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ON SOME MULTIPLICATIVE FUNCTIONS AND VECTOR SPACES OF ARITHMETICAL FUNCTIONS

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Dedicated to Professor Imre Kátai on his 65th birthday

1. Introduction

1.1. Some introductory remarks and comments

Multiplicative functions of modulus less or equal to 1 are of considerable importance in number theory. In this article, we shall study some properties of these functions from the viewpoint of a classical analyst. More precisely, in a first part, we shall consider the space M of the completely multiplicative functions with values in T, the unit circle. It will be shown that there is a natural group topology on it, inspired directly from results of probabilistic number theory, which leads to a precise description of this space and of some of its properties, and we shall determine its dual group (in the sense of Pontriagin). Then, in a second part, we shall consider some spaces of arithmetical functions obtained as a closure of special algebras associated to multiplicative functions with values in a finite subgroup $\mathcal{F}$ of T. It will be shown that these spaces can be viewed essentially as sums of copies of spaces of limit-periodic sequences (in the sense of Besicovitch), and that their elements are characterized by a generalized Fourier series. This second part is also related to a research work of Indlekofer on uniformly summable functions [1], since it will be shown that the spaces of uniformly summable functions built via multiplicative functions with values in a finite subgroup $\mathcal{F}$ of T are exactly the spaces of arithmetical functions obtained as closures of our special algebras associated to multiplicative functions with values in this finite group. And so, a characterization of such spaces of uniformly summable functions will be given.

Most of the proofs in the first part will be detailed, while in the second part, we shall refer sometimes to the analogous statements which correspond in the first part: the reason is that there is no special difficulty to transfer some of the proofs given for completely multiplicative functions of modulus 1 to the case of ordinary multiplicative functions with values in $\mathcal{F} \cup \{0\}$. Moreover, concerning this second part, it is assumed that the reader has some familiarity with the classical theory of multiplicative functions, and their relations with almost periodic functions, as it is presented for instance in the book of W. Schwarz and J. Spilker [4]. This will allow to shorten some of the proofs.

As a last remark, I would like to mention that by choice, the present work will ultimately deal with the ordinary mean value, or first Cesaro mean, for it is a solid tradition in the theory of arithmetical functions. Other choices exist. And this, not only for the mean value, but also for the choice of dealing with a finite group: much more general settings can be considered using the methods developped in the present paper. As written by K.-H. Indlekofer in [2], *a general problem of probabilistic number theory is to find appropriate probability spaces where large classes of arithmetical functions can be considered as random variables.* Among other things, the present article will provide in a simple way a quite precise description of such probability spaces where the class of multiplicative functions with values in a finite subgroup of T can be considered as random variables.

1.2. Acknowledgements

The question of the description of the spaces of uniformly summable functions built via multiplicative functions with values in a finite subgroup of T was initially raised by Pr. E. Saias at the end of a conference of Pr. Indlekofer in Paris (02/09/2000). Recent discussions with some colleagues working on arithmetical functions lead me to reconsider this problem. I thank especially Prs. K.H. Indlekofer, L. Lucht, E. Manstavicius, W. Schwarz.

1.3. Notations and definitions

Z (resp.N, resp.N^*) is the set of the integers (resp. non-negative integers, resp. positive integers).

P is the set of the prime numbers, and p will denote a generic element of P.

Q (resp.Q_+^*) is the set of the rational numbers (resp. positive rational numbers).

C (resp. R) is the set of the complex (resp. real) numbers, and R_d will denote the group of the real numbers equipped with the discrete topology.

T is the set of the complex numbers of modulus 1, and $\mathcal{F}$ a finite subgroup of T.

Let f be an arithmetical function, i.e. a function $N^* \to C$. f is multiplicative function if $f(1) = 1$ and $f(mn) = f(m)f(n)$ when $\gcd(m, n) = 1$. If $f(mn) = f(m)f(n)$ for all m and n, f is called a completely multiplicative function.

$M(C)$ will denote the set of the C-valued multiplicative functions.

M is the set of the complex-valued completely multiplicative functions of modulus 1.

If $a(n)$ is a complex (resp. real) sequence indexed by N^*, we denote by $m(a)$ (resp. $\overline{m}(a)$) the limit (resp. the upper limit), when it exists, of the expression $(1/x) \sum\limits_{n \le x} a(n)$ when x tends to infinity.

p being in P and α in N^*, $p^\alpha \mid n$ (resp. $p^\alpha \nmid n$) means that p^α divides n (resp. p^α does not divide n) while $p^\alpha \| n$ means that p^α divides exactly n, i.e. $p^\alpha | n$ but $p^{\alpha+1} \nmid n$.

1.4. Toolbox

We recall some results on $M(C)$ that we shall need later.

Theorem 1. *Let h be an element of $M(C)$ such that $|h| \le 1$. Then,*

$$\limsup_{x\to\infty} (1/x) \left| \sum_{1 \le n \le x, (n,2)=1} h(n) \right| > 0$$

if and only if there exists τ_h in R such that

$$\sum_{p \in P} \left(1 - \operatorname{Re}\, h(p) p^{-i\tau_h}\right) p^{-1}$$

converge.

