}$ in the sense of the product measure $\otimes_0^\infty \nu$ gives the full shift.

Let be X a compact metrizable space and $T : X \to X$ a continuous map. Recall that the topological dynamical system (X, T) is said to be *minimal* if the only closed subset Y of X such that $Y \subset T^{-1}(Y)$ are the full space X and the empty set (that implies the surjectivity of T). That is classically equivalent to the density of all orbits. For subshifts, the minimality of $\mathcal{F}(x)$ is equivalent to the property of uniform recurrence: for any $s \in \mathbf{N}$ and any $\varepsilon > 0$, the set

$$V_x(s, \varepsilon) := \{n \in \mathbf{N}\,;\ |x(n+j) - x(j)| \le \varepsilon \quad \text{for } j = 0, 1, \ldots, q_s - 1\}$$

has bounded gaps.

A necessary and sufficient condition for unique ergodicity of $(X, \mathcal{B}, T)$ is the uniform convergence of the sequence $(S_N(f))_N$, where $S_N(f) = N^{-1} \sum\limits_{n<N} f \circ T^n$ for any $f \in \mathcal{C}(X, \mathbf{C})$. If it is the case, there is still uniform convergence if f is $\mu-$continuous, that is if the set of the points of discontinuity of f has measure zero with respect to the unique invariant measure of $(X, \mathcal{B}, T)$.

Let $(G, \cdot)$ be a compact metrizable group and $\zeta = (\zeta(n))_{n\in\mathbf{N}} \in G^{\mathbf{N}}$. By definition, the first forward difference sequence of ζ is the sequence $\Delta\zeta$ given by $\Delta\zeta(n) := \zeta(n+1) \cdot \zeta(n)^{-1}$. For short, $\Delta\zeta$ will be called a *difference sequence*. Notice that the sequence ζ can be recovered from $\Delta\zeta$ and $\zeta(0)$. Indeed, the flows $\mathcal{F}(\zeta)$ and $\mathcal{F}(\Delta\zeta)$ are also strongly related as we will see in Section 5.

All the sequences we will look at take their values in a subgroup G of the unit circle $\mathbf{U}$. If $X = \mathbf{U}$, there is a simple criterion of unique ergodicity for $\mathcal{F}(x)$, which derives from the folklore.

Lemma 1. *For $\underline{m} = (m_0, \dots, m_s) \in \mathbf{Z}^{s+1}$, define the character $\chi_{\underline{m}}$ of $\mathbf{U}^{\mathbf{N}}$ by $\chi_{\underline{m}}(x) = x_0^{m_0} x_1^{m_1} \cdots x_s^{m_s}$. Let $\xi \in \mathbf{U}^{\mathbf{N}}$; then $\mathcal{F}(\xi)$ is uniquely ergodic if and only if, for any $s \in \mathbf{N}$ and for any $\underline{m} \in \mathbf{Z}^{s+1}$,*

$$N \mapsto \frac{1}{N} \sum_{n<N} \chi_{\underline{m}}(\sigma^{n+j}\xi)$$

converges uniformly in j when N tends to infinity.

1.4. Ostrowski α-sequences

Definition 1. *A sequence $f : \mathbf{N} \to \mathbf{R}$ is said to be Ostrowski α-additive if $f(0) = 0$ and*

$$f(m) = \sum_{j\geq 0} f(e_j(m)q_j) , \quad m \in \mathbf{N}.$$

Let $\mathbf{U}$ denote the unit circle in $\mathbf{C}$. An unimodular Ostrowski α-multiplicative sequence is a map $\zeta : \mathbf{N} \to \mathbf{U}$, such that $\zeta(0) = 1$ and

$$\zeta(m) = \prod_{j\geq 0} \zeta(e_j(m)q_j) , \quad m \in \mathbf{N}. \tag{4}$$

A typical example of Ostrowski α-additive sequence is given by the *sum-of-digits* s_α defined by $s_\alpha(m) = \sum_{j\geq 0} e_j(m)$. It is known [10, 11, 12, 14, 15] that such sequences, with additional properties, are well-uniformly distributed modulo 1. In particular, this is the case for the sequences $n \mapsto \rho s_\alpha(n)$where ρ is an irrational number (see [10, 11, 12]). For usual definitions and properties of uniform distribution, see [34].

1.5. Survey

The classical notions of q-multiplicative or q-additive sequences f have been intensively explored and a vast amount of literature is devoted to the study of their statistical and harmonical properties, which have strongly motivated this paper.

Let us first cite the seminal works of Bush [7] and Gelfond [26], then papers on special q-multiplicative sequences [6, 42, 43, 16] and their generalizations, largely studied by J. Coquet in his thesis [9] and in many subsequent other works. Of particular interest for us are [8, 30, 45], which are mainly related to spectral properties. The ergodic structures of sequences derived from the sum-of-digits function $n \mapsto s_q(n)$ is exploited by Kakutani in [29], while Kamae introduces in [31] cyclic extensions of the q-adic Kakutani machine in connection with the sum-of-digits to base q. Basic tools to investigate dynamical properties of sequences are furnished by [13] and applications to q-multiplicative sequences can be found in [41, 40, 37, 38]. This latter approach inspires our investigations on the elucidation of dynamical structures behind α-multiplicative sequences. Notice that only few methods are known concerning the understanding of dynamical systems arizing from subsequences, except for those obtained by induction. A typical example is the sequences $n \to s_q(n^2)$ evocated by Gelfond [26], which seems to be out of the scope of ergodic tools. Nevertheless, the distribution approach, even in the case of restriction to the set of prime numbers, has been already explored by I. Kátai [32, 33].

In Section 2, we recall the basic notions of α-odometer which is the extension of the map $m \mapsto m+1$ to a suitable compactification of $\mathbf{N}$ derived from the α-expansion. This is a special but explicit case of the construction given in [25]. Section 3 identifies the α-odometer with the rotation $R_\alpha : t \mapsto t+\alpha$ on the one-dimensional torus $\mathbf{T} = \mathbf{R}/\mathbf{Z}$ by finitary isomorphism. We shall freely identify $\mathbf{T}$ and its Haar measure $h_{\mathbf{T}}$ with the interval $[0,1)$ (but also with $[-\alpha, 1-\alpha)$ for natural reasons issuing from our study) and its normalized Lebesgue measure λ, the rotation R_α being now the addition of α (mod 1). This yields unique ergodicity of the α-odometer and an explicit description of its invariant measure. We then retrieve interesting results of [50]. Section 4 is devoted to inductions of the α-odometer and study in details the return times to cylinders. These results will be used in Section 7. For reading convenience, Sections 2 and 3 are self contained and do assume a previous reading neither of [25] nor of [50], which are more general.

Sections 5 and 6 are interested in flows coming from Ostrowski α-multiplicative sequences. For such a sequence ζ, Section 5 gives the dynamical structure of the flow $\mathcal{F}(\Delta\zeta)$ associated to the difference sequence $\Delta\zeta$. It is constant or almost topologically conjugate to the odometer. Section 6 is then devoted to the study of the shift $\mathcal{F}(\zeta)$ generated by ζ itself when $\Delta\zeta$ is not constant. It is proved that $\mathcal{F}(\zeta)$ is minimal and its possible structures are studied in details with respect to the notion of almost topological isomorphism. For this purpose, we introduce the so-called topological essential values; they form a subgroup of $\mathbf{U}$ which turns out to play a fundamental role. Section 7 is devoted to metrical classification of $\mathcal{F}(\zeta)$ and unique ergodicity is discussed with criterions and examples. Our main tools are issuing from [2, 13, 40, 41] and [48]. The final section gives various applications and examples; in particular when α has bounded partial quotients with $\zeta(\mathbf{N})$ finite. If α has unbounded partial quotients, pathological examples are exhibited: the flow $\mathcal{F}(\zeta)$ is still minimal but can be viewed, from a metrical point of view, as the union of a finite or uncountable copies of the odometer. We end with the study of flows arising from the sum-of-digits function.

2. The α-odometer

Let α be a fixed irrational real number in the interval $[0, 1/2]$. The aim of this section is to introduce the *Ostrowski α-odometer*, that is, a natural compactification of the set of the α-expansions of the non-negative integers endowed with the adic machine performing the addition of 1. For more details on the odometer for general numeration systems see [25] and its extensions [3] and [4].

2.1. The α-compactification of N

Let E_j be the set of integers b such that $0 \leq b < q_{j+1}/q_j$, *i.e.* $E_j = \{0, \cdots, a_{j+1}\}$ for $j \geq 1$ and $E_0 = \{0, \cdots, a_1 - 1\}$ endowed with the discrete topology, and let E be the infinite compact product space $\prod_{j=0}^{\infty} E_j$. Note that in all what follows, $\mathbf{N}$ denotes the set of non-negative integers, *i.e.* $0 \in \mathbf{N}$. The set

$$\begin{aligned}\mathcal{K}_\alpha &= \{(x_n)_{n\geq 0} \in E\ ;\ \forall j \geq 0 : x_0 q_0 + \cdots + x_j q_j < q_{j+1}\} = \\ &= \{(x_n)_{n\geq 0} \in E\ x_0 \leq a_1 - 1 \quad \text{and} \\ &\qquad\qquad \forall j \geq 1 :\ x_j \leq a_{j+1}\ \&\ [x_j = a_{j+1} \Rightarrow x_{j-1} = 0]\},\end{aligned}$$

is a compact subspace of E. Elements of E are viewed as sequences or infinite strings of symbols. A finite string $b = b_0 b_1 \ldots b_n$ will be said *Ostrowski α-admissible* (or simply *admissible*) if the infinite string $b0^\omega$ obtained by putting $b_j = 0$ for all $j > n$ belongs to $\mathcal{K}_\alpha$. The string $0^{(n)}$ (for $n \in \mathbf{N}$) is defined inductively by $0^{(0)} = \wedge$ (the empty string) and $0^{(n)} = 00^{(n-1)}$. The natural injection $n \mapsto \check{n}$ of $\mathbf{N}$ into $\mathcal{K}_\alpha$ given by $\check{m} := (e_j(m))_{j\geq 0}$ will be used to identify $\mathbf{N}$ with its image $\check{\mathbf{N}}$ which is a dense subset of $\mathcal{K}_\alpha$. If the finite string

$b = b_0 b_1 \dots b_n$ is admissible, then there is a unique integer N, namely $N(b) = \sum_{j=0}^{n} b_j q_j$, such that $\check{N} = b0^\omega$. For any finite string $b = b_0 b_1 \dots b_n$, $|b| = n+1$ will denote its length. By convention, $|\wedge| = 0$.

The addition of 1 in $\mathbf{N}$ gives rise to the transformation $\tau : \check{\mathbf{N}} \to \check{\mathbf{N}}$, called *Ostrowski α-adding machine*, and given by

$$\tau\big(e_0(N)e_1(N)e_2(N)\dots\big) := e_0(N+1)e_1(N+1)e_2(N+1)\dots$$

In the sequel, we currently identify n with $\check{n}$. In order to extend the adding machine to $\mathcal{K}_\alpha$, one needs to understand how the adding machine works with respect to the carries. Hence it is useful to study, for $x = (x_n)_{n\ge 0} \in \mathcal{K}_\alpha$, the set

$$D(x) := \{j \in \mathbf{N}\ ;\ 1 + x_0 q_0 + \dots + x_j q_j = q_{j+1}\}\,.$$

For this purpose, we need some special strings: for $m \ge 1$, let Q_m be the string of length m corresponding to the α-expansion of $q_m - 1$ and let $Q_0 = \wedge$.

Lemma 2. *Let $x \in \mathcal{K}_\alpha$. Assume that $D(x)$ is finite (but not empty) and put $m := \max D(x)$. Then $D(x) = \{0, 2, 4, \dots, 2n\}$ or $D(x) = \{1, 3, 5, \dots, 2n+1\}$ according as m is even or odd. Moreover, $x = Q_{m+1}x_{m+1}x_{m+2}\dots$, with*

$$Q_{m+1}0^\omega = (q_{m+1}-1)\check{} = \begin{cases} (a_1-1)0^\omega & \text{if } m = 0,\\[2ex] (a_1-1)(0a_3)\dots(0a_{m+1})0^\omega & \text{if } m = 2\ell,\ \ell \ge 1,\\[2ex] (0a_2)\dots(0a_{m+1})0^\infty & \text{if } m = 2\ell+1,\ \ell \ge 0,\end{cases}$$

and $x_{m+1}x_{m+2} \ne 0a_{m+3}$. Furthermore, $x_{m+1} < a_{m+2}$.

Proof. Assume that $D(x) = \{d_0, \dots, d_k\}$, with $d_0 < \dots < d_k$ and put $m = d_k$. By definition of $D(x)$, we have

$$x = Q_{m+1}x_{m+1}x_{m+2}\dots,$$

where no $Q_{m'}$ with $m' > m+1$ is a prefix of x. It only remains to verify the description of Q_{m+1} claimed above, which is proved by induction using the recurrence relation $q_{m+1} - 1 = a_{m+1}q_m + (q_{m-1} - 1)$, from which we derive

$$\begin{aligned} q_{2\ell} - 1 &= a_{2\ell}q_{2\ell-1} + a_{2\ell-2}q_{2\ell-3} + \dots + a_2 q_1,\\ q_{2\ell+1} - 1 &= a_{2\ell+1}q_{2\ell} + a_{2\ell-1}q_{2\ell-2} + \dots + (a_1-1)q_0. \end{aligned} \tag{5}$$

Lemma 3. *The set $D(x)$ is infinite if and only if*

$$x = 0a_2 0a_4 0a_6 \cdots \quad (D(x) = \{1, 3, 5, \cdots\}), \quad \text{or}$$
$$x = (a_1 - 1)0a_3 0a_5 \cdots \quad (D(x) = \{0, 2, 4, \cdots\}).$$

Proof. First notice that the infinite strings $0a_2 0a_4 0a_6 \cdots$ and $(a_1 - {} - 1)0a_3 0a_5 \cdots$ are in $\mathcal{K}_\alpha$. If $x = 0a_2 0a_4 0a_6 \cdots$, then $D(x) = \{1, 3, 5, \cdots\}$, and if $x = (a_1 - 1)0a_3 0a_5 \cdots$, then $D(x) = \{0, 2, 4, \cdots\}$. Conversely, let $d \in D(x)$; then by Lemma 2, the word $x_0 \ldots x_d$ is equal to $(0a_2) \ldots (0a_{d+1})$ if d is odd, and $(a_1 - 1)(0a_3) \ldots (0a_{d+1})$ if d is even. The two possible expressions for $D(x)$ follow.

2.2. Definition and first ergodic properties of the α-odometer

Definition 2. *For any $x = x_0 x_1 x_2 \ldots$ in $\mathcal{K}_\alpha$, let $m(x) := \sup D(x)$ if $D(x)$ is not empty and $m(x) = -1$ otherwise and let $\tau : \mathcal{K}_\alpha \to \mathcal{K}_\alpha$ be the map defined by*

$$\tau(x) := \begin{cases} 0^{(m+1)}(x_{m+1} + 1)x_{m+2} \ldots & \textit{if } m = m(x) \ \ \textit{is finite}, \\ 0^\omega & \textit{if } m(x) = +\infty\,. \end{cases}$$

The dynamical system $(\mathcal{K}_\alpha, \tau)$ is called the Ostrowski α-odometer.

The restriction of τ on $\check{\mathbf{N}}$ is just the adding machine and by Lemma 3 we have

$$\tau^{-1}(0^\omega) = \{0a_2 0a_4 0a_6 \cdots\ ,\ (a_1 - 1)0a_3 0a_5 \cdots\}\,.$$

The next theorem follows from the general theory of odometer given in [25]. Here we give a more direct and easier proof.

Theorem 1. *The map $\tau : \mathcal{K}_\alpha \to \mathcal{K}_\alpha$ is continuous and surjective, and the corresponding flow $(\mathcal{K}_\alpha, \tau)$ is minimal.*

Proof. With the notations of Definition 2 the map $x \mapsto m(x)$ from $\mathcal{K}_\alpha$ onto the compact space $\mathbf{N} \cup \{-1, +\infty\}$ is continuous by Lemma 2. It follows from Definition 2 and the product topology that τ is continuous too.

Since the orbit of 0^ω under τ, that is $\check{\mathbf{N}}$, is dense in $\mathcal{K}_\alpha$ and τ is continuous, the minimality of τ will be proved if it is shown that 0^ω lies in the orbit closure of any point $x \in \mathcal{K}_\alpha$. But this is a straightforward consequence of the fact that the sequence $\ell \mapsto m(\tau^\ell(x))$ is unbounded.

Notice that the surjectivity follows from the density of $\tau(\check{\mathbf{N}})$, the compacity of $\mathcal{K}_\alpha$ and the continuity of τ, but it is also a consequence of the minimality.

The preimages of 0^ω play a somehow disturbing role in the understanding of τ, what will be confirmed in Section 3. For this reason, we introduce a further subset of $\mathcal{K}_\alpha$, which we shall often encounter in the sequel:

$$\mathcal{K}_\alpha^\infty = \mathcal{K}_\alpha \setminus \{\tau^{-n}(0^\omega)\,;\, n \geq 1\}.$$

Notice that $\mathcal{K}_\alpha^\infty$ is stable by τ and that the restriction of τ to $\mathcal{K}_\alpha^\infty$ is injective, but not onto $\mathcal{K}_\alpha^\infty$. According to the definitions of Subsection 1.2 there is a further natural subset of $\mathcal{K}_\alpha$ to introduce, namely

$$\mathcal{K}_\alpha^\bullet = \mathcal{K}_\alpha^\infty \setminus \check{\mathbf{N}}.$$

Readily $\mathcal{K}_\alpha^\bullet$ is a dense G_δ subset of $\mathcal{K}_\alpha$. Thus $(\mathcal{K}, \mathcal{K}_\alpha^\bullet, \tau)$ is an almost topological dynamical system. For an occurrence of $\mathcal{K}_\alpha^\infty$ and $\mathcal{K}_\alpha^\bullet$ in a more general setting we refer to [4].

3. The Ostrowski α-expansion

We have expanded in the previous section natural integers with respect to a numeration scale given by the denominators of the continued fraction expansion of the irrational number α. It is also possible to expand real numbers with respect to the basis $(q_n\alpha - p_n)_{n\geq 0}$. The aim of this section is to study the properties of this α-expansion of real numbers, called *Ostrowski α-expansion* – that has been studied and sometimes rediscovered by several authors [18, 24, 28, 35, 36, 52, 50, 51] in order to get distribution results for the sequence $(n\alpha)_{n\geq 0}$ – and then to prove the isomorphism with the rotation of angle α on the one-dimensional torus.

3.1. Definition, α-order

Definition 3. *Let $I(\alpha)$ be the interval $[-\alpha, 1-\alpha]$. We define the map $\varphi : \mathcal{K}_\alpha \to I(\alpha)$ by*

$$\varphi(x) := \sum_{n=0}^{\infty} x_n(q_n\alpha - p_n). \tag{6}$$

The inequalities

$$x_n|q_n\alpha - p_n| < a_{n+1}/q_{n+1} < 1/q_n$$

show that the series in (6) converges normally. Thus φ is well defined and continuous. We will justify in the proof of Proposition 1 that $\varphi(x) \in I(\alpha)$.