This is the well-known Halász's Theorem [4]. Another useful result is

Theorem 2. *Let h be an element of $M(C)$ such that $|h| \le 1$. Then,*

$$\lim_{x\to+\infty} (1/x) \sum_{1 \le n \le x,\ (n,2)=1} h(n)$$

exists and is not zero if and only if

$$\sum_{p \in P} (1 - h(p)) p^{-1}$$

converge.

More precisely,

$$\lim_{x\to+\infty} x^{-1}\sum_{1\le n\le x} h(n) = \lim_{x\to+\infty}\prod_{p\le x}\left(1-\frac{1}{p}\right)\sum_{0\le k} h(p^k)p^{-k}.$$

This result is due to Delange [4].

Another needed result of Delange is the following (Delange Extended Theorem [4]):

Theorem 3. *Let h be an element of $M(C)$ such that $|h|\le 1$. Then, the condition*

$$\sum_{p\in P}(1-\mathrm{Re}\ h(p))p^{-1}$$

converge, implies that

$$\lim_{x\to+\infty}\left(x^{-1}\sum_{1\le n\le x} h(n)-\prod_{p\le x}\left(1-\frac{1}{p}\right)\sum_{0\le k} hp^k p^{-k}\right)=0.$$

Now, we recall the following result of Ruzsa [3]:

Theorem 4. *Let $\mathcal{G}$ be an abelian group and f a $\mathcal{G}$-valued additive arithmetical function. Then, given any a in $\mathcal{G}$, the set of the integers n such that $f(n)=a$ has a density.*

Another useful result for our purpose in the following statement of Weil [5]:

Theorem 5. *Let G be a topological abelian group and g a closed subgroup of G such that G/g is discrete. Assume that any element a of g has a n-th root for any n in N^*, i.e. there exists some b such that $b^n=a$ (we shall say that g is divisible). Then, G is isomorphic to $(G/g)\times g$.*

And we can recall that the dual group (in the sense of Pontriagin) of the product of two abelian groups is the product of the dual groups of these groups.

2. The space M

2.1. M as a topological group

We recall that M is the set of the complex-valued completely multiplicative functions of modulus 1.

Proposition 6. *M is an abelian group.* This is evident.

Proposition 7. *On M we define a topology in the following way: for f and g in M, let c be defined by*

$$c(f,g) = \sqrt{\sum_{p\in P} |f(p)-g(p)|^2 . p^{-1}}$$

if this quantity is finite, $c(f,g) = +\infty$ if not. Then, the non-negative real-valued function d defined on $M \times M$ by $d(f,g) = c(f,g)(1+c(f,g))^{-1}$ is a distance on M, invariant by translation.

Proof. First of all $d(f,g)$ is non-negative real-valued and since $c(f,g) = = c(g,f)$, it is clear that $d(f,g) = d(g,f)$. Moreover, $d(f,g)$ is equal to 0 if and only if $c(f,g) = 0$, which gives that for all p in P we have $f(p) = g(p)$, and since f and g are completely multiplicative, this implies that $f = g$.

We remark now that if f, g, h are in M, we have

$$\begin{aligned} c(hf,hg) &= \sqrt{\sum_{p\in P} |h(p)f(p)-h(p)g(p)|^2 . p^{-1}} = \\ &= \sqrt{\sum_{p\in P} |h(p)(f(p)-g(p))|^2 . p^{-1}} = \\ &= \sqrt{\sum_{p\in P} |h(p)||f(p)-g(p)|^2 . p^{-1}} = \\ &= \sqrt{\sum_{p\in P} |f(p)-g(p)|^2 . p^{-1}} = \\ &= c(f,g), \end{aligned}$$

since $|h(p)| = 1$ for all p in P, and this gives the translation invariance. It remains to check that the triangular inequality holds. We have, if f, g, h are in M

$$c(f,h) \leq c(f,g) + c(g,h),$$

by the Cauchy inequality.

Now, we remark that the function $x \to x(1+x)^{-1}$ is increasing on the positive axis, and so, if x, y, z are non-negative real numbers such that $z \leq x+y$, we have

$$\begin{aligned} z(1+z)^{-1} &\leq (x+y).(1+(1+y))^{-1} \leq \\ &\leq x(1+x+y)^{-1} + y(1+x+y)^{-1} \leq \\ &\leq x(1+x)^{-1} + y(1+y)^{-1}, \end{aligned}$$

and as a consequence, we have

$$c(f,h).(1+c(f,h))^{-1} \leq c(f,g).(1+c(f,g))^{-1} + c(g,h).(1+c(g,h))^{-1},$$

i.e.

$$d(f,h) \leq d(f,g) + d(g,h).$$

We denote by M' the subset of M defined by

$$M' = \{f \in M,\ d(f,1) < 1\}.$$

Theorem 8. *M' is a subgroup of M, open, complete and separable for the topology defined by d. As a consequence, the quotient $S = M/M'$ is discrete, and so, there exists a set of representatives of S such that $M = \underset{s \in S}{\cup} s.M'$, the union being disjoint.*

Corollary 9. *M is a complete metric group.*

Proof.

1) It is evident that M' is open since it is an open ball of center 1 and radius 1.

2) We prove that M' is a subgroup of M.

To do that, it is sufficient to prove that given f and g two elements of M', the product $f\overline{g}$ is in M'.

By hypothesis, f and g are in M' and so we have $d(f,1) < 1$ and $d(g,1) < < 1$. Replacing $d(f,1)$ by $c(f,1)(1+c(f,1))^{-1}$, we get that

$$c(f,1) < +\infty,$$

i.e.

$$c(f,1) = \sqrt{\sum_{p \in P} |f(p)-1|^2.p^{-1}} < +\infty,$$

and this gives that

$$\sum_{p\in P} |f(p)-1|^2.p^{-1} < +\infty.$$

Remark that the condition $f \in M'$ is equivalent to

$$\sum_{p\in P} |f(p)-1|^2.p^{-1} < +\infty.$$

Similarly, we have

$$\sum_{p\in P} |g(p)-1|^2.p^{-1} < +\infty,$$

and since $|g(p)-1| = |\overline{g}(p)-1|$, we get that $\overline{g}$ is in M'.

Now, we remark that $\overline{g}(p)f(p)-1$ can be written as $\overline{g}(p)f(p)-1 = = \overline{g}(p)(f(p)-1)+\overline{g}(p)-1$, and so, we get that

$$\begin{aligned}|\overline{g}(p)f(p)-1| &\leq |\overline{g}(p)(f(p)-1)+\overline{g}(p)-1| \leq \\ &\leq |\overline{g}(p)|.|f(p)-1| + |\overline{g}(p)-1| \leq \\ &\leq |f(p)-1| + |g(p)-1|,\end{aligned}$$

since $|g(p)| = 1$ and $|\overline{g}(p)-1| = |g(p)-1|$.

Now, since we have the inequality

$$(|f(p)-1| + |g(p)-1|)^2 \leq 2\left(|f(p)-1|^2 + |g(p)-1|^2\right),$$

we get that

$$|\overline{g}(p)f(p)-1|^2 \leq 2\left(|f(p)-1|^2 + |g(p)-1|^2\right),$$

and since the two series

$$\sum_{p\in P} |f(p)-1|^2.p^{-1} \text{ and } \sum_{p\in P} |g(p)-1|^2.p^{-1}$$

converge, the series

$$\sum_{p\in P} |\overline{g}(p)f(p)-1|^2.p^{-1}$$

is also convergent, and as a consequence, the product $f\overline{g}$ is in M'.

3) We prove the separability of M'.

Let f be an element of M'. $f \in M'$ is equivalent to

$$\sum_{p\in P} |f(p)-1|^2.p^{-1} < +\infty.$$

Denoting by V the set of the roots of unity, we can choose for any given $\varepsilon > 0$, elements v_p of V such that $|v_p - f(p)|^2 \leq \varepsilon.2^{-(p+1)}$. Now, we define an integer K by the condition

$$K = \min\left\{ m \in N^*; \sum_{p\in P, p>m} |f(p)-1|^2.p^{-1} < (\varepsilon/2) \right\}.$$

We get that

$$\sum_{p\in P, p<K} |f(p)-v_p|^2.p^{-1} \leq \sum_{p\in P, p<K} \varepsilon.2^{-(p+1)}.p^{-1} \leq \varepsilon/2.$$

This gives us that the completely multiplicative function $v_K(n)$ defined by $v_K(p) = v_p$ if $p \leq K$, 1 if not, verifies the condition $d(f, v_K) \leq \varepsilon$, and so, since V^K is countable for all K in N^*, the set $\underset{K\in N^*}{\cup} V^K$ defines a countable family of elements of M' dense in M' for the topology defined by d.

4) It remains to prove that M' is complete.

Let f_k be a Cauchy sequence of elements of M'. Then, for a fixed p in P, the sequence $f_k(p)$ is a Cauchy sequence and so, $f(p) = \lim_{k\to+\infty} f_k(p)$ exists and $|f(p)| = 1$. To finish the proof, remark that the sequence $(1 - f_k(p))\sqrt{p^{-1}}$ can be viewed as the Fourier coefficients of the function $F_k(t)$ with Fourier series

$$\sum_{p\in P} \left((1 - f_k(p))\sqrt{p^{-1}}\right) \exp 2i\pi pt,$$

which is in $L^2([0,1])$. Since F_k is a Cauchy sequence in this space, it has a limit in it and it is immediate that the Fourier coefficients of this limit are

$$(1 - f(p))\sqrt{p^{-1}},\ p \in P,$$

which gives that

$$\sum_{p\in P} |f(p)-1|^2.p^{-1} < +\infty,$$

and so, f is in M'.

2.2. Dual group of M

We recall that the dual group of a topological group G is the set of the continuous characters of G, i.e. the set of the group homomorphisms of G with values in T, continuous for the topology of G.

We first prove the following result.