In order to get an understanding of the map φ, we introduce a notion of α-order $\prec$ related to the relation $q_n\alpha - p_n = (-1)^n|q_n\alpha - p_n|$. By definition, $\prec$ is the binary relation on $\mathcal{K}_\alpha$ given by $x \prec y$ if and only if either $x = y$ or there exists an index k (possibly 0) such that $x_j = y_j$ for all $j < k$ and $(-1)^k x_k < (-1)^k y_k$. Clearly $\prec$ is a total order on $\mathcal{K}_\alpha$. For $n \geq 0$, let

$$A(n) := (0a_{n+2})(0a_{n+4})\dots ;$$

the relations (5) yield the following helpful equation for $n \geq 1$ (the last equality remaining true for $n = 0$):

$$\varphi\big(0^{(n-1)}(1)(a_{n+1} - 1)A(n+1)\big) = \varphi\big(0^{(n)}A(n)\big) = -(q_n\alpha - p_n). \tag{7}$$

Let $b = b_0 \dots b_n$ be α-admissible. We denote by $[b]$ the cylinder set

$$[b] := \{x \in \mathcal{K}_\alpha \ ; \ x_0 \dots x_n = b\}$$

and define

$$b^- := \begin{cases} b(a_{n+2} - 1)A(n+2) & \text{if } n \text{ is even and } b_n \neq 0, \\ ba_{n+2}A(n+2) & \text{if } n \text{ is even and } b_n = 0, \\ bA(n+1) & \text{if } n \text{ is odd;} \end{cases}$$

$$b^+ := \begin{cases} bA(n+1) & \text{if } n \text{ is even,} \\ b(a_{n+2} - 1)A(n+2) & \text{if } n \text{ is odd and } b_n \neq 0, \\ ba_{n+2}A(n+2) & \text{if } n \text{ is odd and } b_n = 0. \end{cases}$$

Since we will need these expressions several times, we compute from (7) the values of φ at $b^\pm$.

$$\begin{aligned} \varphi\big(bA(n+1)\big) =& \varphi(b0^\omega) - (q_{n+1}\alpha - p_{n+1}), \\ \varphi(b(a_{n+2} - 1)A(n+2)) =& \varphi(b_0 \dots b_{n-1}(b_n - 1)0^\omega) - (q_{n+1}\alpha - p_{n+1}) \\ & \text{if } \ b_n \neq 0, \\ \varphi\big(ba_{n+2}A(n+2)\big) =& \varphi(b0^\omega) - (q_n\alpha - p_n) \ \text{ if } \ b_n = 0. \end{aligned} \tag{8}$$

Let further $\mathcal{S}_n$ be the set of strings $b := b_0 \dots b_n \in E_0 \times \dots \times E_n$ which are α-admissible. Then $\mathcal{S}_n$ has cardinal q_{n+1} and the family $\mathcal{F}_n := \{[b] \, ; \, b \in \mathcal{S}_n\}$ is a covering of $\mathcal{K}_\alpha$ by compact subsets, called *α-partition* of rank n. We recall the usual notation $\sigma(\pi_0, \dots, \pi_n)$ for the boolean algebra generated by $\mathcal{F}_n$. Note that $\mathcal{F}_{n+1}$ is a refinement of $\mathcal{F}_n$.

Proposition 1. *The map φ is increasing with respect to $\prec$, i.e. for all x and y in $\mathcal{K}_\alpha$,*

$$x \prec y \Rightarrow \varphi(x) \leq \varphi(y).$$

For any admissible distinct words b and b' we have $\varphi([b]) = [\varphi(b^-), \varphi(b^+)]$ and $\varphi([b]) \cap \varphi([b'])$ is either empty or a singleton.

Proof. We first notice that by definition of b^- and b^+ we have $b^- \prec x \prec \prec b^+$ for any $x \in [b]$. Recall now that $q_n\alpha - p_n = (-1)^n |q_n\alpha - p_n|$. Then (3) yields

$$\varphi([b]) \subset [\varphi(b^-), \varphi(b^+)]\,.$$

In particular, $\varphi(b^-) < \varphi(b) < \varphi(b^+)$ and, for $b = \wedge$ and by (8),

$$-\alpha = \varphi(A(0)) \leq \varphi(x) \leq \varphi((a_1 - 1)A(1)) = 1 - \alpha \tag{9}$$

for all $x \in \mathcal{K}_\alpha$: the map φ takes its values in $I(\alpha)$.

Let $x \prec y$ two distinct elements of $\mathcal{K}_\alpha$. There exists a unique word b such that $x_0x_1 \ldots x_n = y_0y_1 \ldots y_n = b$ and $(-1)^{n+1}x_{n+1} < (-1)^{n+1}y_{n+1}$. Then, we have $x \prec (bx_{n+1})^+$ and $(by_{n+1})^- \prec y$. Assume for instance that n is even: $y_{n+1} < x_{n+1}$ and we have by (8), noticing that $x_{n+1} \neq 0$,

$$\begin{aligned}\varphi(y) - \varphi(x) &\geq \varphi\big((by_{n+1})^-\big) - \varphi\big((bx_{n+1})^+\big) \geq \\ &\geq \varphi(by_{n+1}A(n+2)) - \varphi(bx_{n+1}(a_{n+3} - 1)A(n+3)) \geq \\ &\geq (y_{n+1} - x_{n+1} + 1)(q_{n+1}\alpha - p_{n+1}) \geq 0.\end{aligned}$$

The case n odd is similar. It follows that if b and b' are distinct strings of $\mathcal{S}_n$, the intersection $\varphi([b]) \cap \varphi([b'])$ is either empty or reduced to one element. Moreover, any positive integer m fulfills $\varphi(\tilde{m}) = m\alpha \pmod 1$; thus $\varphi(\mathcal{K}_\alpha)$ is dense in $I(\alpha)$. Since it is compact, $\varphi(\mathcal{K}_\alpha) = I(\alpha)$ and $\varphi([b]) = [\varphi(b^-), \varphi(b^+)]$.

3.2. The canonical Ostrowski expansion

Proposition 1 ensures that any real number $x \in I(\alpha)$ admits a representation given by the series (6), called Ostrowski α-expansion and that this representation is unique, except if $x = \varphi(b^\pm)$ for some b in $\mathcal{S}_n$ for some n. For fixed n, a direct application of (8) and $\varphi(b) = (b_0 + b_1q_1 + \cdots + b_nq_n)\,\alpha \pmod 1$ shows that

$$\begin{aligned}\{\varphi(b^-) \,(\mathrm{mod}\ 1)\,;\, b\ \in \mathcal{S}_n\} &= \{\varphi(b^+)\,(\mathrm{mod}\ 1)\,;\, b\ \in \mathcal{S}_n\} = \\ &= \{-m\alpha\,(\mathrm{mod}\ 1)\,;\, 1 \leq m \leq q_{n+1}\}.\end{aligned} \tag{10}$$

That remark and Proposition 1 yield the following

Proposition 2. *Let $t_1 = -\alpha$ and for any integer $n \geq 2$, let t_n be the point in $I(\alpha)$ congruent to $-n\alpha \pmod 1$. Let $t_1 = \xi_0 < \xi_1 < \ldots < \xi_{q_{n+1}} = 1-\alpha$ be the increasing indexation of the set of points $\{1-\alpha\} \cup \{t_n\,;\, n = 1, 2, \ldots, q_{n+1}\}$ and let $\mathcal{I}_n$ be the covering of $I(\alpha)$ given by the set of intervals $[\xi_k, \xi_{k+1}]$, $0 \leq$ $\leq k \leq q_{n+1} - 1$.*

(i) The map φ establishes a 1-1-correspondence between $\mathcal{F}_n$ and $\mathcal{I}_n$.

(ii) The real numbers of $I(\alpha)$ having more than one preimage by φ are exactly the points t_m, $m \geq 2$. Explicitly, for $q_n \leq m < q_{n+1}$ and $b = b_0 \ldots b_n =$ $= (q_{n+1} - m)\breve{}$, there are two preimages, namely

$$(11)\qquad \begin{aligned} -m\alpha &= \varphi\big(bA(n+1)\big) = \\ &= \varphi\big(b_0 \ldots b_{n-1}(b_n+1)(a_{n+2}-1)A(n+2)\big). \end{aligned}$$

After Lesca (see [35]), we conventionally define the *canonical* expansion of $-m\alpha \mod 1$ as the expansion (11) which terminates by an infinite string of the form $A(2\ell) = (0a_{2\ell+2})(0a_{2\ell+4})\ldots$ In other words, for every n and every $b \in \mathcal{S}_n$, b^- is always the canonical α-expansion of $\varphi(b^-)$. Notice that for $m = -1$ the expressions (11) give $-\alpha$ and $1-\alpha$, as shown by (9). We thus have proved the following.

Proposition 3. (J. Lesca) *Every real number ξ, with $-\alpha \leq \xi \leq 1-\alpha$, has a unique canonical Ostrowski α-expansion, i.e.*

$$(12)\qquad \xi = \sum_{n=0}^{+\infty} x_n(q_n\alpha - p_n),$$

$$\textit{where}\quad 0 \leq x_0 \leq a_1 - 1,\ 0 \leq x_k \leq a_{k+1},\ \textit{for}\quad k \geq 2,$$

$$x_k = 0\quad \textit{if}\quad x_{k+1} = a_{k+2},$$

$x_{2k} \neq a_{2k+1}$, for infinitely many k (this last condition being removed for $\xi =$ $= 1-\alpha$).

Remark 1. The expansions (12) have alternate signs. In [27] S.Ito shows that every real number ξ, with $-\alpha \leq \xi < 1-\alpha$, has a unique positive expansion

$$(13)\qquad \xi = \alpha + \sum_{n=1}^{+\infty} y_n|q_n\alpha - p_n|,$$

where $0 \leq y_k \leq a_k$ for each $k \geq 1$, $y_{k+1} = 0$ if $y_k = a_k$ and $y_k \neq a_k$ for infinitely many k. Both expansions are extensively studied in [50] as coming from a *Markov compactum* in Vershik's terminology.

3.3. Metric properties of the dynamical system $\mathcal{K}_\alpha$

Let $\psi : I(\alpha) \to \mathcal{K}_\alpha$ be the map which associate to a real number ξ in $I(\alpha)$ its canonical expansion, i.e. $\psi(\xi) = x$ if and only if the series $\varphi(x)$ is the canonical expansion of ξ.

Proposition 4. *The map $\psi : I(\alpha) \to \mathcal{K}_\alpha$ is a right inverse of φ and continuous at each point of $I(\alpha)^* := I(\alpha) \setminus \{t_m\,;\, m \geq 2\}$. Moreover, ψ is right-continuous everywhere.*

Proof. By construction, $\varphi \circ \psi = Id$. Let $\xi \in I(\alpha)$ and $x = x_0 \dots x_n \dots = \psi(\xi)$. For $n \geq 0$, let $U_n = [x_0 \dots x_n]$ and $V_n = \varphi(U_n)$. Then $\psi(V_n) \subset U_n$ for each n. If $\xi \in I(\alpha)^*$, $(V_n)_n$ is a basis of neighborhoods of ξ, hence the continuity of ψ at ξ. The same argument shows that ψ is right-continuous at any point.

Remark 2. The alternative choice of (11) for the canonical expansion would have yielded to left-continuity of ψ.

Let R_α denote the translation of angle α on $[-\alpha, 1-\alpha)$, *i.e.* $R_\alpha(t) := t + \alpha \pmod 1$. In the next theorem, we retrieve results from the folklore of the nineties, and collected in [50].

Theorem 2. *The flow $(\mathcal{K}_\alpha, \tau)$ is strictly ergodic. If μ_α is the unique $\tau-$invariant probability then $\Phi : x \mapsto \varphi(x) \pmod 1$ (see (6)) realizes a finitary isomorphism between $(\mathcal{K}_\alpha, \tau, \mu_\alpha)$ and the translation $([-\alpha, 1-\alpha), R_\alpha, \lambda)$. Moreover, if $||\cdot||$ denotes the distance to the nearest integer and an α-admissible string $b = b_0 \dots b_n$, then*

$$\mu_\alpha([b]) = \varphi(b^+) - \varphi(b^-) = \begin{cases} ||q_n\alpha|| + ||q_{n+1}\alpha|| & \text{if } b_n = 0, \\ ||q_n\alpha|| & \text{if } b_n \neq 0, \end{cases} \tag{14}$$

for any admissible word $b = b_0 \dots b_n$. Furthermore, if π_n is the random variable defined on the probability space $(\mathcal{K}_\alpha, \mu_\alpha)$ by $\pi_n((x_0, x_1, \dots)) := x_n$, then the sequence of random variables $(\pi_n)_{n\geq 0}$ forms an inhomogeneous Markov chain with transition probabilities (with $n \geq 1$)

$$\mu_\alpha(\pi_n = a \mid \pi_{n-1} = a') = \tag{15}$$

$$= \begin{cases} \dfrac{||q_n\alpha|| + ||q_{n+1}\alpha||}{||q_{n-1}\alpha|| + ||q_n\alpha||} & \text{if } a = a' = 0, \\ \dfrac{||q_n\alpha|| + ||q_{n+1}\alpha||}{||q_{n-1}\alpha||} & \text{if } a = 0 \text{ and } a' \geq 1, \\ \dfrac{||q_n\alpha||}{||q_{n-1}\alpha|| + ||q_n\alpha||} & \text{if } a \geq 1 \text{ and } a' = 0, \\ \dfrac{||q_n\alpha||}{||q_{n-1}\alpha||} & \text{if } 1 \leq a \leq a_{n+1} - 1 \text{ and } a' \geq 1, \\ 0 & \text{otherwise,} \end{cases}$$

and distribution given, for $n \geq 0$, by

$$\mu_\alpha(\pi_n = a) = \begin{cases} q_n(||q_n\alpha|| + ||q_{n+1}\alpha||) & \text{if } a = 0, \\ q_n||q_n\alpha|| & \text{if } 1 \leq a \leq a_{n+1} - 1, \\ q_{n-1}||q_n\alpha|| & \text{if } a = a_{n+1}. \end{cases} \tag{16}$$

Proof. By construction, Φ is continuous and verifies $\Phi(\tau(\check{n})) = \Phi(\check{n}) + \alpha \pmod 1$ for all positive integers. Hence, by continuity we obtain $\Phi \circ \tau = R_\alpha \circ \Phi$. From Proposition 1 and 2, the inverse of the restriction $\Phi_{|\mathcal{K}_\alpha^\infty}$ is given on $I(\alpha)^*$ by ψ.

Let μ be a Borel measure of probability on $\mathcal{K}_\alpha$ such that $\mu \circ \tau^{-1} = \mu$. First notice that since by construction τ does not have any cycle, μ is non atomic. Let $\nu := \mu \circ \Phi^{-1}$, then $\nu \circ R_\alpha^{-1} = \mu \circ (\Phi \circ \tau)^{-1} = (\mu \circ \tau^{-1}) \circ \Phi^{-1} = \mu \circ \Phi^{-1} = \nu$. Hence, by unique ergodicity of R_α, the measure ν is the Lebesgue measure λ. By Proposition 2 and for all $b \in \mathcal{S}_n$, $\varphi^{-1}([\varphi(b^-), \varphi(b^+)])$ is constituted by $[b]$ and one or two more elements, hence $\mu([b]) = \varphi(b^+) - \varphi(b^-)$. Thus $\mu_\alpha := \mu$ is well and univocal defined. Formulas (14) are then immediate consequences of (8) and of the classical equalities $||q_n\alpha|| = |q_n\alpha - p_n|$. The system of equations (3) ensures that for given (a, a') and finite words W and W' of length n and n' respectively, the word $Wa'aW'$ is admissible if and only if Wa', $0^{(n)}a'a$ and $0^{(n+1)}aW'$ are themselves admissible. Let $b_0b_1 \dots b_{n-2}a'a$ be an admissible word. Then, for any $N > n$,

$$\frac{\#\{m < q_N \; ; \; \check{m} \in [b_0b_1 \dots b_{n-2}a'a]\}}{\#\{m < q_N \; ; \; \check{m} \in [b_0b_1 \dots b_{n-2}a']\}} = \frac{\#\{m < q_N \; ; \; e_{n-1}(m)e_n(m) = a'a\}}{\#\{m < q_N \; ; \; e_{n-1}(m) = a'\}},$$

hence $\mu_\alpha(\pi_n = a \mid \pi_{n-1} = a', \pi_{n-2} = b_{n-2}, \dots, \pi_0 = b_0) = \mu_\alpha(\pi_n = a \mid \pi_{n-1} = a')$ by unique ergodicity of $(\mathcal{K}_\alpha, \tau)$ and passing N to infinity. Taking arbitrary $b_0b_1 \dots b_{n-2}$, one deduces formulas (15) from (14) without any computation. Formulas (16) follow from the summation $\mu(\pi_n = a) = \sum_b \mu([ba])$, where the summation is taken over the strings $b = b_0b_1 \dots b_{n-1}$ such that ba is admissible.

We have established the uniqueness of $\mu_\alpha := \mu$ and given its value on cylinder sets. It remains to show that we have an isomorphism in the sense of 1.2. That is done by considering the almost topological dynamical systems $(\mathcal{K}, \mathcal{K}_\alpha^\bullet, \tau)$ and $([-\alpha, 1-\alpha), [-\alpha, 1-\alpha) \setminus (\mathbf{Z}\alpha + \mathbf{Z}), R_\alpha)$: since the exceptional sets are countable and both measure non-atomic, $\mathcal{K}_\alpha^\bullet$ and $[-\alpha, 1-\alpha) \setminus (\mathbf{Z}\alpha + \mathbf{Z})$ are dense G_δ subsets with full measure of $\mathcal{K}_\alpha$ and $[-\alpha, 1-\alpha)$ respectively.

Remark 3. Summing the relations (16) over a for fixed n gives the classical formula

$$q_{n+1}||q_n\alpha|| + q_n||q_{n+1}\alpha|| = 1$$

which ensures

$$a_{n+1}||q_n\alpha|| + ||q_{n+1}\alpha|| = ||q_{n-1}\alpha||\,.$$

We will use then.

4. Induction of the α-odometer

The second family of dynamical system we look at is that obtained by induction of the α-odometer on cylinders. It turns out that the induction is still a rotation on $\mathbf{T}$, the angle of which can be computed explicitly. For this purpose, we first study the sequence of return times to a cylinder. This sequence can be entirely described in terms of continued fractions of real numbers strongly related with α. Notice that since the α-odometer is a rotation and since the cylinders correspond by Φ to intervals of length belonging to $\mathbf{Z}\alpha$ (mod 1), the situation is related with bounded remainder sets, for the induction on which we refer to [22].

4.1. Return times

Let n be a positive integer and let $C := [b_0 b_1 \dots b_{n-1}] \in \mathcal{F}_n$. Recall that for $x \in C$ the first return time to C is $r(x) = \min\{s \geq 1; \tau^s(x) \in C\}$. The sequence of (consecutive) return times is recursively defined as the increasing sequence of positive integers $\big(r_k(x)\big)_{k\geq 1}$, where

$$r_1(x) = r(x) \quad \text{and} \quad r_k(x) = r\big(\tau^{r_{k-1}(x)}x\big).$$

We denote by $M_n = \big[0^{(n)}\big]$ the cylinder corresponding to $b_0 = b_1 = \ldots = = b_{n-1} = 0$ and we first deal with $C = M_n$ – the integer n remaining fixed until further notice. Then $r(x)$ is equal to q_{n-1} (resp. q_n) whenever $x_n = a_{n+1}$ (resp. $x_n \neq a_{n+1}$). Thus we can look at the sequence $\big(r_k(x)\big)_{k\geq 1}$ as an infinite word on a two letters alphabet $\big\{q_{n-1}, q_n\big\}$, $W(x)$ say.