Proposition 10. *The dual group of M' is isomorphic to the group Q_+^* of the positive rational numbers.*

Proof. For all p in P, we define a class F_p of elements f_p of M' by $f_p(p)$, is in T, $f_p(q) = 1$ for all prime numbers q different of p. It is clear that F_p is a subgroup of M' and as a set, is exactly the circle T, for the topology defined on M' by the distance d induces on F_p the same topology as the ordinary topology on the circle T, and so, the group F_p can be viewed as the circle T with the ordinary topology, and as a consequence, a character X on F_p can be written as $X(f_p) = f(p)^{n_p}(= f(p^{n_p}))$, where n_p is an integer. Now, we remark that if f is in M', $d(1, f)$ is smaller than 1 and so, $c(1, f)$ is bounded. This gives us that the series

$$\sum_{p\in P} |f(p) - 1|^2.p^{-1}$$

is convergent, and it is clear that the sequence h_k of elements of M' defined by

$$h_k(n) = \prod_{p\leq k} f_p(n)$$

tends to f in M'. As a consequence, if X is a character of M', we must have

$$X(f) = \lim_{k\to+\infty} X(h_k) = \lim_{k\to+\infty} X\left(\prod_{p\leq k} f_p\right)$$

by continuity of X, and so, since the finite product $\prod_{p\leq k} f_p(n)$ is in M', we must have

$$X(f) = \lim_{k\to+\infty} \prod_{p\leq k} X(f_p)$$

in M'. This gives us that a character X is defined by a sequence of integers $\{n_p(X)\}_{p\in P}$, with the condition that the convergence of the series

$$\sum_{p\in P} |f(p) - 1|^2.p^{-1}$$

will imply that

$$\prod_{p\leq k} f(p)^{n_p(X)}$$

has a limit when k tends to infinity. This leads to the conclusion that only a finite number of the $n_p(X)$ are different of 0. For assume that there is a character X such that an infinite number of the $n_p(X)$ are different of 0. We denote by P' the set of the primes such that the $n_p(X)$ are different of 0. Now, we shall construct an element h of M' such that X will be not defined for the value of the argument h. To do that, we begin by selecting an infinite subset P'' of P' such that

$$\sum_{p\in P''} p^{-1} < +\infty.$$

We define h as a completely multiplicative function in the following way:

if p is a prime not in P'', $h(p)$ has the value 1,
if p is a prime in P'', $h(p)$ has the value $\exp(i\pi/n_p(X))$.

By construction, h is in M' since for m in N, we have

$$\begin{aligned}\sum_{p\in P,\ p\leq m} |h(p)-1|^2.p^{-1} = & \sum_{p\in P-P'',\ p\leq m} |h(p)-1|^2.p^{-1}+ \\ & + \sum_{p\in P'',\ p\leq m} |h(p)-1|^2.p^{-1} = \\ & = \sum_{p\in P'',\ p\leq m} |h(p)-1|^2.p^{-1},\end{aligned}$$

and this series converges since it is bounded by

$$\sum_{p\in P'',\ p\leq m} 2p^{-1}$$

which converge by definition of P''. This gives us that h is in M'.

Now, we remark that the sequence of functions h'_m defined by

$$h'_m(n) = \prod_{p\in P,\ p\leq m, p^\alpha\|n} h(p)^\alpha$$

is in M' since

$$\sum_{p\in P,\ p\leq m} |h(p)-1|^2.p^{-1} < +\infty,$$

and moreover, that for the metrizable group topology defined on M', we have $\lim_{k\to+\infty} h'_m = h$. As a consequence, since the character X is continuous, the limit, $m \to +\infty$, of $X(h_m)$ will exist. But we have

$$\begin{aligned} X(h'_m) &= \prod_{p\in P,\ p\leq m} h(p)^{n_p(X)} = \\ &= \prod_{p\in P'',\ p\leq m} h(p)^{n_p(X)} = \\ &= \prod_{p\in P'',\ p\leq m} (\exp(i\pi/n_p(X)))^{n_p(X)} \end{aligned}$$

and this sequence is oscillating, taking the values 1 and -1 alternatively. So, there is no limit and as a consequence, we get that for characters X of M', only a finite number of the $n_p(X)$ are different of 0.

Now, returning to the expression of $X(f)$, which is

$$X(f) = \prod_{p\in P} f(p^{n_p(X)}),$$

to this character X we can associate in a unique way the positive rational number

$$\alpha(X) = \prod_{p\in P} p^{n_p(X)}.$$

It is immediate that the function $X \longmapsto \alpha(X)$ is a group isomorphism $\widehat{M'} \to$ $\to Q^*_+$. For let X and X' be in $\widehat{M'}$. We have for all f in M',

$$\begin{aligned} X(f) &= \prod_{p\in P} f(p^{n_p(X)}), \\ X'(f) &= \prod_{p\in P} f\left(p^{n_p(X')}\right), \end{aligned}$$

where the integers $n_p(X)$ and $n_p(X')$ are independent of f. Now, since X and X' are in $\widehat{M'}$, we have $(X.X')(f) = X(f).X'(f)$.

Replacing $X(f)$ and $X'(f)$ by their value, we get that

$$\begin{aligned}(X.X')(f) &= \prod_{p\in P} f\left(p^{n_p(XX')}\right) = \\ &= \left(\prod_{p\in P} f\left(p^{n_p(X)}\right)\right) \times \left(\prod_{p\in P} f\left(p^{n_p(X')}\right)\right) = \\ &= \prod_{p\in P} f\left(p^{n_p(X)+n_p(X')}\right).\end{aligned}$$

Since this holds for all f, we get that

$$\begin{aligned}\alpha(XX') &= \prod_{p\in P} p^{n_p(X.X')} = \prod_{p\in P} p^{n_p(X)+n_p(X')} = \\ &= \left(\prod_{p\in P} p^{n_p(X)}\right) \times \left(\prod_{p\in P} p^{n_p(X')}\right) = \\ &= \alpha(X).\alpha(X').\end{aligned}$$

This gives us that $X \longmapsto \alpha(X)$ is a group homomorphism. It is an isomorphism for $\alpha(X) = 1$ if and only if all the $n_p(X)$ are equal to 0, i.e. $X = 1$.

Let h be in M, and assume that

$$\limsup_{x\to+\infty}(1/x)\left|\sum_{1\le n\le x,\ (n,2)=1} h(n)\right| > 0.$$

By Halász's theorem, we know that there exists τ_h in R such that

$$\sum_{p\in P}\left(1 - \mathrm{Re}\ h(p)p^{-i\tau_h}\right)p^{-1}$$

converge.

We shall denote the space of such functions by $M!$.

As in the case of M', we can prove that $M!$ is a group, for if f and g are in $M!$, then, setting $\tau_f - \tau_g = \tau_{f\overline{g}}$, we get that $f.\overline{g}$ is in $M!$.

Now, remark that M' is an open subgroup of $M!$, and so, $M!/M'$ is discrete. In fact, it is immediate that $M!/M'$ is isomorphic to R_d, and that we have $M! \simeq M' \times R_d$. As a consequence, we get that $\widehat{M}!$, the dual of $M!$, is isomorphic to $\widehat{M'} \times \widehat{R_d}$. Now, we have proved above that $\widehat{M'}$ is isomorphic to Q_+^*, the set of the positive rational numbers. And $\widehat{R_d}$ is well known, since it is $\mathcal{B}(R)$, the Bohr compactification of the real line. So, up to an isomorphism, we have determined the dual group of $M!$: in fact, $\widehat{M!} \simeq Q_+^* \times \mathcal{B}(R)$.

Now, we remark that $M!$ is an open subgroup of M, since first of all, $M!$ is a subgroup of M, and second, we have $M! \simeq \bigcup_{r \in R_d} (r, M')$, with M' open in M. As a consequence, $M/M!$ is discrete. This gives us that if $\mathcal{S}$ is a set of representatives of $M/M!$, we have $M = \bigcup_{\sigma \in \mathcal{S}} \sigma M!$, the union being disjoint. Moreover, we remark that $M!$ is divisible, since for a given k in N^*, the element of M defined by

$$f_k(p) = \exp(i \arg f(p)/k) \quad \text{if} \quad \pi \geq \arg f(p),$$
$$f_k(p) = \exp(i(\arg f(p) - 2\pi)/k) \quad \text{if} \quad \pi \leq \arg f(p),$$

with the real parameter $\tau_{f_k} = \tau_f/k$, is in $M!$ and so, we have, by the Weil theorem, $M \simeq (M/M!) \times M!$, which gives us that $M \simeq (M/M!) \times M' \times R_d$ since $M! \simeq M' \times R_d$. As a consequence, we get that $\widehat{M} \simeq (\widehat{M/M!}) \times \widehat{M'} \times \widehat{M'} \times \widehat{R_d}$, i.e. $\widehat{M} \simeq (\widehat{M/M!}) \times Q_+^* \times \mathcal{B}(R)$, since we have $\widehat{M!} \simeq Q_+^* \times \mathcal{B}(R)$. So, to conclude, we have the following result:

Theorem 11. *The dual group $\widehat{M}$ of M can be written as $\widehat{M} \simeq \mathcal{K} \times Q_+^* \times$ $\times \mathcal{B}(R)$, where $\mathcal{K}$ is a compact group, dual of the discrete group $(\widehat{M/M!})$, $\mathcal{B}(R)$ is the Bohr compactification of the real line, and Q_+^* is the multiplicative group of the positive rational numbers.*

3. Multiplicative functions with values in a finite subgroup of T and related sets of functions

Let $\mathcal{F}$ be a finite subgroup of T of order Ω. F will denote the set of the multiplicative functions with values in $\mathcal{F} \cup \{0\}$, and F_c the set of the elements of M such that for all p, $f(p)$ belongs to $\mathcal{F}$. Moreover, we shall denote by F'_c the group $M' \cap F_c$.

3.1. Completely multiplicative functions with values in a finite subgroup of T

Theorem 12. *F'_c is an open subgroup of F_c, complete and metrizable; and as a consequence, the quotient $U = F_c/F'_c$ is discrete, and so, there exists* *a set of representative of U such that $F_c = \underset{s\in U}{\cup} s.F'_c$, the union being disjoint. Moreover, F_c is a complete metric group.*

The dual group of F'_c is $Q^*_+/Q^{*\Omega}_+$, where $Q^{*\Omega}_+$ is the group of the rationals which are an exact Ω-power of elements of Q^*_+.

Proof. It is immediate that F_c is a subgroup of M, and so, it is metrical for the distance induced by the distance d on M. F'_c, as a subgroup of F_c, is open since

$$F'_c = \{f \in F_c \mid d(1, f) < 1\}.$$

This implies that there exists a set of representative of U such that $F_c = = \underset{s\in U}{\cup} s.F'_c$, the union being disjoint.