First, we claim that the word $W := W(0^\omega)$ is given by the limit of its sequence of prefixes $(W_k)_{k\geq n}$ obtained by the initial values and the recursive formulas

$$\begin{cases} W_n = q_n, \\ W_{n+1} = (q_n)^{a_{n+1}} q_{n-1}, \\ W_k = (W_{k-1})^{a_k} W_{k-2} \quad \text{for any } k \geq n+2. \end{cases} \tag{17}$$

Formulas (17) follow from the interpretation of W_k (for $k \geq n$) as the word of the return times to M_n up hitting M_k. Notice furthermore that W_k is a prefix of W_{k+1}, hence the existence of $\lim W_k$.

We now look at the frequency of letters in W. Define $|W_k|$ and $|W_k|_0$ to be the length of W_k and the number of occurrences of q_{n-1} in W_k, respectively. Both sequences satisfy the recurrence relations $u_k = a_k u_{k-1} + u_{k-2}$; moreover, $|W_n|_0 = 0$, $|W_{n+1}|_0 = |W_n| = 1$ and $|W_{n+1}| = a_{n+1} + 1$, which show that the quotient $|W_k|_0/|W_k|$ is equal to the finite continued fraction $[0; a_{n+1} + {} + 1, a_{n+2}, a_{n+3}, \ldots, a_k]$. Consequently, $\lim |W_k|_0/|W_k|$ exists and is equal to $\rho_n = [0; a_{n+1} + 1, a_{n+2}, a_{n+3}, \ldots]$. We claim that

$$\lim_N \frac{1}{N} \#\big\{k < N\,;\, r_{k+j}(0^\omega) = q_{n-1}\big\} = \rho_n, \tag{18}$$

the limit being uniform in j. In fact, W can be written as a concatenation of copies of W_{k-1} and W_k for any $k \geq n$ and $|W_k|$ tends to infinity. Hence the result by a standard argument.

In a second step, we are interested in the return times $W(x)$ still to M_n, but from an arbitrary point x. Let $x = 0^{(n)} x_n x_{n+1} \ldots \in M_n \cap \mathcal{K}_\alpha^\infty$. Split x as follows:

$$\begin{aligned} x = 0^{(n)} B_0 B_1 \ldots B_{m_1-1} x_{n+2m_1} B_{m_1} \ldots B_{m_2-1} x_{n+2m_2+1} \cdots \\ x_{n+2m_{\ell-1}+\ell-2} B_{m_{\ell-1}} \cdots B_{m_\ell-1} x_{n+2m_\ell+\ell-1} \cdots \end{aligned} \tag{19}$$

where $(m_\ell)_\ell$ is a non-decreasing sequence of natural integers and the B_*'s are blocks of length 2 of the form $0a_\star$, that is, more precisely, $B_j = 0a_{n+2j+\ell+1}$ for $m_{\ell-1} \leq j < m_\ell$. By convention, $m_0 = 0$ and the string $B_{m_{\ell-1}} \cdots B_{m_\ell-1}$ is empty if $m_{\ell-1} = m_\ell$. Then we have

$$W(x) = \tag{20}$$

$$\begin{aligned} W_n W_{n+2m_1}^{a_{n+2m_1+1} - x_{n+2m_1} - 1} W_{n+2m_1-1} W_{n+2m_2+1}^{a_{n+2m_2+2} - x_{n+2m_2+1} - 1} \\ W_{n+2m_2} W_{n+2m_3+2}^{a_{n+2m_3+3} - x_{n+2m_3+2} - 1} W_{n+2m_3+1} \cdots \end{aligned}$$

By convention, W_{n-1} is the word of length one $W_{n-1} = q_{n-1}$; if $x_n = a_{n+1}$, then $m_1 = 0$ and the formula above gives $W(x) = W_n W_n^{-1} W_{n-1} \ldots$, which has to be conventionally understood as $W(x) = W_{n-1} \ldots$ No further negative power can appear. Finally, note that for $m_j = x_j = 0$ for all j, one retrieves $W(0^\omega) = W$. Furthermore, any prefix of $W(x)$ is a subword of W and the limit (18) also holds for x in place of 0^ω (those last facts are also direct consequences of the study of the special case $x = 0^\omega$ and do not need the explicit expressions given above).

We deal now with a general cylinder. Let $x \in \mathcal{K}_\alpha^\infty$, $C = [x_0 \ldots x_{n-1}]$ and $x' = 0^{(n)} x_n x_{n+1} \ldots$. If $x_{n-1} = 0$, then $x_0 \ldots x_{n-1} \xi_n \xi_{n+1} \ldots$ belongs to $\mathcal{K}_\alpha$ if and only if $0^{(n)} \xi_n \xi_{n+1} \ldots$ does. Therefore, $W_C(x) = W(x')$ (where W_C is the infinite word of return times to C). If $x_{n-1} \neq 0$, then $x_0 \ldots x_{n-1} \xi_n \xi_{n+1} \ldots$ belongs to $\mathcal{K}_\alpha$ if and only if $\xi_n \neq a_{n+1}$ and $0^{(n)} \xi_n \xi_{n+1} \ldots \in \mathcal{K}_\alpha$. The returns of x' to M_n which yield a_{n+1} as n-th coordinate are coded by q_n, followed by q_{n-1}. In that case, one returns twice to M_n until hitting C. Since there are never two consecutive q_{n-1} in $W(x')$, $W_C(x)$ is an infinite word on the two letters alphabet $\{q_{n-1} + q_n, q_n\}$ obtained by projection from $W(x')$ first by the substitution $q_n q_{n-1} \mapsto q_{n-1} + q_n$, then by keeping the remaining q_n's.

Using (18), we have the following proposition.

Proposition 5. *Assume $x \in \mathcal{K}_\alpha^\infty$. Let n be a positive integer and let C be the cylinder set $[x_0 x_1 \ldots x_{n-1}]$. Then the sequence of return times of x to C is given by an infinite word $W_C(x) = w_{C,1}(x) w_{C,2}(x) \ldots$ on a two letters alphabet, namely $\{q_{n-1}, q_n\}$ if $x_{n-1} = 0$ and $\{q_n, q_{n-1} + q_n\}$ otherwise. The word $W = W_{[0^{(n)}]}(0^\omega)$ is obtained as the limit of the words W_k, $k \geq n$ defined by (17) and all the other words $W_C(x)$ can be explicitly described in terms of prefixes of W. Furthermore, if $x_{n-1} = 0$ (resp. $x_{n-1} \neq 0$), we have*

$$\lim_N \frac{1}{N} \#\{k < N\,;\, w_{C,k+j}(x) = q_{n-1}\} = [0; a_{n+1}+1, a_{n+2}, \ldots, a_{n+k}, \ldots],$$

$$\lim_N \frac{1}{N} \#\{k < N\,;\, w_{C,k+j}(x) = q_{n-1} + q_n\} = [0; a_{n+1}, a_{n+2}, \ldots, a_{n+k}, \ldots],$$

the limit being uniform in j and x.

The relation

$$[0; a_{n+1}+1, a_{n+2}, \ldots, a_{n+k}, \ldots] + [0; 1, a_{n+1}, a_{n+2}, \ldots, a_{n+k}, \ldots] = 1$$

yields expressions for the frequency of q_n in $W_C(x)$, namely

$$\begin{aligned} &[0; 1, a_{n+1}, a_{n+2}, \ldots, a_{n+k}, \ldots] &&\text{if} \quad x_{n-1} = 0, \\ &[0; 1, a_{n+1} - 1, a_{n+2}, \ldots, a_{n+k}, \ldots] &&\text{if} \quad x_{n-1} \neq 0 \text{ and } a_{n+1} \geq 2, \\ &[0; a_{n+2}+1, a_{n+3}, \ldots, a_{n+k}, \ldots] &&\text{if} \quad x_{n-1} \neq 0 \text{ and } a_{n+1} = 1. \end{aligned}$$

4.2. Induced α-odometer

Let X be a compact subset of $\mathcal{K}_\alpha$ with nonempty interior. By minimality of $(\mathcal{K}_\alpha, \tau)$, the first return time $r(x) = \min\{r \geq 1\,;\, \tau^r(x) \in X\}$ exists for every x ($r(\cdot)$ is even bounded). Then the transformation τ induces on X a transformation T by $T(x) = \tau^{r(x)}(x)$.

For any α-admissible string $b = b_0 b_1 \ldots b_{n-1}$, let us take $X = [b]$ and assume for a while that $b_{n-1} = 0$. Then $r(x) \in \{q_{n-1}, q_n\}$. Write

$$X_1 = [b_0 b_1 \ldots b_{n-2} 0 a_{n+1}] \quad \text{and} \quad X_2 = \bigcup_{e=0}^{a_{n+1}-1} [b_0 b_1 \ldots b_{n-2} 0 e].$$

Let $I = \varphi(X)$, $I_1 = \varphi(X_1)$ and $I_2 = \varphi(X_2)$. By Proposition 1, I, I_1 and I_2 are intervals such that $I = I_1 \cup I_2$, and $I_1 \cap I_2$ is a singleton. By Theorem 2 and the discussion in Subsection 4.1, T is isomorphic to the application $T' : I \to I$ where $T'_{|I_1}$ is the translation of $q_{n-1}\alpha \pmod 1$ and $T'_{|I_2}$ is the translation of $q_n \alpha \pmod 1$. Since τ is surjective, T is surjective, too, hence T' as well. Thus T' is a two intervals exchange transformation, hence a translation of β which, after renormalization to the unit interval, is given by

$$\beta = \begin{cases} \dfrac{\lambda(I_1)}{\lambda(I)} \left(= \dfrac{\mu_\alpha(X_1)}{\mu_\alpha(X)} \right) & \text{if } n \text{ is even;} \\ 1 - \dfrac{\lambda(I_1)}{\lambda(I)} & \text{otherwise.} \end{cases}$$

By ergodicity of $(\mathcal{K}_\alpha, \tau, \mu_\alpha)$, β is also equal to the frequency of q_{n-1} in W_X. Note finally that the case $b_{n-1} \neq 0$ is similar with $X_1 = [b_0 b_1 \ldots b_{n-1}(a_{n+1} - 1)]$ and $X_2 = X \setminus X_1$ and we have proved the following proposition.

Proposition 6. *According to the definitions above, the induced transformation of τ on the α-admissible cylinder $[b_0 b_1 \ldots b_{n-1}]$ is isomorphic to the rotation of angle $[0; a_{n+1} + 1, a_{n+2}, \ldots, a_{n+k}, \ldots]$ if $b_{n-1} = 0$ and angle $[0; a_{n+1}, a_{n+2}, \ldots, a_{n+k}, \ldots]$ if $b_{n-1} \neq 0$.*

5. Differences of α-multiplicative sequences

From now on, we say for short that the sequence $\zeta : \mathbf{N} \to C^*$, is α-multiplicative if it is Ostrowski α-multiplicative and unimodular (see Definition 1). The letter ζ will always denote such a sequence. For given ζ, the aim of the rest of the paper is the study of the topological structure and ergodic properties of the flow $\mathcal{F}(\zeta)$.

5.1. Difference flows

In this subsection we deal with the difference sequence $\Delta\zeta$. The aim is to describe the flow $\mathcal{F}(\Delta\zeta)$. We prove in particular that it is strictly ergodic.

Lemma 4. *Let ζ be an α-multiplicative sequence. Let s be a non-negative integer. Then there exists a countable family $\mathcal{B}_s$ of pairwise disjoint clopen (closed and open) sets of $\mathcal{K}_\alpha$ such that $n \mapsto \big(\Delta\zeta(n), \Delta\zeta(n+1), \dots, \Delta\zeta(n+s)\big)$ is constant on $B \cap \mathbf{N}$ for any $B \in \mathcal{B}_s$ and*

$$\mathbf{N} \subset \bigcup_{B\in\mathcal{B}_s} B = \mathcal{K}_\alpha \setminus \bigcup_{k=1}^{s+1} \tau^{-k}(0^\omega).$$

Proof. We use in the sequel the notation of Lemma 2. Assume first $s = 0$ and let

$$\mathcal{B}_1 = \big\{[Q_r jk]\ ;\ r \geq 0,\ 0 \leq j \leq a^*_{r+1} - 1 \text{ and } k \leq a_{r+2} - 1\big\},$$

where $a^*_k = a_k$ if $k \geq 2$ and $a^*_1 = a_1 - 1$. By Lemma 2, $\mathcal{B}_1$ forms a partition of $\mathcal{K}_\alpha \setminus \tau^{-1}(0^\omega)$. We deduce from $\tau([Q_r jk]) = [0^{(r)}(j+1)k]$ that

$$\Delta\zeta(n) = \frac{\zeta((j+1)q_r)}{\zeta(q_r - 1)\zeta(jq_r)} \tag{21}$$

whenever $\check{n} \in [Q_r jk]$, which proves Lemma 4 for $s = 0$. Notice that in the latter, the integer k does not occur, but we introduce the digit k to point out in a convenient manner the constraint $(j,k) \neq (0, a_{r+2})$.

For arbitrary $[b] \in \mathcal{B}_1$, $s \geq 1$ and $k \leq s$, the sequence $n \mapsto \Delta\zeta(n+k)$ is constant on $\tau^{-k}([b]) \cap \mathbf{N}$; thus $n \mapsto \big(\Delta\zeta(n), \Delta\zeta(n+1), \dots, \Delta\zeta(n+s)\big)$ is constant on the intersection of $\mathbf{N}$ with $\bigcap\limits_{0\leq k\leq s} \tau^{-k}[b_k]$ for any $(b_0, b_1, \dots, b_s)$ with $[b_k] \in \mathcal{B}_1$ for $0 \leq k \leq s$. Since $[b_k]$ is clopen, $\tau^{-k}[b_k]$ is clopen, too (hence a finite union of cylinders). Let

$$\mathcal{B}_s := \left\{ \bigcap_{k=0}^{s} \tau^{-k}[b_k]\ ;\ \forall k = 0 \dots s, b_k \in \mathcal{B}_1 \right\}.$$

Then

$$\bigcup_{B\in\mathcal{B}_s} B = \bigcap_{k=0}^{s} \tau^{-k}\left(\mathcal{K}_\alpha \setminus \tau^{-1}(0^\omega)\right) = \mathcal{K}_\alpha \setminus \bigcup_{k=1}^{s+1} \tau^{-k}(0^\omega)$$

and $\mathcal{B}_s$ answers the question.

Corollary 1. *The sequence $n \mapsto \Delta\zeta(n)$ has a unique continuous extension to $\mathcal{K}_\alpha \setminus \tau^{-1}(0^\omega)$.*

Clearly this extension, denoted in the sequel by $(\Delta\zeta)\tilde{}$, is unique and takes the constant value $\dfrac{\zeta((j+1)q_r)}{\zeta(q_r-1)\zeta(jq_r)}$ on each cylinder $[Q_r jk]$ of the family $\mathcal{B}_1$. We extend $(\Delta\zeta)\tilde{}$ at any point w in $\tau^{-1}(0^\omega)$ by continuity if it is possible, otherwise by setting $(\Delta\zeta)\tilde{}(w) = \ell_w$, where ℓ_w is the limit point of $(\Delta\zeta)\tilde{}$ at w which has the smallest positive argument. In all cases, we get a μ_α-continuous function.

Theorem 3. *Let ζ be an α-multiplicative sequence and let $\Delta\zeta = (\zeta(n+ +1)/\zeta(n))_{n\in\mathbf{N}}$. Then the flow $\mathcal{F}(\Delta\zeta)$ is strictly ergodic.*

Proof. We first prove that $\mathcal{F}(\Delta\zeta)$ is uniquely ergodic. Let $s \in \mathbf{N}$ and $\underline{m} = (m_0, \ldots, m_s) \in \mathbf{Z}^{s+1}$. Put $x = (\Delta\zeta(n))_n$ with the notation of Lemma 1 and define the sequence

$$\eta(n) = \chi_{\Delta\zeta,\underline{m}}(n) = \Delta\zeta(n)^{m_0}\Delta\zeta(n+1)^{m_1}\cdots\Delta\zeta(n+s)^{m_s}.$$

By Lemma 4, η can be extended to a continuous function on $\mathcal{K}_\alpha \setminus \overset{s+1}{\underset{k=1}{\cup}} \tau^{-k}(0^\omega)$, constant on B for any $B \in \mathcal{B}_s$, taking there a value η_B. Since the sets $\tau^{-k}(0^\omega)$ are finite, η can be ultimately extended to a function $\tilde{\eta}$ defined on $\mathcal{K}_\alpha$, continuous everywhere with finitely many possible exceptional points. Hence $\tilde{\eta}$ is $\mu-$continuous and

$$\lim_N \frac{1}{N}\sum_{n<N} \eta(n+j) = \sum_{B\in\mathcal{B}_s} \eta_B \mu_\alpha(B) = \int_{\mathcal{K}_\alpha} \tilde{\eta}(x)\,\mathrm{d}\mu_\alpha(x),$$

the limit being uniform in j. By Lemma 1, this shows unique ergodicity of $\mathcal{F}(\Delta\zeta)$. The following general Lemma 5 below implies that $\mathcal{F}(\Delta\zeta)$ is minimal.

Remark 4. The integral above can be explicitly computed in some simple cases. Set $\underline{m} = (1)$. Then $\chi_{\Delta\zeta,\underline{m}} = \Delta\zeta$, which is constant on the cylinders $[Q_r jk]$, $j \leq a^*_{r+1} - 1$, $k \leq a_{r+2} - 1$ of $\mathcal{B}_1$, where it takes the value $\dfrac{\zeta((j+1)q_r)}{\zeta(q_r-1)\zeta(jq_r)}$. Therefore, the mean values $\frac{1}{N} \sum\limits_{n<N} \Delta\zeta(n)$ converge to

$$\begin{aligned}\int\limits_{\mathcal{K}_\alpha} (\Delta\zeta)\tilde{}\,\mathrm{d}\mu_\alpha &= \sum_{r=0}^{\infty} \sum_{k=0}^{a_{r+2}-1} \sum_{j=0}^{a^*_{r+1}-1} \frac{\zeta((j+1)q_r)}{\zeta(q_r-1)\zeta(jq_r)} \mu_\alpha([Q_r jk]) = \\ &= \sum_{r=0}^{\infty} ||q_r\alpha|| \sum_{j=0}^{a^*_{r+1}-1} \frac{\zeta((j+1)q_r)}{\zeta(q_r-1)\zeta(jq_r)}.\end{aligned} \tag{22}$$

Lemma 5. *Let X and Y be compact metrizable spaces and let $T: X \to X$ be a continuous map such that the flow (T, X) is minimal. Let $\{B_i\,;\ i \in \mathcal{I}\}$ be a countable family of open pairwise disjoint subsets of X such that $\overline{\bigcup_{i\in\mathcal{I}} B_i} = X$. Then for any map $\Gamma : X \to Y$ which take constant value on each set B_i and for each point $x \in X$ such that $T^n(x) \in \bigcup_{i\in\mathcal{I}} B_i$ for all $n \in \mathbf{N}$, the flow $\mathcal{F}(\gamma_x)$ associated with the sequence $\gamma_x = (\Gamma(T^n x))_{n\geq 0} (\in Y^{\mathbf{N}})$ is minimal. Moreover, if $x' \in X$ also satisfies $T^n(x') \in \bigcup_{i\in\mathcal{I}}$ for all $n \in \mathbf{N}$, then $\mathcal{O}_{\gamma_x} = \mathcal{O}_{\gamma_{x'}}$.*

Proof. Let $x \in X$ such that $T^n x \in \bigcup_{i\in\mathcal{I}} B_i$ for all integers $n \geq 0$ and any $k \geq 0$, let $i(k)$ be such that $T^k x \in B_{i(k)}$. We have to show that the sequence γ_x is uniformly recurrent or equivalently that for all open set V in $Y^{\mathbf{N}}$ such that $V \cap \mathcal{O}_{\gamma_x} \neq \emptyset$, the set $\{k \in \mathbf{N}\,;\ \sigma^k(\gamma_x) \in V\}$ has bounded gaps. Let V be such an open set, then there exists $n \geq 0$ satisfying $\sigma^n(\gamma_x) \in V$ and consequently there are open sets $V_0, \cdots, V_s$ in Y such that $V_0 \times \ldots \times V_s \times Y^{\mathbf{N}} \subset V$ and $\Gamma(T^{n+j}x) \in$ $\in V_j$ for $0 \leq j \leq s$. Let $W = B_{i(n)} \cap T^{-1}B_{i(n+1)} \cap \ldots \cap T^{-s}B_{i(n+s)}$. Notice that W is a nonempty open set and that

$$\{k \in \mathbf{N}\,;\ T^k x \in W\} \subset \{k \in \mathbf{N}\,;\ \sigma^k(\gamma_x) \in V\}.$$

By minimality of (T, X), the set $\{k \in \mathbf{N}\,;\ T^k x \in W\}$ has bounded gaps, and so is for the set $\{k \in \mathbf{N}\,;\ \sigma^k(\gamma_x) \in V\}$ as expected.

To complete the proof, let $x' \in X$ such that $T^n(x') \in \bigcup_{i\in\mathcal{I}} B_i$ for all $n \geq 0$. By density of $\{T^k x\,;\ k \in \mathbf{N}\}$ in X and continuity of T, there exists an increasing sequence of natural numbers $(n_j)_j$ such that $(\sigma^{n_j}(\gamma_x))_j$ converges to $\gamma_{x'}$. Thus $\gamma_{x'} \in \mathcal{O}_{\gamma_x}$ and finally $\mathcal{O}_{\gamma_x} = \mathcal{O}_{\gamma_{x'}}$.

Corollary 2. *For any $x \in \mathcal{K}_\alpha$, $\mathcal{O}(\Delta\zeta) = \mathcal{O}(n \mapsto \Delta\zeta\tilde{}(\tau^n x))$.*

Proof. Choose in the above lemma $X = \mathcal{K}_\alpha$, $Y = \mathbf{U}$, $\Gamma = \Delta\zeta\tilde{}$, with the family $\mathcal{B}_1$. If $x \in \mathcal{K}_\alpha^\infty$, the equality of the corollary is just $\mathcal{O}(\Delta\zeta) = \mathcal{O}_{\gamma_{0^\omega}} = \mathcal{O}_{\gamma_x}$. If $x \in \tau^{-k}(0^\omega)$ for a given $k \geq 1$, by construction $\gamma_{0^\omega} = \sigma^k \gamma_x$. It remains to prove that $\gamma_x \in \mathcal{O}_{\gamma_{0^\omega}}$. Let $w = \tau^{k-1}(x)$ and let $(m_j)_j$ be an increasing sequence of integers converging to w in $\mathcal{K}_\alpha$, such that $\lim_j \Delta\zeta(\tau(\check{m}_j)) = \Delta\zeta\tilde{}(w)$. Using the continuity of $\Delta\zeta\tilde{}$ at each point $\tau^n(x)$, $n \in \mathbf{N} \setminus \{k-1\}$, we obtain $\lim_j \sigma^{m_j-k+1}\gamma_0 = \gamma_x$.

We can now refine Theorem 3.

Theorem 4. *The flow $\mathcal{F}(\Delta\zeta)$ is either constant or almost topological isomorphic to $(\mathcal{K}_\alpha, \tau)$.*

We first establish a general lemma.

Lemma 6. *Let h denote the Haar measure on $\mathbf{T}$ and let V be a nonempty open set in $\mathbf{T}$ such that:*

1. *$h(\partial V) = 0$ (where ∂V denotes the boundary of V),*
2. *$h(V) < 1$,*
3. *V splits the torsion points of $\mathbf{T}$, i.e. for all torsion points $r \neq 0$, if $V + r = V$ h-a.e., then $r = 0$.*

Then the map

$$\begin{aligned} \vartheta : \mathbf{T} &\longrightarrow \{0,1\}^{\mathbf{N}} \\ x &\longmapsto \vartheta_x := (\mathbf{1}_V(x + n\cdot\alpha))_{n\geq 0} \end{aligned}$$

(where $\mathbf{1}_V(\cdot)$ is the indicator function of V) is continuous at each point in $E = \mathbf{T} \setminus \bigcup_{n\geq 0} R_\alpha^{-n}(\partial V)$ and its restriction on E is one-to-one.

Proof. The continuity property of ϑ follows from that of R_α. Choose x and y in E and assume $\vartheta_x = \vartheta_y$. Equivalently, for all integers n, the equalities $\mathbf{1}_V(x + n\cdot\alpha) = \mathbf{1}_V(y + n\cdot\alpha)$ hold. The condition $h(\partial V) = 0$ ensures that $\mathbf{1}_V$ is $h-$continuous. Then unique ergodicity of $(\mathbf{T}, R_\alpha)$ yields

$$\int_{\mathbf{T}} |\mathbf{1}_V(x+t) - \mathbf{1}_V(y+t)|\, h(dt) = \lim_{N\to\infty} \frac{1}{N} \sum_{n<N} |\mathbf{1}_V(x+n\alpha) - \mathbf{1}_V(y+n\alpha)| = 0.$$

Thus $V + x = V + y \quad h-$a.e. Since $1 < h(V) < 0$, $x - y$ cannot be irrational and the last condition finally gives $x = y$.

Proof of the Theorem. Let

$$\begin{aligned} \Xi_\zeta : \mathcal{K}_\alpha &\longrightarrow \mathcal{O}_{\Delta\zeta} \\ x &\longmapsto \gamma_x = \big(\Delta\zeta\tilde{}(\tau^n x)\big)_{n\geq 0}. \end{aligned} \tag{23}$$

By Corollaries 1 and 2, the application Ξ_ζ is well defined, continuous at each point in $\mathcal{K}_\alpha^\infty$, and $\mathcal{O}_{\gamma_x} = \mathcal{O}_{\Delta\zeta}$ for all $x \in \mathcal{K}_\alpha$. Moreover, we have a commutative diagram expressed by the relation $\sigma \circ \Xi_\zeta = \Xi_\zeta \circ \tau$.

Assume that $\mathcal{F}(\Delta\zeta)$ is not constant. By Theorem 2 and Lemma 4, there exists $v \in \mathbf{U}$ such that

$$V = \bigcup_{\substack{B \in \mathcal{B}_1 \\ \Delta\zeta(B) = \{v\}}} \varphi(B \cap \mathcal{K}_\alpha^\infty)$$

is a nonempty open subset of $]-\alpha, 1-\alpha[$ with non-full measure. Furthermore, V is a finite or countable union of disjoint open subintervals of the form $]t_n, t_m[$ which eventually accumulate to $-\alpha$ or $1-\alpha$ (notation of Proposition 2). Thus $\partial V \subset \{t_m;\ m \geq 1\} \cup \{1-\alpha\}$ and it has measure zero. Therefore, the assumptions of Lemma 6 are fulfilled (after identifying $\mathbf{T}$ with $[-\alpha, 1-\alpha)$), $\Xi_\zeta \circ \psi$ is one-to-one on $I(\alpha)^*$ and Ξ_ζ is one-to-one on $\mathcal{K}_\alpha^\infty$.

We claim that the inverse bijection of Ξ_ζ (restricted to $\mathcal{K}_\alpha^\infty$), which is defined on $\Sigma := \Xi_\zeta(\mathcal{K}_\alpha^\infty)$, is continuous. Recall that if X and Y are metric spaces, D a dense subset of X and $f : X \to Y$, then f is continuous on X if and only if for any $x \in X$ and any sequence $(d_n)_n$ of D converging to x, we have $\lim f(d_n) = f(x)$. Let $x \in \mathcal{K}_\alpha^\infty$; by construction $\Xi_\zeta(x) = \lim\limits_{\check{n} \to x} (\Delta\zeta(n), \Delta\zeta(n+1), \ldots)$. Let $(m_j)_j$ be an increasing sequence of integers such that $\big(\Xi_\zeta(\check{m}_j)\big)_j$ converges to $\Xi_\zeta(x)$. We have to show that $\lim\limits_j \check{m}_j = x$. By compacity of $\mathcal{K}_\alpha$, we can assume that $y := \lim\limits_j(\check{m}_j)$ exists. If $y \in \mathcal{K}_\alpha^\infty$, continuity and injectivity of Ξ_ζ ensure that $y = x$. If $\tau^k(y) = 0^\omega$ for some k, then $\sigma^k(\lim_j \Xi_\zeta(\check{m}_j)) = \sigma^k x = \Delta\zeta$, hence $\tau^k x = 0^\omega$ (by injectivity, since $0^\omega \in \mathcal{K}_\alpha^\infty$), which is not allowed. Thus Ξ_ζ realizes an homeomorphism between $\mathcal{K}_\alpha^\infty$ and Σ.

We prove now that both $\mathcal{K}_\alpha^\infty$ and Σ ($\subset \mathcal{O}_{\Delta\zeta}$) are countable intersections of dense open subsets of compact spaces. It is obvious for $\mathcal{K}_\alpha^\infty$. For Σ, note that the elements of $\mathcal{O}_{\Delta\zeta} \setminus \Sigma$ are exactly the $\lim\limits_j \sigma^{n_j}(\Delta\zeta)$, where $(n_j)_j$ is an increasing sequence of integers with $\lim\limits_j \check{n}_j \in \tau^{-k}(0^\omega)$ for some $k \geq 1$. Thus

$$\mathcal{O}_{\Delta\sigma} \setminus \Sigma = \bigcup_{k \geq 1} \sigma^{-k}(\Delta\zeta).$$

Let $w \in \tau^{-1}(0^\omega)$ and let $L_w(\zeta)$ be the set of $z \in \mathbf{U}$ such that there exists an increasing sequence of integers $(m_k)_k$ which converges in $\mathcal{K}_\alpha$ to w and $\lim\limits_k \Delta\zeta(m_k) = z$. It follows by construction that

$$\lim_k \sigma^{m_k}(\Delta\zeta) = (z, \Delta\zeta(0), \Delta\zeta(1), \Delta\zeta(2), \ldots).$$

Reciprocally, any sequence $(z, \Delta\zeta) := (z, \Delta\zeta(0), \Delta\zeta(1), \Delta\zeta(2), \ldots)$ with $z \in \in \bigcup_{w\in\tau^{-1}(0^\omega)} L_w$ belongs to $\mathcal{O}_{\Delta\zeta}$, so that

$$\sigma^{-1}(\Delta\zeta) = \bigcup_{w\in\tau^{-1}(0^\omega)} \{(z, \Delta\zeta)\,;\ z \in L_w(\zeta)\}$$

and more generally, for any $k \geq 2$:

$$\sigma^{-k}(\Delta\zeta) =$$

$$= \bigcup_{v\in\tau^{-k}(0^\omega)} \{(\Delta\zeta(v), \ldots, \Delta\zeta(\tau^{k-2}v), z, \Delta\zeta(0), \Delta\zeta(1), \ldots)\,;\ z \in L_{\tau^{k-1}v}\}.$$

In particular, $\mathcal{O}_{\Delta\xi} \setminus \Sigma$ is a countable union of compact sets of empty interior. The underlying almost topological dynamical systems are $(\mathcal{K}_\alpha^\bullet, \tau)$ and its image $(\Xi_\zeta(\mathcal{K}_\alpha^\bullet), \sigma)$, hence Ξ realizes an almost topological isomorphism between $(\mathcal{K}_\alpha, \tau)$ and $(\mathcal{O}_{\Delta\zeta}, \sigma)$.

For the sequel we need to extend Ξ_ζ^{-1} at all points of $\mathcal{O}_{\Delta\zeta}$ (and use abusively the same notation). In fact, if $\xi \in \sigma^{-k}(\Delta\zeta)$ we have

$$\xi = (\Delta\zeta(v), \ldots, \Delta\zeta(\tau^{k-2}v), z, \Delta\zeta(0), \Delta\zeta(1), \ldots)$$

with $\tau^k(v) = 0^\omega$. There are two possible values for v; we fix one of them by assuming $\tau^{k-1}(v) = 0a_20a_40a_6\cdots$. Now we put $\Xi_\zeta^{-1}(\xi) = v$.

Theorem 4 has two interesting corollaries. The first one will be useful in the next sections and follows actually from Theorems 2 and 4. The second one extends to α-multiplicative sequences the following result: a sequence f is both p- and q-multiplicative for coprime integers p and q, if and only if, Δf is constant (and consequently there exists $\theta \in \mathbf{U}$ such that $f(n) = \theta^n$ for all $n \in \mathbf{N}$).

Corollary 3. *Let (m_k) be an increasing sequence of integers. Then the following propositions are equivalent:*

1. *The sequence $(\sigma^{m_k}(\Delta\zeta))_k$ converges to $\Delta\zeta$ in $\mathcal{O}_{\Delta\zeta}$.*
2. *The sequence $(m_k{\cdot}\alpha)_k$ converges to 0 in $\mathbf{T}$.*
3. The sequence $(m_k)_k$ converges to 0^ω in $\mathcal{K}_\alpha$.

Corollary 4. *A sequence ζ is both α- and β-multiplicative for distinct irrational α and β in $[0, 1/2]$ if and only if $\Delta\zeta$ is constant.*

Proof. If $\Delta\zeta$ is constant, then $\zeta = \zeta(1)^n$ for any integer $n \geq 0$. Therefore ζ is α-multiplicative for any α. Conversely, if $\Delta\zeta$ is not constant and if

ζ is α-multiplicative, then Theorem 4 ensures that $\mathcal{F}(\Delta\zeta)$ is isomorphic to $([-\alpha, 1-\alpha), R_\alpha)$, which is isomorphic to $([0,1), R_\beta)$ only for $\beta \in \{\alpha, 1-$ $-\alpha\}$.

6. α-multiplicative flows

6.1. Relation between $\mathcal{F}(\Delta\zeta)$ and $\mathcal{F}(\zeta)$

We first need to recall some basic facts about skew products. Assume that G is a compact metrizable group and let h_G be its Haar measure. We identify $G^{\mathbf{N}}$ with $G^{\mathbf{N}} \times G$ by means of the map $J : G^{\mathbf{N}} \to G^{\mathbf{N}} \times G$ defined by

$$J(g_0, g_1, g_2, \ldots) := (\Delta g, g_0), \tag{24}$$

where $g = (g_0, g_1, \ldots)$ and $\Delta g := (g_1 g_0^{-1}, g_2 g_1^{-1}, \ldots)$. Readily J is an homeomorphism of inverse map

$$J^{-1}((h_0, h_1, \ldots), \gamma) = (\gamma, h_0\gamma, h_1 h_0 \gamma, h_2 h_1 h_0 \gamma, \ldots)$$

and we let the reader to verify that the full shift $\mathcal{F}(G)$ is topologically conjugate under J to the skew product

$$\mathcal{F}(G) \square_{\pi_0} G := (G^{\mathbf{N}} \times G, \sigma'),$$

where $\pi_0 : G^{\mathbf{N}} \to G$ is the first projection and

$$\sigma'((g_0, g_1, \ldots), \gamma) := (\sigma(g_0, g_1, \ldots), \pi_0(g_0, g_1, \ldots)\gamma) = ((g_1, g_2, \ldots), g_0\gamma).$$

For notational convenience, we shall omit π_0 in the sequel and note $\mathcal{F}(G) \square G$ for the skew product given by π_0. Moreover, for each $\gamma \in G$, let $g \mapsto g \cdot \gamma$ denote the G-action on $G^{\mathbf{N}}$ given by

$$(g_0, g_1, g_2, \ldots) \cdot \gamma := (g_0\gamma, g_1\gamma, g_2\gamma, \ldots).$$

We refer to [13] and [41] for more information on the subject in a more general setting. Note that the map $g \mapsto \Delta g$ is continuous, surjective, commutes with the underlying shifts, and that for any $g \in G^{\mathbf{N}}$,

$$\Delta(\mathcal{O}_g) = \mathcal{O}_{\Delta g}.$$

Lemma 7. *Let $g \in G^{\mathbf{N}}$. Then $J(\mathcal{O}_g) = \mathcal{O}_{\Delta g} \times G$ if and only if*

$$\forall\, \gamma \in G,\ g \cdot \gamma \in \mathcal{O}_g. \tag{25}$$

If it holds, then the flow $\mathcal{F}(g)$ is conjugate (by restriction of J to $\mathcal{O}_g$) to $\mathcal{F}(\Delta g) \square G = (\mathcal{O}_{\Delta g} \times G, \sigma')$.

Proof. The inclusion $J(\mathcal{O}_g) \subset \mathcal{O}_{\Delta g} \times G$ is trivial. The characterization of the inverse inclusion is based on the observation that $\Delta g = \Delta(g \cdot \gamma)$ for all $g \in G^{\mathbf{N}}$ and all $\gamma \in G$, hence $J(g \cdot \gamma) = (\Delta g, g_0 \gamma)$. Assume (25). Then

$$J(\mathcal{O}_g) \supset J(\{g \cdot \gamma\,;\, \gamma \in G\}) = \{\Delta g\} \times G,$$

hence $J(\mathcal{O}_g) = \mathcal{O}_{\Delta g} \times G$. Reciprocally, assume the equality $J(\mathcal{O}_g) = \mathcal{O}_{\Delta g} \times G$ to hold. Then, $J^{-1}(\Delta g, g_0 \gamma) = g \cdot \gamma \in \mathcal{O}_g$, and (25) follows. The last part of the lemma is a consequence of its first part and of the discussion above.

Example 1. We have already met the case where $\Delta\zeta$ is constant, say equal to θ. This corresponds to $\zeta(n) = \theta^n$. Let $G = \mathbf{U}(\theta)$ be the closed subgroup of $\mathbf{U}$ generated by θ; Property (25) is verified and $F(\Delta\zeta) = (\{(\theta, \theta, \theta, \ldots)\}, \sigma)$ is trivial. We then retrieve a classical result from Lemma 7 which says that $\mathcal{F}(\zeta)$ is topologically conjugate by means of the first projection $\mathcal{O}_\zeta \to G$ to the translation $x \mapsto \theta x$ on G. Note that this translation is uniquely ergodic.

If $\Delta\zeta$ is not constant, the possible existence of a continuous extension of ζ on $\mathcal{K}_\alpha$ will play an important role in the study of the flow $\mathcal{F}(\zeta)$. The next lemma is interested in that topological property.

Lemma 8. *Let ζ be an α-multiplicative sequence. Then the following propositions are equivalent:*

(i) $\displaystyle\lim_{k\to\infty} \sup_{n \in M_k} \left|\zeta(n) - 1\right| = 0.$

(ii) The sequence $(\zeta \circ S_k)_k$ is uniformly convergent, where

$$S_k(x) := e_0(x)q_0 + \cdots + e_{k-1}(x)q_{k-1}$$

for $x \in \mathcal{K}_\alpha$.

(iii) The sequence ζ can be extended to a continuous function on $\mathcal{K}_\alpha$.

Proof. (i) $\Rightarrow$ (ii) by Cauchy's criterion.