Now, F'_c is complete for if f_k is a Cauchy sequence in F'_c, hence in M', it has a limit f in M', and since $f_k^\Omega = 1$, we must have $f^\Omega = 1$, which means that f is in F'_c.

The determination of the dual group of F'_c can be done as in the case of the dual group of M': as above, for all p in P, we define a class F_p of elements f_p of F'_c by $f_p(p)$ is in $\mathcal{F}$, $f_p(q) = 1$ for all prime numbers q different of p. F_p can be identified to the group $\mathcal{F}$ with the discrete topology, and so, a character X on F_p can be written as $X(f_p) = f(p)^{n_p}$, where n_p is an integer mod Ω. Now, since the sequence h_k of elements of F'_c defined by

$$h_k(n) = \prod_{p\le k} f_p(n)$$

tends to f in F'_c, if X is a character of F'_c, we must have

$$X(f) = \lim_{k\to+\infty} X(h_k) = \lim_{k\to+\infty} X\left(\prod_{p\le k} f_p\right)$$

by continuity of X, i.e.

$$X(f) = \lim_{k\to+\infty} \prod_{p\le k} X(f_p)$$

in F'_c. So, a character X is defined by a sequence of integers $\{n_p(X)\mathrm{mod}\Omega\}_{p\in P}$, with the condition that the convergence of the series

$$\sum_{p\in P} |f(p) - 1)|^2.p^{-1}$$

will imply that

$$\prod_{p\le k} f(p)^{n_p(X)}$$

has a limit when k tends to infinity, and the same kind of argument as in the case of M' will give that only a finite number of the $n_p(X)$ are different of 0.

3.2. Multiplicative functions with values in $\mathcal{F}\cup\{0\}$

We shall now consider the set F of the multiplicative functions with values in $\mathcal{F}\cup\{0\}$.

3.2.1. Existence of a mean value

First of all, we give a result on the existence of the mean value of any element of F. We shall prove that

Proposition 13. *If f is in F, it has an arithmetical mean value $m(f)$.*

Proof. 1) If f is in F_c, we have

$$(1/x)\sum_{n\le x} f(n) = \sum_{a\in\mathcal{F}} \left((1/x)a \sum_{n\le x,\ f(n)=a} 1 \right),$$

and by Ruzsa theorem,

$$\lim_{x\to+\infty} (1/x) \sum_{n\le x,\ f(n)=a} 1$$

exists.

So, $m(f)$ exists.

Now, remark that by Halász theorem, there is no possibility to have

$$\sum \mathrm{Re}\ (1 - f(p)p^{-it})p^{-1} < +\infty$$

with $t\ne 0$. For if one has

$$\sum \mathrm{Re}\ (1 - f(p)p^{-it})p^{-1} < +\infty,$$

then,

$$2\mathrm{Re}\left(1-(f(p)p^{-it})^{\Omega}\right) = \left|1-(f(p)p^{-it})^{\Omega}\right|^2 \le \left|1-f(p)p^{-it}\right|^2 .\Omega^2 =$$
$$= \Omega^2 . \sum 2\mathrm{Re}\left(1-f(p)p^{-it}\right),$$

and as a consequence, since $f(p)^{\Omega} = 1$, we shall have

$$\sum \mathrm{Re}\left(1 - p^{-it\Omega}\right) p^{-1} < +\infty,$$

and this is not possible if $t \neq 0$.

2) If f is in F, we define f_c in F_c by $f_c(p) = f(p)$ if $f(p) \neq 0, = 1$ if not. We remark that we have $f(n) = f_c(n).f'(n)$, where f' is in F and is defined by $f'(p^k) = \overline{f_c}(p)^k.f(p^k)$. Remark that $f'(p) = 1$ or 0 and $|f'(n)| = |f(n)|$.

We have two cases.

First case: $\sum(1-f'(p))p^{-1}$ is not finite. In this case, since $|f'(p)| = |f(p)|$, Delange theorem gives that $m(|f|)$ exists and is equal to 0.

Second case: $\sum(1 - f'(p))p^{-1}$ is finite. We remark that $\sum(1 - f(p))p^{-1}$ can be written as

$$\sum(1 - f(p))p^{-1} = \sum_{f(p)\neq 0} (1 - f(p))p^{-1} + \sum_{f(p)=0} (1 - f(p))p^{-1},$$

and since we have

$$\sum_{f(p)\neq 0} (1 - f(p))p^{-1} = \sum(1 - f_c(p))p^{-1}$$

for $f_c(p) = 1$ if $f(p) = 0$, and

$$\sum_{f(p)=0} (1 - f(p))p^{-1} = \sum(1 - f'(p))p^{-1}$$

for $1 - f'(p) = 0$ if $f(p) \neq 0$, we get that

$$\sum(1 - f(p))p^{-1} = \sum(1 - f_c(p))p^{-1} + \sum(1 - f'(p))p^{-1},$$

and this gives us that

$$\sum(1 - f(p))p^{-1} = \sum(1 - f_c(p))p^{-1} + \textit{ an absolutely convergent series.}$$

Now, if $\sum(1 - f_c(p))p^{-1}$ converges, $\sum(1 - f(p))p^{-1}$ converges, too, and so, f has a mean value by Delange theorem. Since there is not $t \neq 0$ such that

$$\sum \mathrm{Re}(1 - f_c(p)p^{-it})p^{-1} < +\infty,$$

the same holds for f, and so, if $\sum(1-f_c(p))p^{-1}$ is not convergent, $m(f)$ exists and is equal to 0.