(ii) $\Rightarrow$ (iii): for given k, the function $\zeta \circ S_k$ is constant on the cylinders of the α-partition $\mathcal{F}_{k-1}$ (see Subsection 3.1). Thus $\zeta \circ S_k$ is continuous and its uniform limit is continuous, too.

(iii) $\Rightarrow$ (i): assume that $\limsup \displaystyle\sup_{n \in M_k} \left|\zeta(n) - 1\right| > 0$. Then there exist increasing sequences $(n_j)_j$ and $(k_j)_j$ of positive integers and a real constant

κ such that $n_j < q_{k_j}$, $\check{n}_{j+1} \in M_{k_j+1}$ and $|\zeta(n_j) - 1| \geq \kappa > 0$ for all j. By construction, for all $(j, k, \ell) \in \mathbf{N}^3$ with $j \neq k$, we have $e_\ell(n_j)e_\ell(n_k) = 0$. Let $x \in \mathcal{K}_\alpha$ defined by $S_{k_m}(x) = \left(\sum_{j=0}^{m} n_j\right)\check{}$ for any index m. Then the sequence $(\zeta \circ S_k(x))_k$ does not converge and ζ can not be extended to a continuous function at the point x.

The flow $\mathcal{F}(\zeta)$ when ζ has a continuous extension on $\mathcal{K}_\alpha$ is of a particular interest.

Theorem 5. *Let $f : \mathcal{K}_\alpha \to \mathbf{U}$ be a continuous map such that the sequence $\zeta : n \mapsto f(n)$ is α-multiplicative. Then the flow $\mathcal{F}(\zeta)$ is topologically conjugate to $\mathcal{F}(\Delta\zeta)$ – the conjugation map being realized with Δ (restricted to $\mathcal{F}(\zeta)$). Moreover, the map $\Xi_\zeta : \mathcal{K}_\alpha \to \mathcal{O}_{\Delta\zeta}$ (see (23)) is continuous, surjective (and realizes an almost topological isomorphism of the underlying flows).*

Proof. It is clear that Δ is a continuous epimorphism of flows from $\mathcal{F}(\zeta)$ onto $\mathcal{F}(\Delta\zeta)$. In order to prove that Δ is one-to-one on $\mathcal{O}_\zeta$ let us first notice that due to the continuity of f we have

$$\mathcal{O}_\zeta = \{(f(\tau^n x))_{n\geq 0}\,;\; x \in \mathcal{K}_\alpha\}.$$

Let ξ and ξ' be in $\mathcal{O}_\zeta$ and suppose that $\Delta(\xi) = \Delta(\xi')$. There exist x and x' in $\mathcal{K}_\alpha$ such that $\xi = (f(\tau^n x))_n$ and $\xi' = (f(\tau^n x'))_n$. Assume $x \in \mathcal{K}_\alpha^\infty$; if $x' \in \mathcal{K}_\alpha^\infty$, using (23) and the proof of Theorem 4, we obtain $\Xi^{-1}(\Delta\xi) = x = x'$. If $x' \in \tau^{-k'}(0^\omega)$ for an integer $k' \geq 1$, by construction $\sigma^{k'}(\Delta(\xi')) = \Delta\zeta$, hence $\Xi^{-1}(\sigma^{k'}(\Delta(\xi'))) = 0^\omega = \tau^{k'}x' = \tau^{k'}x$ in contradiction with the assumption on x. Finally, suppose there are positive integers k and k' verifying $x \in \tau^{-k}(0^\omega)$ and $x' \in \tau^{-k'}(0^\omega)$. We may assume $k' \geq k$ and now $\Xi^{-1}(\sigma^{k'}\xi') = \Xi^{-1}(\Delta\zeta) = 0^\omega = \tau^{k'}(x)$; consequently $k = k'$ and $\xi_n = \xi'_n$ for any $n \geq k$, but we also have $\xi_n = \xi'_n$ if $0 \leq n < k$ by the product formula

$$\begin{aligned}(\Delta\xi)_n \cdots (\Delta\xi)_{k-1} &= \frac{f(\tau^{n+1}x)}{f(\tau^n x)}\,\frac{f(\tau^{n+2}x)}{f(\tau^{n+1}x)} \cdots \frac{f(0^\omega)}{f(\tau^{k-1}x)}\\ &= f(\tau^n x)^{-1} = \xi_n^{-1}.\end{aligned}$$

We have proved that Δ is a continuous and bijective map between the compact spaces $\mathcal{O}_\zeta$ and $\mathcal{O}_{\Delta\zeta}$, hence it is an homeomorphism as expected, which commutes with the corresponding shifts. The last part of the theorem is clear.

Remark 5. If f separates the two points of $\tau^{-1}(0^\omega)$, then the map $x \mapsto (f(\tau^n x))_n$ from $\mathcal{K}_\alpha$ to $\mathcal{O}_\zeta$ realizes a topological isomorphism between the α-odometer and the flow $\mathcal{F}(\zeta)$.

Remark 6. Let G be the closed subgroup of $\mathbf{U}$ generated by $\zeta(\mathbf{N})$. Under the assumption of Theorem 4, the skew product $\mathcal{F}(\Delta\zeta)\square G$ is certainly not minimal. In fact, the image of $\mathcal{O}_\zeta$ by J is a nontrivial shift invariant compact subspace of $\mathcal{F}(\Delta\zeta)\square G$, and according to a result of H. Furstenberg [23] the first projection p on $\mathcal{O}_{\Delta\zeta}$ is a continuous coboundary, namely $p(y) = c(\sigma y)/c(y)$, where $c(\cdot)$ is the continuous map defined by

$$c(y) = (\Delta^{-1}(y))_0 \quad (y \in \mathcal{O}_{\Delta\zeta}).$$

6.2. Topological essential values

From now on, we fix $G = G(\zeta)$ (the closed subgroup of $\mathbf{U}$ generated by all the values of ζ), $g := \zeta \in G(\zeta)^{\mathbf{N}}$, and apply the general discussion above to this case. For $k \in \mathbf{N}$, recall that $M_k = [0^{(k)}]$, and set $\zeta(M_k) = \{\zeta(n)\,;\, \check{n} \in M_k\}$.

Definition 4. *The elements in*

$$G_1(\zeta) = \bigcap_k \overline{\zeta(M_k)}. \tag{26}$$

are called topological essential values of ζ.

The following proposition collects facts about $G_1(\zeta)$ and $\mathcal{F}(\zeta)$, which are bounded up with each other.

Proposition 7. *The set $G_1(\zeta)$ of topological essential values is a compact subgroup of $G(\zeta)$, and*

$$\begin{aligned} G_1(\zeta) &= \{\gamma \in G(\zeta)\,;\, \zeta \cdot \gamma \in \mathcal{O}_\zeta\} = \\ &= \{\gamma \in G(\zeta)\,;\, (\Delta\zeta, \gamma) \in J(\mathcal{F}(\zeta))\} = \\ &= \{\gamma \in G(\zeta)\,;\, \mathcal{O}_\zeta{\cdot}\gamma = \mathcal{O}_\zeta\}. \end{aligned}$$

The flow $\mathcal{F}(\zeta)$ is surjective. The closed orbit $\mathcal{O}_\zeta$ has isolated points if and only if $\zeta(n) = \rho^n$ for every n and some root of unity ρ. Furthermore, the characters χ of $G(\zeta)$ such that $\chi \circ \zeta$ admits a continuous extension to $\mathcal{K}_\alpha$ are exactly those whose restriction to $G_1(\zeta)$ is trivial. Eventually, the following propositions are equivalent:

1. *$G_1(\zeta) = G(\zeta)$.*
2. *For any $z \in G(\zeta)$, $\zeta{\cdot}z \in \mathcal{O}_\zeta$ (Condition 25) of Lemma 7).*
3. *For any non-negative integer k, the set $\zeta(M_k)$ is dense in $G(\zeta)$.*
4. *For any character χ of $G(\zeta)$, if $\chi \circ \zeta$ can be extended to a continuous function on $\mathcal{K}_\alpha$, then χ is the trivial character.*

Proof. Obviously, $G_1(\zeta)$ is a compact subset of $G(\zeta)$ containing 1. Let be $(z, z') \in G_1(\zeta)^2$. Due to the particular structure of the decreasing sequence of compact sets $\overline{\zeta(M_k)}$, we are able to select two increasing sequences of integers $(n_j)_j$ and $(k_j)_j$ such that , $n_j \in M_{k_j}$ with $n_j < q_{k_{j+1}-1}$, $\lim_j \zeta(n_{2j}) = z$ and $\lim_j \zeta(n_{2j+1}) = z'$. By construction, $\zeta(n_{2j}+n_{2j+1}) = \zeta(n_{2j})\zeta(n_{2j+1}) \in \zeta(M_{k_{2j}})$. Passing to the limit leads to $zz' \in G_1(\zeta)$. Thus $G_1(\zeta)$ is a submonoid of $\mathbf{U}$. Since $G_1(\zeta)$ is compact, it is either equal to $\mathbf{U}_\ell$ (the group of ℓ-th roots of unity) some ℓ or to $\mathbf{U}$.

The flow $\mathcal{F}(\zeta)$ is surjective if and only if $\zeta \in \sigma(\mathcal{O}_\zeta)$. For this purpose, we prove that there exists an increasing sequence of integers $(m_k)_k$ such that $\sigma^{m_k}\zeta$ tends to ζ. Using the above method, we can build an increasing sequence of integers $(m_k)_k$ with $m_k \in M_k$, and $\lim_k \zeta(m_k) = 1$. But $\zeta(n+m_k) = \zeta(n)\zeta(m_k)$ if $0 \leq n < q_k$, proving that $\lim_k \sigma^{m_k}\zeta = \zeta$. Consequently, O_ζ has isolated points if and only if it is periodic, that is if $\zeta(n) = \rho^n$ for every n and some root of unity ρ.

For $\gamma \in G(\zeta)$, $\zeta{\cdot}\gamma$ belongs to $\mathcal{O}_\zeta$ if and only if there exists an increasing sequence $(m_k)_k$ of integers such that $\sigma^{m_k}\zeta$ tends to $\zeta{\cdot}\gamma$. By Corollary 3, that is equivalent with $\check{m}_k$ tends to 0^ω and $\zeta(m_k)$ tends to γ, hence the equality $G_1(\zeta) = \{\gamma \in G(\zeta)\,;\, \zeta \cdot \gamma \in \mathcal{O}_\zeta\}$. The second equality then follows from the commutation $J(\sigma^{m_k}\zeta) = (\sigma^{m_k}\Delta\zeta, m_k)$. The third equality is a straightforward consequence of the first one.

Let χ be a character on $G(\zeta)$. Lemma 8 applied to the α-multiplicative sequence $\chi \circ \zeta$ shows that

$$\chi\left(\bigcap_k \overline{\zeta(M_k)}\right) = \bigcap_k \overline{\chi \circ \zeta(M_k)} = \{1\}$$

if and only if $\chi \circ \zeta$ can be extended to a continuous function on $\mathcal{K}_\alpha$, hence the characterization of these characters.

For $g = \zeta \in G(\zeta)^{\mathbf{N}}$ Condition (25) becomes: for any $z \in G(\zeta)$ there exists an increasing sequence $(m_j)_j$ of integers such that $(\sigma^{m_j}\zeta)_j$ tends to $\zeta{\cdot}z$ when j tends to infinity. The equivalence between the conditions is then immediate from the first part of the proposition.

6.3. The structure of the α-multiplicative flow $\mathcal{F}(\zeta)$

We are ready to give a full description of the flow $\mathcal{F}(\zeta)$. The next theorem deals with the easy case where $G_1(\zeta) = G(\zeta)$ and follows easily from the above study except for the minimality.

Theorem 6. *Let ζ be an α-multiplicative sequence such that $\Delta\zeta$ is not constant. The flow $\mathcal{F}(\zeta)$ is topologically conjugate (by J) to $(\mathcal{F}(\Delta\zeta)\times G,\sigma')$ if and only if for any integer $k\geq 0$, $\zeta(M_k)$ is dense in $G(\zeta)$. In that case, $\mathcal{F}(\zeta)$ is a minimal flow almost isomorphic to $(\mathcal{K}_\alpha\times G(\zeta),T_\zeta)$, with*

$$\begin{aligned} T_\zeta : \mathcal{K}_\alpha\times G(\zeta) &\longrightarrow \mathcal{K}_\alpha\times G(\zeta)\\ (x,\gamma) &\longmapsto (\tau x, \gamma\Delta\tilde{\zeta}(x))\,, \end{aligned}$$

the almost isomorphism from $(\mathcal{F}(\Delta\zeta)\times G,\sigma')$ to $\mathcal{F}(\zeta)$ being given by

$$H(x,\gamma) = J^{-1}(\Xi_\zeta(x),\gamma).$$

Proof. We only prove the minimality of $F(\zeta)$. Choose $\varepsilon>0$ and a positive integer s. By density of $\zeta(M_k)$ in $G(\zeta)$ for any integer k and precompacity of $\mathbf{U}$, there exists a finite subset $A=A(s,\varepsilon)$ of M_{s+1} such that

$$\forall\, z\in G(\zeta),\quad \exists\, a\in A\,:\, |z-\zeta(a)|\leq\varepsilon. \tag{27}$$

Choose an integer t such that $e_k(a)=0$ for all $a\in A$ and all $k\geq t$. Let $m\in M_{s+t+1}$. Taking $z=\overline{\zeta(m)}$ in (27) shows that there exists $a_m\in A$ with $|\zeta(a_m)-\overline{\zeta(m)}|\leq\varepsilon$. Hence, for all integers j, $0\leq j<q_s$, one gets $|\zeta(j)-\zeta(j+a_m+m)|\leq\varepsilon$, so that

$$\{a_m+m\,;\, m\in M_{s+t+1}\}\subset V_\zeta(s,\varepsilon).$$

But A is bounded and M_{s+t+1} has bounded gaps (see Section 4); therefore $V_\zeta(s,\varepsilon)$ has bounded gaps, too.

We now extend the above result to the general case.

Theorem 7. *Let ζ be an $\alpha-$multiplicative sequence such that $\Delta\zeta$ is not constant and let $G_1(\zeta)$ be the compact group of topological essential values of ζ. Then: (i) there exists a continuous map $f_0:\mathcal{K}_\alpha\to\mathbf{U}$ such that the sequence $\zeta_0: n\mapsto f_0(n)$ is α-multiplicative, $\zeta_1:=\zeta/\zeta_0$ takes its values in $G_1(\zeta)$ and $G_1(\zeta_1)=G_1(\zeta)$;*

(ii) the flow $\mathcal{F}(\zeta)$ is minimal, almost topologically conjugate to $(\mathcal{F}(\Delta\zeta)\times$ $\times G_1(\zeta),\sigma'')$ with

$$\sigma''(g,\gamma) = \left(\sigma(g), \gamma g_0\frac{f_0(\Xi_\zeta^{-1}(g))}{f_0(\Xi_\zeta^{-1}(\sigma g))}\right).$$

In particular, if $\Delta\zeta_1$ is constant, equal to θ, then $\mathcal{F}(\zeta)$ is almost conjugate to the direct product $\mathcal{F}(\zeta_0)\times(\mathbf{U}(\theta),z\mapsto z\theta)$;

(iii) assume that $\zeta = \zeta_1\zeta_0$ *with* ζ_i α*-multiplicative,* $\Delta\zeta_1$ *not constant, and* ζ_0 *can be extended to a continuous map* $f_0 : \mathcal{K}_\alpha \to \mathbf{U}$*. Then* $G_1(\zeta) = G_1(\zeta_1)$*, and the flow* $\mathcal{F}(\zeta)$ *is almost topologically conjugate to the flow* $\mathcal{F}(\zeta_1)$*;*

(iv) Let $\zeta = \zeta_1\zeta_0$ *be a factorization as in (iii), then* $\mathcal{F}(\zeta)$ *is almost topologically conjugate to* $(\mathcal{F}(\Delta\zeta_1) \times G_1(\zeta), \sigma')$*.*

Proof. *(i)* If $G(\zeta) = G_1(\zeta)$, the function $f_0 = 1$ answers the question. Thus we may assume $G(\zeta) \neq G_1(\zeta)$. Therefore $G_1(\zeta) = \mathbf{U}_\ell$ for some $\ell \geq 1$. Since $G_1(\zeta)$ is the decreasing intersection of the closed sets $\overline{\zeta(M_k)}$ in a compact space, those tend to $G_1(\zeta)$ for the Hausdorff distance. Consequently, there exists an integer k such that

$$\forall n \in M_k,\ \exists \zeta_1(n) \in \mathbf{U}_\ell : \ |\zeta(n) - \zeta_1(n)| < \frac{1}{3}|\exp(i\pi/\ell) - 1|, \tag{28}$$

the choice of $\zeta_1(n)$ being unique. For $n \in M_k$ and $k_1 > k$, let $n_1 = e_k(n)q_k + {} + \cdots + e_{k_1-1}(n)q_{k_1-1}$ and $n_2 = n - n_1$. Then

$$|\zeta_1(n) - \zeta_1(n_1)\zeta_1(n_2)| \leq |\zeta_1(n) - \zeta(n)| + |\zeta_1(n_1) - \zeta(n_1)| + |\zeta_1(n_2) - \zeta(n_2)|,$$

showing that $\zeta_1(n) = \zeta_1(n_1)\zeta_1(n_2)$ by (28). Finally extend ζ_1 to $\mathbf{N}$ by $\zeta_1(n) = \zeta_1(e_k(n)q_k + e_{k+1}(n)q_{k+1} + \cdots)$. Then ζ_1 is $\alpha-$multiplicative and Lemma 8 shows that $\zeta_0 = \zeta\zeta_1^{-1}$ is extendable to a continuous function f_0 on $\mathcal{K}_\alpha$. Moreover, by construction, $\zeta_1(\mathbf{N}) = G_1(\zeta) = G_1(\zeta_1) = G(\zeta_1) = \mathbf{U}_\ell$.

(ii) The minimality of $\mathcal{F}(\zeta)$ can be derived from a suitable modification of the proof given in Theorem 6. Let ζ_1 and ζ_0 be as in *(i)* and choose $\varepsilon > 0$ and $k_0 \geq k_1$ such that the inequality $k \geq k_0$ implies $|1 - \zeta_0(u)| \leq \varepsilon/2$ for any $u \in M_k$. Fix an integer $s \geq 1$ and a subset $A = \{n_0, \dots, n_{\ell-1}\}$ of M_{s+1} such that the Hausdorff distance between $\zeta_1(A)$ and $G_1(\zeta)$ is less than $\varepsilon/2$. Choose an integer $t \geq k_0$ such that $e_k(a) = 0$ for all $a \in A$ and $k \geq t$. For any $m \in M_{s+t+1}$ there exists $a_m \in A$ with $|\zeta_1(a_m) - \overline{\zeta_1(m)}| \leq \varepsilon/2$. Therefore, for any integer $j \in \{0, \dots, q_s - 1\}$ we can write

$$\begin{aligned}|\zeta(j) - \zeta(j + a_m + m)| = |\zeta(j) - \zeta(j)\zeta(a_m)\zeta(m)| = |1 - \zeta(a_m)\zeta(m)| \leq \\ \leq |1 - \zeta_1(a_m)\zeta_1(m)| + |1 - \zeta_0(a_m + m)| \leq \varepsilon,\end{aligned}$$

so that the inclusion

$$\{a_m + m\,;\ m \in M_{s+t+1}\} \subset V_\zeta(s, \varepsilon),$$

holds and implies that $V_\zeta(s, \varepsilon)$ has bounded gaps.