3.2.2. Distance on F

We define a real-valued function $\delta(f,g)$ on $F\times F$ by $\delta(f,g)=\gamma(f,g)(1+$ $+\gamma(f,g))^{-1}$, where

$$\gamma(f,g)=\sqrt{\sum_{p\in P,\ k>0}|f(p^k)-g(p^k)|^2.p^{-k}}.$$

We have the following statement.

Proposition 14. *δ is a distance on F and F is a complete metric group.*

Proof. Use the same argument as in the study of the distance d on M.

3.2.3. Relation between the distance and the mean value on F

The following result will explicit the relation between the distance δ and the mean value m.

Proposition 15. *The set of the f in F such that $\delta(1,f)<1$ (resp. $\delta(1,f)=1$) and the set of the f in F such that*

$$\lim_{x\to+\infty}(1/x)\sum_{n\leq x,\ 2\not| n}f(n)\neq 0$$

(resp. $\lim_{x\to+\infty}(1/x)\sum_{n\leq x,\ 2\not| n}f(n)=0$) are the same.

Proof. We recall that $\delta(f,g)=\gamma(f,g)(1+\gamma(f,g))^{-1}$, where

$$\gamma(f,g)=\sqrt{\sum_{p\in P,\ k>0}|f(p^k)-g(p^k)|^2.p^{-k}}.$$

If $\delta(1,f)<1$, this means that

$$\gamma(f,1)=\sqrt{\sum_{p\in P,\ k>0}|f(p^k)-1|^2.p^{-k}}$$

is a finite quantity, and by the extended theorem of Delange, we get that

$$\lim_{x\to+\infty}\left(x^{-1}\sum_{1\leq n\leq x,\ 2\not| n}f(n)-\prod_{p\leq x,\ p\neq 2}\left(1-\frac{1}{p}\right)\sum_{0\leq k}f(p^k)p^{-1}\right)=0.$$

But the theorem on the existence of the mean value gives that

$$\lim_{x\to+\infty} x^{-1} \sum_{1\le n\le x,\ 2\nmid n} f(n)$$

exists, and so, we get that the sequence

$$\prod_{p\le x,\ p\ne 2} \left(1-\frac{1}{p}\right) \sum_{0\le k} f(p^k)p^{-1}$$

is convergent and its limit exists and is not 0 since all the terms are different of 0.

Now, if $\delta(f,1)=1$, we get that

$$\sum_{p\in P,\ k>0} |f(p^k)-1|^2.p^{-k}$$

is not finite, and since

$$\lim_{x\to+\infty} x^{-1} \sum_{1\le n\le x,\ 2\nmid n} f(n)$$

exists, we get that this limit must be 0, which is also the limit of the product

$$\prod_{p\le x,\ p\ne 2} \left(1-\frac{1}{p}\right) \sum_{0\le k} f(p^k)p^{-1}.$$

3.2.4. A decomposition of F

We shall denote by F' (resp.K) the set of the f in F such that $\delta(1,f)<1$ (resp. $m(|f|)=0$).

Remark 1. *F' is a semigroup and F'_c is a subsemigroup of F'.*

(This is a simple consequence of the Cauchy inequality.)

We have the following result.

Theorem 16. *There exists a set of representatives U of the quotient group F_c/F'_c such that $F = K \cup \left(\bigcup_{s\in U} s.F'\right)$, the union being disjoint.*

Proof. If $m(|f|)=0$, by definition, f is in K. So, we assume now that $m(|f|)\ne 0$.

Let f be an element of F. We associate to f two elements of F in the following way:

f_1 is in F_c and is defined by $f_1(p) = f(p)$ if $f(p) \neq 0, = 1$ if not.

f_2 is in F' and is defined by $f_2(p^k) = \overline{f_1(p^k)}.f(p^k)$.

f_2 is in F' because $f_2(p) = |f(p)|$ and since $m(|f|) \neq 0$, we have

$$\sum (1 - |f(p)|)p^{-1} < +\infty.$$

We have clearly the identity $f = f_1 f_2$, for

$$f(p^k) = (f(p)^k) \times \left(\overline{f(p)^k} f(p^k)\right) = f_1(p^k) \times \overline{f_1(p^k)}.f(p^k).$$

Now, since f_1 is in F_c, there exist unique elements s in U and f' in F'_c such that $f_1 = s.f'$. Hence we get that $f = s.f'.f_2,\ s \in U,\ f' \in F'_c,\ f_2 \in F'$, and since F'_c is a subsemigroup of F', we get that f can be written as $f = s.h$, with $s \in U,\ h \in F'$. Moreover, this decomposition is unique, due to the way in which it is obtained. It suffices to remark that if we have $f = sh = s'h'$, we get that $h = \overline{s}s'h'$. But h and h' are in F', and so, we get that $|f|\overline{s}s'$ is in F', which implies that $\overline{s}s'$ is in F'_c (the role of the p such that $f(p) = 0$ here is secondary for $\sum_{f(p)=0} p^{-1}$ converge), and so, that $\overline{s}s' = 1$ which gives that $h = h'$.