The flow $\mathcal{F}(\zeta)$ is topologically conjugate to the compact σ'-invariant subset $J(\mathcal{O}_\zeta)$ in $\mathcal{F}(\Delta\zeta)\times G$ throw the map $J : G(\zeta)^{\mathbf{N}} \to G(\zeta)^{\mathbf{N}} \times G(\zeta)$ given in (24). Introduce the almost automorphism

$$A : (g,\gamma) \mapsto \big(g, \gamma\overline{f_0(\Xi_\zeta^{-1}(g))}\big)$$

of $\mathcal{F}(\Delta\zeta)\square_{\pi_0} G := (G^{\mathbf{N}} \times G, \sigma')$ to obtain $A\circ J(\mathcal{F}(\zeta)) = \mathcal{F}(\Delta\zeta)\times G_1(\zeta)$. Then $\mathcal{F}(\zeta)$ is almost conjugate by $A\circ J$ to $(\mathcal{O}_{\Delta\zeta}\times G_1(\zeta), \sigma'')$ with

$$\sigma''(g,\gamma) = \left(\sigma(g), \gamma g_0 \frac{f_0(\Xi_\zeta^{-1}(g))}{f_0(\Xi_\zeta^{-1}(\sigma g))}\right). \tag{29}$$

If $\Delta\zeta_1$ is constant, equal to θ, then $\mathcal{O}_{\Delta\zeta} = \mathcal{O}_{\Delta\zeta_0}\cdot\theta$ and a short computation gives $\sigma''(g,\gamma) = (\sigma(g), \gamma\theta)$. Identifying $\mathcal{O}_{\Delta\zeta_0}$ with $\mathcal{O}_{\Delta\zeta}$ by $\xi \mapsto \xi\cdot\theta$, the transformation (29) viewed on $\mathcal{O}_{\Delta\zeta_0}\times G_1(\zeta)$ is topologically conjugate by $(g,\gamma)\mapsto(\Delta^{-1}(g),\gamma)$ (Theorem 5) to the direct product $\mathcal{F}(\zeta_0)\times\mathcal{F}(n\mapsto\theta^n)$.

(iii) The equality $G_1(\zeta) = G_1(\zeta_1)$ is a straightforward consequence of the definition of essential topological values. Let $Q : \mathcal{O}_{\zeta_1} \to \mathcal{O}_\zeta$ be defined as follows. Let ξ be in $\mathcal{O}_{\zeta_1}$. If $\Delta\xi \in Y_1 := \mathcal{O}_{\Delta\zeta_1} \setminus \bigcup_{k\geq 1} \sigma^{-k}(\Delta\zeta_1)$ there exists a unique $x \in \mathcal{K}_\alpha^\infty$ such that $\Xi_{\zeta_1}(x) = \Delta\xi$ and put

$$Q(\xi) = (f_0(\tau^n x)\xi(n))_{n\geq 0}. \tag{30}$$

Thus we have proved the almost isomorphism we claimed.

Nevertheless it is possible to extend Q to the whole closed orbit, what we do now. If $\Delta\xi \notin Y_1$, there exists $k\geq 1$ such that $\sigma^k\Delta\xi = \Delta\zeta_1$ and we know from the proof of Theorem 4 that $\Delta\xi$ has the form

$$(\Delta\tilde{\zeta_1}(v), \ldots, \Delta\tilde{\zeta_1}(\tau^{k-2}v), z, \Delta\zeta_1(0), \Delta\zeta_1(1), \ldots)$$

with $v \in \tau^{-k}(0^\omega)$ selected as $v = \Xi_{\zeta_1}^{-1}(\Delta\xi)$, and $z \in L_{\tau^{k-1}(v)}(\zeta_1)$. Then, put

$$Q(\xi) = (f_0(\tau^n v)\xi(n))_n \quad (\in \mathcal{O}_\zeta), \tag{31}$$

so that $\sigma^k Q(\xi) = (z, z\zeta(1), z\zeta(2), z\zeta(3), \ldots)$. The map Q commutes with the corresponding shifts, is continuous at each point of $\Delta^{-1}(Y_1)$ and one-to-one on $\Delta^{-1}(Y_1)$. By interchanging ζ_1 and ζ, and after replacing f_0 by $\overline{f_0}$ we can define an analogous map $P : \mathcal{O}_\zeta \to \mathcal{O}_{\zeta_1}$ which is continuous and one-to-one on $\Delta^{-1}(Y)$ with $Y := \mathcal{O}_{\Delta\zeta} \setminus \bigcup_{k\geq 1}\sigma^{-k}(\Delta\zeta)$. In fact $Q(\Delta^{-1}(Y_1)) = \Delta^{-1}(Y)$ and $P\circ Q(\xi) = \xi$ for any $\xi\in\Delta^{-1}(Y_1)$. This ends the proof of *(iii)*.

The assertion *(iv)* follows from *(iii)* and Theorem 6.

Remark 7. Property *(i)* in Theorem 7 shows that $L_u(\zeta) = f_0(u)G_1(\zeta)$ for $u \in \tau^{-1}(0^\omega)$. This property is particularly interesting when $G_1(\zeta) = \mathbf{U}_\ell$. In that later case, if $\zeta = \zeta_1'\zeta_0'$ is another factorization with $G(\zeta_1') = G_1(\zeta)$ and ζ_0' extendable to a continuous map on $\mathcal{K}_\alpha$, then ζ_1'/ζ_1 only depend on a finite numbers of digits. In other words, there exists k such that ζ_1'/ζ_1 is constant on M_k.

Example 2. Let $\zeta(n) = \theta^{s_\alpha(n)}$ with $\theta \in \mathbf{U}$ fixed. Clearly $G(\zeta) = G_1(\zeta) = = \mathbf{U}(\theta)$, where $\mathbf{U}(\theta)$ (the closed subgroup of $\mathbf{U}$ generated by θ).

Example 3. Choose an integer $\ell \geq 2$ and let ζ be the α-multiplicative sequence defined by $\zeta(bq_0) = e^{2i\pi b\beta}$ for $0 \leq b \leq a_1 - 1$ and $\zeta(bq_k) = e^{2i\pi b/\ell}$ for $k \geq 1$ and $0 \leq b \leq a_{k+1}$. If β is irrational, we have $G(\zeta) = \mathbf{U}$, but $G_1(\zeta) = \mathbf{U}_\ell$ and $\mathcal{F}(\zeta)$ is topologically conjugate to $\mathcal{F}(n \mapsto e^{2i\pi s_\alpha(n)/\ell})$ and almost conjugate to $(\mathcal{K}_\alpha \times \mathbf{U}_\ell, T_\alpha)$, with $T_\alpha(x,\gamma) = (\tau(x), \gamma e^{2i\pi w(x)/\ell})$, where $w(x) = \lim\limits_{\tilde{n}\to x}(s_\alpha(n+1) - s_\alpha(n))$, but $w(u) = 0$ if $\tau(u) = 0^\omega$.

Example 4. It is easy to construct ζ with $G(\zeta) = \mathbf{U}$ and $G_1(\zeta) = \{1\}$. But $G_1(\zeta) = \{1\}$ means that ζ can be extended to a continuous map on $\mathcal{K}_\alpha$. Hence $\mathcal{F}(\zeta)$ is almost conjugate to the translation R_α.

7. Metric properties of $\mathcal{F}(\zeta)$

7.1. Metrical isomorphisms

We first look at particular cases.

Example 5. Assume that ζ is not constant and only depends on a finite number of coordinates, i.e. $\zeta(M_k) = \{1\}$ for $k \geq k_0$. Theorem 5 says that $\mathcal{F}(\zeta)$ is almost topologically conjugate to the translation R_α. We can say a bit more. Let H_k be the word $\zeta(0)\zeta(1)\ldots\zeta(q_k - 1)$. Then

$$\forall k \geq k_0 + 2\,, \quad H_k = (H_{k-1})^{a_k} H_{k-2}. \tag{32}$$

This recursion formula is exactly that of (17). Thus we can apply the discussion of Subsection 4.1, which shows that any subword of $H = \lim H_k$ occurs in H with bounded gaps and that this word has a uniform frequency. By Theorem IV.12 and Corollary IV.14 of [46], this gives another proof of the strict ergodicity of $\mathcal{F}(\zeta)$.

Example 6. Assume that $M_0 = \{\zeta(n)\,;\, n \in \mathbf{N}\}$ is finite. There exists ℓ such that $\zeta(M_k) \subset \mathbf{U}_\ell$ for $k \geq k_0$. We denote $H \mapsto H \cdot \gamma$ the substitution $h_0 \dots h_m \cdot \gamma = (h_0\gamma) \dots (h_m\gamma)$. Then (32) becomes

$$(33)\quad H_k = (H_{k-1})(H_{k-1}\cdot\zeta(q_{k-1}))\cdots(H_{k-1}\cdot\zeta((a_k-1)q_{k-1}))(H_{k-2}\cdot\zeta(a_kq_{k-1}))$$

for $k \geq k_0 + 2$. Suitable choices of $\zeta(\varepsilon q_k)$ lead to both possibilities for $\mathcal{F}(\zeta)$: strict ergodicity, minimal (**in any case**) but not uniquely ergodic. We will give a few explicit examples in Section 8.

We have shown that if ζ and ζ_1 are α-multiplicative sequences with $n \mapsto \zeta(n)/\zeta_1(n)$ extendable to a continuous map on $\mathcal{K}_\alpha$, then both flows $\mathcal{F}(\zeta)$ and $\mathcal{F}(\zeta_1)$ are almost topologically conjugate. This leads to corresponding isomorphism in the metric sense by introducing invariant Borel measures on these flows. Let $\Lambda(\zeta)$ be the set of Borel probability measures on $\mathcal{O}_\zeta$ which are invariant under the shift action. The set $\Lambda(\zeta)$ is convex and will be endowed in the sequel with the weak topology with respect to which $\Lambda(\zeta)$ is known to be compact. The set of extremal points of $\Lambda(\zeta)$ is exactly the set of σ-invariant ergodic measures for the flow $\mathcal{F}(\zeta)$. The following theorem exhibits the best result we could expect.

Theorem 8. *Let ζ and ζ_1 be α-multiplicative sequences. Assume that neither $\Delta\zeta$ nor $\Delta\zeta_1$ is constant, and that there exists a continuous map $f_0 : \mathcal{K}_\alpha \to \mathbf{U}$ such that $\zeta(n) = \zeta_1(n)f_0(n)$ for all integers $n \geq 0$. Let $Q : \mathcal{F}(\zeta_1) \to \to \mathcal{F}(\zeta)$ be the almost isomorphism defined in the proof of Theorem 7 by (30) and (31). Then the map $\nu \mapsto \nu\circ Q^{-1}$ from $\Lambda(\zeta_1)$ to $\Lambda(\zeta)$ is an homeomorphism.*

Proof. Define

$$X(\zeta) := \mathcal{O}_\zeta \setminus \{\xi\,;\, \exists\, k \in \mathbf{N}, \sigma^k\Delta\xi = \Delta\zeta \text{ or } \sigma^k\Delta\zeta = \Delta\xi\}.$$

From the above study, the restriction $Q_{|X(\zeta_1)}$ of Q on $X(\zeta_1)$ is an homeomorphism from $X(\zeta_1)$ to $X(\zeta)$ which commutes with the shifts.

Let us show that $\nu(X(\zeta)) = 1$ for any measure ν in $\Lambda(\zeta)$. If it is not the case, by shift invariance and denombrability, we certainly have $\nu(\Delta^{-1}(\Delta\xi)) > > 0$. The measure $\nu\circ\Delta^{-1}$ is an invariant probability for the flow $\mathcal{F}(\Delta\zeta)$ which is uniquely ergodic (Theorem 3); hence $\nu\circ\Delta^{-1}(\{\Delta\xi\}) = 0$, giving a contradiction. A consequence of this result, both applied to ζ and ζ_1, is that the map $\nu \mapsto \mapsto \nu \circ Q^{-1}$ is bijective. In order to show its continuity, due to the fact that the spaces $\Lambda(\zeta)$ and $\Lambda(\zeta_1)$ are metrizable, it is enough to prove that for any continuous map $f : \mathcal{O}_\zeta \to \mathbf{R}$, if the sequence $(\nu_n)_n$ in $\Lambda(\zeta_1)$ weekly converges to ν, then $\lim_n \nu_n \circ Q^{-1}(f) = \nu \circ Q^{-1}(f)$. In fact, Q is continuous at any point of $X(\zeta_1)$, thus Q and $f \circ Q$ are ν-continuous and the expected limit follows by a standard argument in the Riemann integration theory. The continuity of $\nu \mapsto \nu \circ Q^{-1}$ is proved and by compacity, this map is also an homeomorphism.

To complete Theorem 8 we pay attention to the case $G_1(\zeta) = G(\zeta)$ (see Theorem 6).

Theorem 9. *Assume that $\Delta\zeta$ is not constant and that $G_1(\zeta) = G(\zeta)$. Let further $U : \xi \mapsto (\Xi_\zeta^{-1}(\Delta\xi), \xi_0)$ be the almost isomorphism from $\mathcal{F}(\zeta)$ to $(\mathcal{K}_\alpha \times G(\zeta), T_\zeta)$ and $\Lambda^{\sqcup}(\zeta)$ be the set of T_ζ-invariant Borel probability measures on $\mathcal{K}_\alpha \times G(\zeta)$ endowed with the weak topology. Then the map $\nu \mapsto \nu \circ U^{-1}$ realizes an homeomorphism between $\Lambda(\zeta)$ and $\Lambda^{\sqcup}(\zeta)$.*

Proof. It is not *a priori* clear that $\Lambda^{\sqcup}(\zeta)$ is weakly compact. In fact, for any $\mu \in \Lambda^{\sqcup}(\zeta)$, its first projection $\mu_{|_1}$ is the unique invariant Borel probability μ_α of the odometer. The set of discontinuity points of T_ζ being contained in $\tau^{-1}(0^\omega) \times G(\zeta)$, it is μ-negligible for any $\mu \in \Lambda^{\sqcup}(\zeta)$. Hence T_ζ is μ-continue and from a standard argument, $\Lambda^{\sqcup}(\zeta)$ is weakly closed. Introducing as above the set $X(\zeta)$, we notice that $U_{|X(\zeta)}$ is an homeomorphism between $X(\zeta)$ and $W(\zeta) = \mathcal{K}_\alpha^\bullet \times G(\zeta)$. Now $\mu(\tau^{-1}(0^\omega) \times G(\zeta)) = 0$ for $\mu \in \Lambda^{\sqcup}(\zeta)$, which implies $\mu(W(\zeta)) = 1$ and so yields bijectivity of $\nu \mapsto \nu \circ U^{-1}$. The continuity of $\nu \mapsto \nu \circ U^{-1}$ and its inverse is proved as above.

Since $\mathcal{F}(\Delta\zeta)$ is uniquely ergodic, constant or almost conjugate to the translation R_α, it is natural to consider the Anzai skew product $R_\alpha \sqcup_{C_\zeta} G(\zeta)$ given by the transformation

$$
\begin{array}{rccc}
A_\zeta : [-\alpha, 1-\alpha) \times G(\zeta) & \longrightarrow & [-\alpha, 1-\alpha) \times G(\zeta) \\
(t, \gamma) & \longmapsto & (R_\alpha(t), \gamma\, C_\zeta(t))
\end{array}
\tag{34}
$$

with $C_\zeta : [-\alpha, 1-\alpha) \to G(\zeta)$ defined by $C_\zeta(t) := \Delta\zeta\tilde{~}(\psi(t))$. Notice that if r-lim denotes the right limit on the torus identified to $[-\alpha, 1-\alpha)$, then

$$
C_\zeta(t) = \underset{n\alpha \to t \pmod 1}{\mathrm{r}-\lim} \zeta(n+1)/\zeta(n),
\tag{35}
$$

except for $t = -\alpha \pmod 1$. The map C_ζ, called ζ-cocycle associated to ζ with respect to the translation R_α, is constant by intervals (having extremities in $\mathbf{Z}\alpha + \mathbf{Z}$) which accumulate to $-\alpha$ and $1-\alpha$. In the sequel, we will simplify the notation $R_\alpha \sqcup_{C_\zeta} G(\zeta)$ into $R_\alpha \sqcup_\zeta G(\zeta)$.

Theorem 10. *Suppose that $\Delta\zeta$ is not constant and $G_1(\zeta) = G(\zeta)$. Then $V : \xi \mapsto (\varphi(\Xi_\zeta^{-1}(\Delta\xi)), \xi_0)$ realizes an almost topological isomorphism from $\mathcal{F}(\zeta)$ to $R_\alpha \sqcup_\zeta G(\zeta)$ and the related map $\nu \mapsto \nu \circ V^{-1}$ is an homeomorphism* *between* $\Lambda(\zeta)$ *and* $\Lambda(\zeta)$ *the space of* A_ζ*-invariant Borel probability measures on* $[-\alpha, 1-\alpha) \times G(\zeta)$.

The proof runs as above; details are left to the reader. Skew products over an irrational translation on the torus were introduced first by H. Anzai [2]; the interest to recognize such a structure for $\mathcal{F}(\zeta)$ is that several criteria are known to characterize the ergodicity of the product measure $\lambda \otimes h_{G(\zeta)}$, a fact which implies the unique ergodicity:

Theorem 11. *Assume that* $\Delta\zeta$ *is not constant and* $G_1(\zeta) = G(\zeta)$. *The flow* $\mathcal{F}(\zeta)$ *is uniquely ergodic if and only if the dynamical system*

$$([-\alpha, 1-\alpha) \times G(\zeta), A_\zeta, \lambda \otimes h_{G(\zeta)}), \tag{36}$$

where A_ζ *is given by (34) and (35), is ergodic.*

Proof. If $\mathcal{F}(\zeta)$ is uniquely ergodic, Theorem 10 says that the dynamical system (36) is ergodic. Reciprocally, if this system is ergodic, the underlying translation R_α being uniquely ergodic, then the skew product is also uniquely ergodic. In fact, this result follows from H. Furstenberg [23] if C_ζ is continuous (but this is generally not the case here) or from [39], the cocycle C_ζ being λ-continuous. It also derives from a general result ([13] Corollary 1).

The next two subsections give criteria for unique ergodicity of $\mathcal{F}(\zeta)$. Firstly, we choose a spectral approach which involves correlation functions and spectral measures. Secondly, we introduce (metrical) essential values of K. Schmidt [48] and compare them with the topological essential values.

7.2. Unique ergodicity and spectral charge on $\{0\}$

In this subsection we extend Theorem 8 of [40] to an α-multiplicative sequence ζ which gave a necessary and sufficient condition for $\mathcal{F}(\zeta)$ to be uniquely ergodic in the case of a $q-$multiplicative sequence. For this purpose, we recall some definitions and state three lemmas. For each $\underline{m} = (m_0, \ldots, m_s)$ in $\mathbf{Z}^{s+1}$, set $|\underline{m}| = \sum\limits_{j=0}^{s} m_j$ and associate the character $\chi_{\underline{m}} : x \mapsto x_0^{m_0} \ldots x_s^{m_s}$ on $\mathbf{U}^{\mathbf{N}}$, and the sequence $\chi_{\zeta,\underline{m}} : k \mapsto \chi_{\underline{m}}(\sigma^k(\zeta))$.

Lemma 9. *The sequence* $\chi_{\zeta,\underline{m}}$ *has a spectral Borel measure* $\Lambda_{\zeta,\underline{m}}$ *(on the torus* $\mathbf{T} := \mathbf{R}/\mathbf{Z}$*), i.e.*

$$\widehat{\Lambda}_{\zeta,\underline{m}}(k) = \int\limits_{\mathbf{T}} e^{2ik\pi t} \Lambda_{\zeta,\underline{m}}(dt) = \lim_N \frac{1}{N} \sum_{n<N} \chi_{\zeta,\underline{m}}(k+n)\overline{\chi_{\zeta,\underline{m}}(n)}.$$

Moreover for any $\nu \in \Lambda(\zeta)$, $\Lambda_{\zeta,\underline{m}}$ is also the spectral measure of $\chi_{\underline{m}}$ restricted to $\mathcal{F}(\zeta)$, i.e.

$$\widehat{\Lambda}_{\zeta,\underline{m}}(k) = \int_{\mathcal{O}_\zeta} \chi_{\underline{m}}(\sigma^k(\xi))\overline{\chi_{\underline{m}}(\xi)}\,\nu(d\xi).$$

Proof. By easy computation we obtain

$$\chi_{\underline{m}}(\sigma^k(\xi))\overline{\chi_{\underline{m}}(\xi)} = \chi_{\underline{m}}(\sigma^{k-1}(\Delta\xi))\ldots\chi_{\underline{m}}(\sigma(\Delta\xi))\chi_{\underline{m}}(\Delta\xi)$$

and so

$$\int_{\mathcal{O}_\zeta} \chi_{\underline{m}}(\sigma^k(\xi))\overline{\chi_{\underline{m}}(\xi)}\,\nu(d\xi) = \int_{\mathcal{O}_{\Delta\zeta}} \chi_{\underline{m}}\circ\sigma^{k-1}(u)\ldots\chi_{\underline{m}}\circ\sigma(u)\chi_{\underline{m}}(u)\,\nu\circ\Delta^{-1}(du).$$

The measure $\nu\circ\Delta^{-1}$ is an invariant probability measure on the flow $\mathcal{F}(\Delta\zeta)$ which is uniquely ergodic. Taking $\Delta\zeta$ as a generic point we obtain the equality

$$\int_{\mathcal{O}_\zeta} \chi_{\underline{m}}(\sigma^k(\xi))\overline{\chi_{\underline{m}}(\xi)}\,\nu(d\xi) = \lim_N \frac{1}{N}\sum_{n<N} \chi_{\zeta,\underline{m}}(k+n)\overline{\chi_{\zeta,\underline{m}}(n)}\,.$$

That proves the lemma.

It is known that for any $\nu \in \Lambda(\zeta)$ one has

$$\sqrt{\Lambda_{\zeta,\underline{m}}(\{t\})} = ||P_t(\chi_{\underline{m}})||_{2,\nu}\,,$$

where $||\cdot||_{2,\nu}$ is the quadratic norm of the Hilbert space $H_{\zeta,\nu} := \mathrm{L}^2(\mathcal{O}_\zeta,\nu)$ and P_t the orthogonal projection onto the proper subspace of $H_{\zeta,\nu}$ corresponding to the possible eigenvalue $e^{2i\pi t}$. The next lemma says explicitly the fact when the flow $\mathcal{F}(\zeta)$ is uniquely ergodic and $t=0$.

Lemma 10. *Assume $\mathcal{F}(\zeta)$ to be uniquely ergodic of unique measure μ_ζ. Then, for any $s \in \mathbf{N}$ and for any $\underline{m} \in \mathbf{Z}^{s+1}$ we have*

$$\Lambda_{\zeta,\underline{m}}(\{0\})^{1/2} = \Big|\int \chi_{\underline{m}}\,d\,\mu_\zeta\Big| = \lim_N \frac{1}{N}\Big|\sum_{n<N}\chi_{\zeta,\underline{m}}(n)\Big|.$$

Lemma 11. *We use the above notations and recall that we look at ζ and $\Delta\zeta$ as elements of $\mathbf{U}^{\mathbf{N}}$.*

1. Let be $M \in M_k \cap \mathbf{N}$ and let be an integer N such that $N + s \le q_k$. Then

$$\sum_{n<N} \chi_{\zeta,\underline{m}}(n+M) - \sum_{n<N} \chi_{\zeta,\underline{m}}(n) = \left(\zeta(M)^{|\underline{m}|} - 1\right) \sum_{n<N} \chi_{\zeta,\underline{m}}(n).$$

2. For any non-negative integer s and any $\underline{m} = (m_0, m_1, \ldots, m_s) \in \mathbf{Z}^{s+1}$, define $\underline{m'} = -(m_0, m_0 + m_1, \ldots, m_0 + m_1 + \cdots + m_{s-1}) \in Z^s$. Then

$$\chi_{\zeta,\underline{m}}(k) = \chi_{\Delta\zeta,\underline{m'}}(k)\zeta(k+s)^{|\underline{m}|}\,.$$

Proof. For any $n < N$ we have

$$\chi_{\zeta,\underline{m}}(n+M) = \zeta(n)^{m_0}\zeta(M)^{m_0} \cdots \zeta(n+s)^{m_s}\zeta(M)^{m_s} = \chi_{\zeta,\underline{m}}(n)\zeta(M)^{m_0+\cdots+m_s}.$$

The verification of **2.** is immediate.

Theorem 12. *The flow $\mathcal{F}(\zeta)$ is uniquely ergodic if and only if for any non-negative integer s and any $\underline{m} \in \mathbf{Z}^{s+1}$, $\Lambda_{\zeta,\underline{m}}(\{0\}) = 0$ or $\zeta^{|\underline{m}|}$ can be extended to a continuous function on $\mathcal{K}_\alpha$.*

Proof. Assume unique ergodicity of $\mathcal{F}(\zeta)$ and that $\Lambda_{\zeta,\underline{m}}(\{0\}) > > 0$. Let $(N_j)_j$ be an increasing sequence of positive integers such that $N_j^{-1}\left|\sum_{n<N_j} \chi_{\zeta,\underline{m}}(n)\right| \ge \kappa > 0$ for some κ and all j. For any j, let be k_j such that $N_j + s \le q_{k(j)}$. Apply Lemma 11 with with $N = N_j$ and $M = M(j) \in M_{k(j)} \cap \mathbf{N}$. We get

$$\left|\frac{1}{N_j}\sum_{n<N_j} \chi_{\zeta,\underline{m}}(n+M(j)) - \frac{1}{N_j}\sum_{n<N_j} \chi_{\zeta,\underline{m}}(n)\right| =$$

$$= \left|\zeta(M(j))^{|\underline{m}|} - 1\right|\left|\frac{1}{N_j}\sum_{n<N_j} \chi_{\zeta,\underline{m}}(n)\right|.$$

Take now the supremum for $M(j) \in M_{k(j)} \cap \mathbf{N}$. For j tending to infinity, the left-hand side of the equality tends to 0 by unique ergodicity. The right-hand side is the product of

$$\sup_{M(j) \in M_{k(j)} \cap \mathbf{N}} \left|\zeta(M(j))^{|\underline{m}|} - 1\right|$$

with a quantity that does not tend to 0 by Lemma 10. Hence, since the sequence $k \mapsto \sup\limits_{M \in M_k \cap \mathbf{N}} \left|\zeta(M)^{|\underline{m}|} - 1\right|$ is decreasing, it tends to 0 and $\zeta^{|\underline{m}|}$ can be extended to a continuous function on $\mathcal{K}_\alpha$ by Lemma 8.

It remains to prove the sufficiency. Let $s \in \mathbf{N}$ and $\underline{m} = (m_0, \ldots, m_s) \in \mathbf{Z}^s$. Assume first that $\zeta^{|\underline{m}|}$ can be extended to a continuous function on $\mathcal{K}_\alpha$. By Lemmas 11 and 4, the sequence

$$n \mapsto \chi_{\zeta,\underline{m}}(n)\zeta(n+s)^{-|\underline{m}|} = \chi_{\Delta\zeta,\underline{m}'}(n)$$

is constant on each $B \in \mathcal{B}_{s-1}$. Therefore $\chi_{\zeta,\underline{m}}$ can be extended to a continuous map on $\bigcup_{B\in\mathcal{B}_{s-1}} B = \mathcal{K}_\alpha \setminus \bigcup_{k=1}^{s} \tau^{-k}(0^\omega)$. As in the proof of Theorem 3, the unique ergodicity of $(\mathcal{K}_\alpha, \tau)$ ensures that $\mathcal{F}(\chi_{\zeta,\underline{m}})$ is uniquely ergodic, too.

Assume now that $\Lambda_{\zeta,\underline{m}}(\{0\}) = 0$. Following the idea of [40], we introduce, for $n = \sum_{j=0}^{\infty} e_j(n)q_j \in \mathbf{N}$,

$$\chi_{\underline{m}}^{(k)}(n) = \chi_{\zeta,\underline{m}}(S_k(\check{n}))\,\zeta\big(e_k(n)q_k + e_{k+1}(n)q_{k+1} + \cdots\big)^{|\underline{m}|}$$

and note that $\chi_{\underline{m}}^{(k)}(n) = \chi_{\zeta,\underline{m}}(n)$ if the successive additions $n \mapsto n+1 \mapsto \cdots \mapsto$ $\mapsto n+s$ never produce a carry at an index at least k, *i.e.* if $S_k(\check{n}) < q_k - s$. Therefrom, and since the smallest return time to a cylinder of length k is q_{k-1} (see Subsection 4.1), we have

$$\begin{aligned}
&\Big|\sum_{n<N} \chi_{\zeta,\underline{m}}(n+j) - \sum_{n<N} \chi_{\underline{m}}^{(k)}(n+j)\Big| \le \\
&\le 2\#\Big\{n<N\,;\, \chi_{\zeta,\underline{m}}(n+j) \ne \chi_{\underline{m}}^{(k)}(n+j)\Big\} \le \\
&\le 2s\left(1 + \frac{N}{q_{k-1}}\right).
\end{aligned}$$

Therefore, Lemma 1 shows that unique ergodicity of $\mathcal{F}(\zeta)$ would be a consequence of the uniform convergence with respect to j of

$$\frac{1}{N}\sum_{n<N} \chi_{\underline{m}}^{(k)}(n+j) = \frac{1}{N}\sum_{r=0}^{q_k-1} \chi_{\zeta,\underline{m}}(r) \sum_{\substack{M\in M_k\cap\mathbf{N}\\ j\le r+M<j+N}} \zeta(M)^{|\underline{m}|}$$

for all integers k (or even for k sufficiently large). Let be $r = e_0(r) + e_1(r)q_1 +$ $+ \cdots + e_{k-1}(r)q_{k-1}$. Since $(\mathcal{K}_\alpha, \tau)$ is uniquely ergodic, the means

$$\frac{1}{N}\#\big\{M \in M_k \cap \mathbf{N}\,;\, j \le r+M < j+N\big\} = \frac{1}{N}\sum_{n<N} \mathbf{1}_{[e_0(r)e_1(r)\cdots e_{k-1}(r)]}(n+j)$$

tend to $\mu_\alpha([r_0 \ldots r_{k-1}])$ uniformly in j (this could also be seen as a consequence of Proposition 5). The trivial upperbound $|\zeta(M)| \le 1$ and (14) yield

$$\frac{1}{N}\left|\sum_{n<N} \chi_m^{(k)}(n+j)\right| \ll$$

$$\ll \sum_{r<q_{k-1}} \chi_{\zeta,\underline{m}}(r)\ \big(\ |q_{k-1}\alpha| + |q_k\alpha|\ \big) + \sum_{q_{k-1}\le r<q_k} \chi_{\zeta,\underline{m}}(r)|q_{k-1}\alpha| \ll$$
$$\ll \frac{1}{q_{k-1}} \sum_{r<q_{k-1}} \chi_{\zeta,\underline{m}}(r)\big(q_{k-1}|q_k\alpha|\ \big) + \frac{1}{q_k}\sum_{r<q_k} \chi_{\zeta,\underline{m}}(r)\ \big(\ q_k|q_{k-1}\alpha|\ \big),$$

which tends uniformly to 0 by the hypothesis $\Lambda_{\zeta,m}(\{0\}) = 0$, and the classical inequalities

$$\limsup \frac{1}{N}\Big|\sum_{n<N} \chi_{\zeta,\underline{m}}(n)\Big| \le \Lambda_{\zeta,\underline{m}}(\{0\})^{1/2}$$

(the sequence $\chi_{\zeta,\underline{m}}$ having a unique spectral measure) and $q_k||q_{k-1}\alpha|| \le 1$.

7.3. Unique ergodicity and essential values

In this subsection we apply techniques of [48] to the skew product $\mathcal{T}(\zeta) :=$ $:= (\mathcal{K}_\alpha \times G(\zeta), T_\zeta, \mu_\alpha \otimes h_{G(\zeta)})$. Recall that T_ζ is given by $T_\zeta(x,\gamma) = \big(\tau x, \gamma\Delta\tilde{\zeta}(x)\big)$ (of course, the method can be also applied to the dynamical system (36)).

Let be for a while G a locally compact metrizable abelian group, the group law being denoted multiplicatively. A measurable map $f : \mathcal{K}_\alpha \to G$ is viewed as the so-called τ-cocycle $(x,n) \mapsto f_n(x)$ defined by

$$f_n(x) = \begin{cases} \prod_{k=0}^{n-1} f(\tau^k x) & \text{if } n \ge 1, \\ \left(\prod_{k=1}^{-n} f(\tau^{-k} x)\right)^{-1} & \text{if } n \le -1, \\ 1_G & \text{if } n = 0. \end{cases}$$

Following [48] (see also the eighth chapter of [1]), $\gamma \in G$ is an essential value of f with respect to τ if for any Borel set B of $\mathcal{K}_\alpha$ with $\mu_\alpha(B) > 0$ and any neighborhood V of γ in G, one has

$$\mu_\alpha\Big(\bigcup_{n\in\mathbf{Z}} \big(B \cap \tau^{-n}B \cap \{x \in \mathcal{K}_\alpha\,;\, f_n(x) \in V\}\big)\Big) > 0\,. \tag{37}$$

Moreover, f is said to have an essential value at infinity if (37) holds for any V such that $A\setminus V$ is compact. This notion at infinity is irrelevant if A is compact. Notice that in case $G = G(\zeta)$, the corresponding products $\Delta\tilde{\zeta}_n$ are well defined and continuous at each point of $\mathcal{K}_\alpha^\bullet$. A measurable map $f_0 : \mathcal{K}_\alpha \to G$ is a τ-coboundary with transfer function $\Im\colon \mathcal{K}_\alpha \to G$ if $\Im$ is measurable and $f_0(x) = \Im(\tau x)/\Im(x)$ for μ_α-almost every $x \in \mathcal{K}_\alpha$. Of course, these definitions can be given for any standard ergodic dynamical system $(X, T, \mathcal{B}, \mu)$ which takes the place of the odometer. Let be $E(f)$ the set of essential values of f and let us recall the following facts (see [48] and [1]):

- The set $E(f)$ is a closed subgroup of G.
- Multiplying f by a τ-coboundary does not affect the set of essential values.
- For any character $\chi : G \to \mathbf{U}$, $\chi(E(f)) \subset E(\chi \circ f)$.
- The cocycle f is a coboundary if and only if f has not an essential value at infinity and $E(f) = \{1\}$.
- There is a (measurable) coboundary f_0 such that the cocycle $f.f_0$ is $E(f)$-valued.

As a consequence of the general result of K. Schmidt asserting that the skew product $(X \times G, T^f, \mathcal{B}(X \times G), \mu \otimes h_G)$ with $T^f(x, g) = (Tx, gf(x))$ is ergodic if and only if $E(f) = G$, one has

Theorem 13. *The dynamical system $\mathcal{T}(\zeta) := (\mathcal{K}_\alpha \times G(\zeta), T_\zeta, \mu_\alpha \otimes h_{G(\zeta)})$ is ergodic if and only if $E(\Delta\zeta) = G(\zeta)$ (in that case, $\mu_\alpha \otimes h_{G(\zeta)}$ is the unique T_ζ-invariant measure ([13] Corollary 1)).*

Definition given by (37) is difficult to handle with in practice. It is possible to exhibit essential values only by taking care of cylinders instead of Borel sets.

Lemma 12. *Let $\mathcal{V}(\gamma)$ denote the set of neighborhoods of $\gamma \in G(\zeta)$. If*

(38) $$\forall V \in \mathcal{V}(\xi),\ \exists \kappa > 0,\ \forall n \in \mathbf{N},\ \forall C \in \sigma(\pi_0, \pi_1, \ldots, \pi_n),\ \exists m \in \mathbf{Z}\,,$$

$$\mu_\alpha\left(C \cap \tau^{-m}(C) \cap \{x \in \mathcal{K}_\alpha\,;\, \Delta\zeta\ \ tilde_m(x) \in V\}\right) \geq \kappa\mu_\alpha(C)$$

holds, then $\xi \in E(\Delta\zeta)$.

Proof. Indeed, let B be a Borel set. For any $\varepsilon > 0$, there exists $n \in \mathbf{N}$ and $C \in \sigma(\pi_0, \pi_1, \ldots, \pi_n)$ such that $\mu_\alpha(B\Delta C) \leq \varepsilon$. Writing

$$C \cap \tau^{-m} C = [(C \cap B) \cap \tau^{-m}(C \cap B)] \cup [(C \setminus B) \cap \tau^{-m}(C \cap B)] \cup [C \cap \tau^{-m}(C \setminus B)]$$

yields $\mu_\alpha(C \cap \tau^{-m} C) \leq \mu_\alpha(B \cap \tau^{-m} B) + 2\varepsilon$, hence

$$\mu_\alpha\left(B \cap \tau^{-m}(B) \cap \{x \in \mathcal{K}_\alpha\,;\, \Delta\zeta_m(x) \in V\}\right) \geq \kappa\mu_\alpha(C) - 2\varepsilon > 0$$

provided that ε is small enough and that C satisfies (38) for m, and ξ is an essential value of $\Delta\zeta$.

Notice that if ζ can be extended to a continuous function f on $\mathcal{K}_\alpha$, then $\Delta\zeta$ is trivially a coboundary (with $\Im = f$) and $E(\Delta\zeta) = \{1\} \neq G(\zeta)$, provided that ζ is not constant. Therefore, $(\mathcal{K}_\alpha \square G(\zeta), T_\zeta)$ is not uniquely ergodic, but contains the uniquely ergodic components $F_\gamma := \{(x, f(x)\gamma), x \in \mathcal{K}_\alpha\}$, each of them being finitary isomorphic to the odometer itself which is finitary isomorphic to $\mathcal{F}(\zeta)$ (see Theorem 5).

Proposition 8. *For any α-multiplicative sequence ζ, any essential value is a topological essential value: $E(\Delta\zeta) \subset G_1(\zeta)$.*

Proof. Choose $\gamma \in E(\Delta\zeta)$, V a closed neigborhood of γ and $B = M_k$ in (37). There is an integer $n_k \in \mathbf{Z}$ such that $\mu_\alpha(B \cap \tau^{-n_k} B \cap \{\Delta\tilde{\zeta}_{n_k} \in V\}) > 0$. But $\Delta\tilde{\zeta}_{n_k}$ is constant on cylinder sets and $B \cap \tau^{-n_k} B$ is a finite union of cylinder sets. Therefore, $B \cap \tau^{-n_k} B$ contains a cylinder set C_k on which $\Delta\tilde{\zeta}_{n_k}$ take a constant value in V and so, there is an integer m_k in C_k, hence in M_k, such that $\Delta\tilde{\zeta}_{n_k}(m_k) \in V$. But $\Delta\tilde{\zeta}_{n_k}(m_k) = \zeta(m_k + n_k)/\zeta(m_k)$ by definition and $k \mapsto \overline{M_k}$ converges to $G_1(\zeta)$ with respect to the Hausdorff distance. Therefore, by a compacity argument, there exists $J \subset \mathbf{N}$ such that the limits

$$\lim_{k\in J} \zeta(m_k) = u, \quad \lim_{k\in J} \zeta(m_k + n_k) = v$$

exist with u and v in $G_1(\zeta)$ and $vu^{-1} = \lim_{k\in J} \zeta(m_k + n_k)/\zeta(m_k) \in V$, proving that γ is an essential topological value.

Theorem 14. *The flow $\mathcal{F}(\zeta)$ is uniquely ergodic if and only if $E(\Delta\zeta) =$ $= G_1(\zeta)$.*

Proof. We distinguish several cases. If $\Delta\zeta$ is constant, equal to θ, then $G_1(\zeta) = \mathbf{U}(\theta) = G(\zeta)$ and we know that $\mathcal{F}(\zeta)$ is isomorphic to the translation $z \mapsto z\theta$ on $\mathbf{U}(\theta)$, which is uniquely ergodic.

Now we assume that $\Delta\zeta$ is not constant, but $\zeta = \zeta_1\zeta_0$ with $\Delta\zeta_1$ constant, equal to θ, and ζ_0 extendable to a continuous map f_0 on $\mathcal{K}_\alpha$. By Theorem 7 part *(ii)*, and Theorem 5, $\mathcal{F}(\zeta)$ is almost conjugate to the direct product $(\mathcal{K}_\alpha \times \mathbf{U}(\theta), (x, \gamma) \mapsto (\tau x, \gamma\theta))$. On the other hand, $\Delta\tilde{\zeta} = \Delta\tilde{\zeta_1}\Delta\tilde{\zeta_0}$ with $\Delta\tilde{\zeta_0} = f_0 \circ \tau / f_0$ implying $E(\Delta\zeta) = E(\Delta\zeta_1) = \mathbf{U}(\theta)$. Theorem 13 asserts that $(\mathcal{K}_\alpha \times \mathbf{U}(\theta), (x, \gamma) \mapsto (\tau x, \gamma\theta), \mu_\alpha \otimes h_{\mathbf{U}(\theta)})$ is ergodic, hence uniquely ergodic and so is $\mathcal{F}(\zeta)$ (notice that this proof includes the case $\theta = 1$).

Finally, assume that $\zeta = \zeta_1\zeta_0$ with $\Delta\zeta$ and $\Delta\zeta_1$ not constant, and ζ_0 extendable to a continuous function on $\mathcal{K}_\alpha$. Then by Theorem 7 part *(iv)*, and Theorem 13, the flow $\mathcal{F}(\zeta)$ is ergodic if and only if $E(\Delta\zeta_1) = G_1(\zeta)$. Since $E(\Delta\zeta) = E(\Delta\zeta_1)$, the result follows.

8. Applications

8.1. $\zeta(\mathbf{N})$ finite and α with bounded partial quotients

If ζ takes finitely many values, the compact sets $\overline{\zeta(M_k)}$ and $\zeta(M_k)$ are equal, there exists k_0 for which

$$\zeta(M_{k_0}) = G_1(\zeta)\,,$$

and we derive the factorization $\zeta = \zeta_1\zeta_0$ of Theorem 7 with $G(\zeta_1) = G_1(\zeta)$ and f_0 only dependent of the first k_0-digits.

Theorem 15. *Assume that $\alpha = [0; a_1, a_2, a_3, \ldots]$ has bounded partial quotients a_j. Let ζ be an α-multiplicative sequence taking finitely many values and such that $G(\zeta) = G_1(\zeta)$ and $\Delta\zeta$ is not constant. Then $E(\Delta\zeta) = G_1(\zeta)$, or equivalently, $\mathcal{F}(\zeta)$ is uniquely ergodic, finitary isomorphic to the skew product $\mathcal{T}(\zeta)$.*

Proof. We may assume that ζ is not extendable to a continuous map on $\mathcal{K}_\alpha$, otherwise by the assumption $G_1(\zeta) = \{1\} = G(\zeta)$, ζ must be trivial and so $\Delta\zeta = 1$. The flow $\mathcal{F}(\Delta\zeta)$ is finitary conjugate to the odometer and it is enough to prove that $\mathcal{T}(\zeta)$ is ergodic, hence to prove that $E(\Delta\zeta) = G_1(\zeta)$. For any $C \in \sigma(\pi_0, \ldots, \pi_n)$ and any integer $\ell \geq 2$ let us show that

$$m_b(n,\ell) := \mu_\alpha\{x \in C\,;\; x_{n+\ell-1}x_{n+\ell}x_{n+\ell+1} = 000\} \geq \kappa\mu_\alpha(C) \tag{39}$$

where κ is a positive constant only dependent of $A := \max\limits_n a_n$. Using the Markov chain structure studied in Theorem 2 and the inequality

$$\mu_\alpha(\pi_m = 0 | \pi_{m-1} = a') \geq \frac{||q_m\alpha|| + ||q_{m+1}\alpha||}{||q_{m-1}\alpha|| + ||q_m\alpha||}$$

one gets

$$m_b(n,\ell) \geq \mu_\alpha(C) \prod_{j=-1}^{+1} \frac{||q_{n+\ell+j}\alpha|| + ||q_{n+\ell+j+1}\alpha||}{||q_{n+\ell-1+j}\alpha|| + ||q_{n+\ell+j}\alpha||} =$$

$$= \mu_\alpha(C)\frac{||q_{n+\ell+1}\alpha|| + ||q_{n+\ell+2}\alpha||}{||q_{n+\ell-2}\alpha|| + ||q_{n+\ell-1}\alpha||}\,.$$

Taking into account the classical inequalities $\frac{1}{q_{m+1}} > ||q_m\alpha|| > \frac{1}{2q_{m+1}}$ and $\frac{q_m}{q_{m+1}} > \frac{1}{A+1}$, we obtain

$$\frac{||q_{n+\ell+1}\alpha|| + ||q_{n+\ell+2}\alpha||}{||q_{n+\ell-2}\alpha|| + ||q_{n+\ell-1}\alpha||} \geq \frac{1}{2}\frac{1}{(A+1)^4}$$

and inequality (39) finally holds with $\kappa = \frac{1}{2(1+A)^3}$.

Let Γ be the set of $\gamma \in G_1(\zeta)$, $\gamma \neq 1$, such that the equality $\zeta(r_m q_m) = \gamma$ holds for infinitely many integers m with $1 \leq r_m \leq a_{m+1}$. This set generates $G_1(\zeta)$. For a given $\gamma \in \Gamma$, choose a sequence of couples of integers (n_k, e_k) with $n_k < n_{k+1}$, $1 \leq e_k \leq a_{n_k+1}$ and $\zeta(e_k q_{n_k}) = \gamma$. For each $C \in \sigma(\pi_0, \ldots, \pi_n)$ take k and ℓ verifying $\ell \geq 2$, $n + \ell = n_k$. Any $x \in C$ with $x_{n_k-1}x_{n_k}x_{n_k+1} = 000$ verifies $x \in \tau^{-e_k q_{n_k}}C$ and $\Delta\tilde{\zeta}_{e_k q_{n_k}}(x) = \zeta(e_k q_{n_k}) = \gamma$. The hypothesis of Lemma 12 is fulfilled, hence $\gamma \in E(\zeta)$.

Theorem 15 can be slightly improved using the same method to obtain the next theorem, whose proof is left to the reader.

Theorem 16. *Assume that α has bounded partial quotients, $\Delta\zeta$ is not constant and $G(\zeta) = G_1(\zeta)$. Furthermore suppose that there is a finite subset B of $\mathbf{U}$ such that for all $n \geq 1$,*

$$\zeta(\{eq_n\,;\ 0 < e \leq a_{n+1}\}) \subset B.$$

Then, $\mathcal{F}(\zeta)$ is uniquely ergodic, finitary isomorphic to the skew product $\mathcal{T}(\zeta)$.

8.2. Cases with unbounded partial quotients

The hypothesis on α in Theorem 15 is essential. The next example builds ζ with $E(\Delta\zeta) = \{1\}$ but $G(\zeta) = G_1(\zeta) = \{-1, 1\}$ in case α has unbounded partial quotients. The construction can be modified to get for $G(\zeta) = G_1(\zeta)$ any closed subgroup of $\mathbf{U}$.

Example 7. Let $\alpha = [0; a_1, a_2, a_3, \ldots]$ with unbounded partial quotients and let $(n_k)_{k\geq 0}$ be an increasing sequence of integers such that $n_{k+1} \geq n_k + 3$, $n_0 \geq 2$, $a_{n_k} \geq 2$ and the series $\sum\limits_{k\geq 0} \frac{1}{a_{n_k+1}}$ converges. Define the α-multiplicative sequence ζ by $\zeta(a_{n_k+1}q_{n_k}) = -1$ and $\zeta(eq_m) = 1$ otherwise. By construction $G_1(\zeta) = \{-1, 1\} = G(\zeta)$. The Borel set

$$E := \{x \in \mathcal{K}_\alpha^\bullet\,;\ \exists\, k = k(x) \geq 0, \forall\, r \geq k,\ x_{n_r} \neq a_{n_r+1}\}$$

being τ-invariant, $\mu_\alpha(E) = 0$ or 1. The event E contains $E^\infty := \bigcap_k E^{(k)}$, where $E^{(k)} := \{x \in \mathcal{K}_\alpha^\bullet\,;\ \forall\, r \le k,\ x_{n_r} \neq a_{n_r+1}\}$. Set $E^{(k)}(a) = \{x \in E^{(k)}; x_{n_k-2} = a\}$. The Markov chain $(\pi_n)_n$ (Theorem 2) allows to write

$$\mu_\alpha(E^{(k)}) = \mu_\alpha(E^{(k-1)}) -$$

$$- \sum_{a'=0}^{a_{n_k-1}} \mu_\alpha(E^{(k-1)}(a'))\mu_\alpha(\pi_{n_k-1} = 0 | \pi_{n_k-2} = a')\mu_\alpha(\pi_{n_k} = a_{n_k+1} | \pi_{n_k-1} = 0) \ge$$

$$\ge \mu_\alpha(E^{(k-1)})(1 - \mu_\alpha(\pi_{n_k} = a_{n_k+1} | \pi_{n_k-1} = 0)).$$

Notice that

$$\begin{aligned}\mu_\alpha(\pi_{n_k} = a_{n_k+1} | \pi_{n_k-1} = 0) &= \frac{||q_{n_k}\alpha||}{||q_{n_k-1}\alpha|| + ||q_{n_k}\alpha||} = \\ &= \frac{||q_{n_k}\alpha||}{(a_{n_k+1}+1)||q_{n_k}\alpha|| + ||q_{n_k+1}\alpha||} \le \\ &\le \frac{1}{a_{n_k+1}+1}\,,\end{aligned}$$

leading to

$$\mu_\alpha(E^{(k)}) \ge \mu_\alpha(E^{(k-1)})\Big(1 - \frac{1}{a_{n_k+1}+1}\Big) \ge \mu_\alpha(E^{(0)}) \prod_{r=0}^{k} \Big(1 - \frac{1}{a_{n_r+1}+1}\Big)\,.$$

The choice of the a_{n_k} implies that the infinite product $\prod_{r\ge 0}\big(1 - \frac{1}{a_{n_r+1}+1}\big)$ converges to a strictly positive number ρ while $\lim_k \mu_\alpha(E^{(k)}) = \mu_\alpha(E^\infty)$. Consequently, $\mu_\alpha(E) \ge \rho\mu_\alpha(E^{(0)}) > 0$. Thus $\mu_\alpha(E) = 1$; then, with the notation of Lemma 8, $\tilde\zeta(x) := \lim_n \zeta(S_n(x))$ exists for μ_α-almost every x. Since those x verify also the relation $\Delta\zeta^\sim(x) = \tilde\zeta(\tau x)/\tilde\zeta(x)$, the map $\Delta\zeta^\sim$ is a coboundary, hence $E(\Delta\zeta) = \{1\}$.

The automorphism $(x, \gamma) \mapsto (x, \gamma\tilde\zeta(x))$ identifies $\mathcal{F}(\zeta)$ with the union of two copies of $\mathcal{T}(1) := (\mathcal{K}_\alpha \times \{1\}, T_1, \mu_\alpha \otimes h_{\{1\}})$, $T_1(x, 1) = (\tau x, 1)$, which is uniquely ergodic. The flow $\mathcal{F}(\zeta)$ furnishes a simple example of minimal flow which admits exactly two ergodic measures.

Example 8. In the above construction we replace $\zeta(a_{n_k+1}q_{n_k}) = -1$ by

$$\zeta(a_{n_k+1}q_{n_k}) = \theta, (\theta \text{ fixed});$$

then $G_1(\zeta) = \mathbf{U}(\theta)$ and by the same arguments given in Example 7, we still have $E(\zeta) = \{1\}$.

Notice that if $\mathbf{U}(\theta) = \mathbf{U}_\ell$, the flow $\mathcal{F}(\zeta)$ is made of ℓ copies of $\mathcal{T}(1)$ (metrically isomorphic to the α-odometer). If $\mathbf{U}(\theta) = \mathbf{U}$, the flow $\mathcal{F}(\zeta)$ is metrically isomorphic to an uncountable union of copies of $\mathcal{T}(1)$.

8.3. The sum-of-digits function

Recall the definition $s_\alpha(x) = \sum_{k\geq 0} e_k(n)$ and fix $z \in \mathbf{U}$. We study the structure of the flow associated to the α-multiplicative sequence ζ_z defined by

$$\zeta_z(n) = z^{s_\alpha(n)}.$$

Proposition 9. *With the above notations, $\Delta\zeta_z$ is constant if and only if z is a root of unity such that*

$$z^{a_1-1} = z \quad \& \quad \forall\, k \geq 2,\, z^{a_k} = z.$$

In that case $\mathcal{F}(\zeta_z)$ is topologically isomorphic to the translation $x \mapsto xz$ on the group of unity generated by z.

Proof. The possible values of the difference map are (see Section 5)

$$\Delta\tilde{\zeta_z}(x) = \begin{cases} z & \text{if } x_0 \neq a_1 - 1; \\ z/z^{(a_1-1)+\ldots+a_{2r+1}} & \text{if } x \in [Q_{2r}jk] \text{ with } 0 \leq j \leq a^*_{2r+1} - 1 \\ & \qquad \text{and } k \leq a_{2r+2}\ (r \geq 0); \\ z/z^{a_2+\ldots+a_{2r+2}} & \text{if } x \in [Q_{2r+1}jk] \text{ with } 0 \leq j \leq a^*_{2r+2} - 1 \\ & \qquad \text{and } k \leq a_{2r+3}\ (r \geq 0). \end{cases}$$

The proposition follows and $\Delta\zeta_z$ takes the constant value z.

Theorem 17. *Assume that $\Delta\zeta_z$ is not constant. Then the group of essential values coincides with the group of topological essential values which is the closed group $\mathbf{U}(z)$ generated by z. As a consequence, $\mathcal{F}(\zeta_z)$ is strictly ergodic and finitary isomorphic to the skew product $\mathcal{T}(\zeta_z)$.*

Proof. If α has bounded partial quotients, apply Theorem 16 to conclude. Otherwise, we may assume $a_m \geq 5$ for infinitely many m. We use Lemma 12. For $n \in \mathbf{N}$ and any $C \in \sigma(\pi_0, \pi_1, \ldots, \pi_n)$, $m \geq n+3$, $0 \leq a \leq a_{m+1}$, set

$C_m(a) = C \cap \{x \in \mathcal{K}_\alpha\,;\; x_m = a\}$ and notice that $C = \bigcup_{b\in\mathcal{S}_{m-1}} ([b] \cap C)$. With $C'_m = \bigcup_{0<a<a_{m+1}-1} C_m(a)$, one gets

$$\begin{aligned}\mu_\alpha(C'_m|C) &= 1 - \mu_\alpha(\pi_m = 0|C) - \mu_\alpha(\pi_m = a_{m+1}|C) \geq \\ &\geq 1 - \frac{||q_m\alpha|| + ||q_{m+1}\alpha||}{||q_{m-1}\alpha|| + ||q_m\alpha||} - \frac{||q_m\alpha|| + ||q_{m+1}\alpha||}{||q_{m-1}\alpha||} \geq \\ &\geq 1 - \frac{4}{a_{m+1}}.\end{aligned}$$

Therefore, for infinitely many $m \geq n+2$ one has both $a_m \geq 5$ and

$$\mu_\alpha(C') \geq \frac{1}{5}\mu_\alpha(C)\,,$$

and by construction $(\Delta\zeta_z)\tilde{}_{q_m}(x) = z$ for any $x \in C'$, giving

$$\mu_\alpha\left(C \cap \tau^{-q_m}(C) \cap \{x \in \mathcal{K}_\alpha\,;\, \Delta\zeta\ \tilde{,}_{q_m}(x) = z\}\right) \geq \frac{1}{5}\mu_\alpha(C).$$

This proves that z is an essential value and finally $G(\zeta_z) = G_1(\zeta_z) = E(\Delta\zeta_z) = = \mathbf{U}(z)$.

We show on this typical example how to retrieve a result of J. Coquet [12] on the distribution of the sum-of-digits.

Corollary 5. *For any $z \in \mathbf{U}$, the sequence $n \mapsto z^{s_\alpha(n)}$ is well distributed in the group $\mathbf{U}(z)$.*

In fact, the case $\Delta\zeta_z$ constant is obvious. Otherwise $\mathcal{F}(\zeta_z)$ is uniquely ergodic and finitary isomorphic to the skew product $\mathcal{T}(\zeta_z)$ so that by Lemma 1, for any non trivial character χ of $\mathbf{U}(z)$,

$$N \mapsto \frac{1}{N}\sum_{n<N} \chi((\sigma^{n+j}\zeta_z)_0)$$

converges uniformly in j to $\int_{\mathbf{U}(z)} \chi(g)h_{\mathbf{U}(z)}(dg)$ $(= 0)$, when N tends to infinity. Since $\chi((\sigma^{n+j}\zeta_z)_0) = \chi(z^{s_\alpha(n+j)})$, the corollary follows.

Remark 8. If we replace $\mathbf{U}(z)$ by any compact monothetic group G generated by z, the conclusion of Theorem 17 is still true and the sequence $n \mapsto z^{s_\alpha(n)}$ is well distributed in the group G again.

Remark 9. Replace $\mathbf{U}(z)$ by $\mathbf{Z}$ in the above study and let Δs_α be the additive difference of s_α, which is extended to $\mathcal{K}_\alpha$ by

$$\Delta s_\alpha\tilde{}(x) = \lim_{\check{n}\to x}(s_\alpha(n+1) - s_\alpha(n)) \quad (\in \mathbf{Z})$$

if $\tau(x) \neq 0^\omega$ and $\Delta s_\alpha(x) = 0$ otherwise. Now, consider the cylindric flow $\mathcal{Z}(s_\alpha) := (\mathcal{K}_\alpha \times \mathbf{Z}, T_{s_\alpha}, h_\alpha \otimes h_{\mathbf{Z}})$ defined by

$$T_{s_\alpha}(x,n) = (\tau x, n + \Delta s_\alpha\tilde{}(x)).$$

A similar proof of that of the above theorem shows that 1 is an essential value of $\Delta s_\alpha\tilde{}$, hence $E(\Delta s_\alpha\tilde{}) = \mathbf{Z}$ and $\mathcal{Z}(s_\alpha)$ is ergodic.

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