3.3. Algebras generated by F

3.3.1. Introduction

Many properties of multiplicative functions which satisfy some growth condition (existence of the mean value of the function and of its modulus, existence of a distribution function etc.) are closely related to some spaces of limit periodic functions. More precisely, we denote by $c_q(n)$ the q-Ramanujan sum, which can be written as

$$c_q(n) = \sum_{(h,q)=1} \exp 2i\pi(hn/q),$$

and for $\lambda \geq 1$, by B^λ_{inv} the space of functions f such that for any $\varepsilon > 0$, there exists a finite set $\{a_1, a_2, \ldots, a_l\}$ of complex numbers such that

$$\limsup_{x \to +\infty} (1/x) \sum_{1 \leq n \leq x} \left| f(n) - \left(\sum_{1 \leq j \leq l} a_j c_j(n) \right) \right|^\lambda \leq \varepsilon.$$

Then, for instance, elements of F' belong to B^{λ}_{inv} for all $\lambda \geq 1$. But the classical Möbius function, or the Liouville function, are not in this kind of space B^{λ}_{inv}. (For more details, see [4].)

The next part of the present article will provide a description of some spaces of arithmetical functions associated to F and related to the B^{λ}_{inv}, $\lambda \geq 1$.

3.3.2. Algebra of sets or algebra functions?

We recall our notations: $\mathcal{F}$ is a finite subgroup of T of order Ω and F is the set of the multiplicative functions with values in $\mathcal{F} \cup \{0\}$. We prove the following essential result:

Theorem 17. *For a given f in F and a in $\mathcal{F} \cup \{0\}$, we denote by $S(f,a)$ the set of the n in N^* such that $f(n) = a$.*

Then we have: the Boole algebra $B(\mathcal{F})$ generated by the family $\{S(f,a);\ f \in \in F,\ a \in \mathcal{F} \cup \{0\}\}$ admits as a finitely additive measure the mean value m, and the complex algebra generated by the characteristic functions of the family $\{S(f,a);\ f \in F,\ a \in \mathcal{F}\cup\{0\}\}$ is identical to the complex algebra $A(F)$ generated by the elements of F.

Proof. Given f in F and a in $\mathcal{F}$, we remark that the characteristic function $I_{S(f,a)}$ of $S(f,a)$ can be written as

$$I_{S(f,a)}(n) = (\Omega^{-1}). \sum_{0 \leq k \leq \Omega-1} (f(n).\overline{a})^k.$$

As a consequence, we have

$$(1/x) \sum_{n \leq x} I_{S(f,a)}(n) = (1/x) \sum_{n \leq x} \left((\Omega^{-1}). \sum_{0 \leq k \leq \Omega-1} (f(n).\overline{a})^k \right) =$$

$$= (\Omega^{-1}). \sum_{0 \leq k \leq \Omega-1} \overline{a}^k. \left((1/x) \sum_{n \leq x} f(n)^k \right),$$

and since f^k is in F, it has a mean value, and so, we have $m(I_{S(f,a)})$ exists and is equal to

$$(\Omega^{-1}). \sum_{0 \leq k \leq \Omega-1} \overline{a}^k.m(f^k).$$

If $a = 0$, then, since $|f| = 1$ or 0, we have $I_{S(f,a)}(n) = 1 - |f(n)|$, and so, $m(I_{S(f,0)}) = 1 - m(|f|)$.

To prove that the Boole algebra $B(F)$ generated by the family $\{S(f,a);\ f \in \in F,\ a \in \mathcal{F} \cup \{0\}\}$ admits the finitely additive measure m, it is necessary to prove that given some W in $B(\mathcal{F})$, the quantity $m(I_W)$ exists. Now, such a W is obtained by finite union and intersection or complementarization of sets $S(f,a)$. Recalling that the characteristic function of a finite union can be expressed as a linear form with integral coefficients of characteristic functions of finite intersections, we shall concentrate on this last case only.

First, we recall that the characteristic function $I_{S(f,a)}$ of $S(f,a)$ can be written as

$$I_{S(f,a)}(n) = (\Omega^{-1}).\sum_{0\le k\le \Omega-1} (f(n).\overline{a})^k$$

if $a \neq 0$, $= (1-|f(n)|)$ if $a = 0$.

This can be shortened in the following form.

$$I_{S(f,a)}(n) = |f(n)|\left((\Omega^{-1}).\sum_{0\le k\le \Omega-1} (f(n).\overline{a})^k\right) + (1-|f(n)|).$$

So, a set written as $S(f,a)\cap S(f',a')$ has a characteristic function given by

$$I_{S(f,a)}(n).I_{S(f',a')}(n) =$$

$$= \left(|f(n)|\left((\Omega^{-1}).\sum_{0\le k\le \Omega-1} (f(n).\overline{a})^k\right) + (1-|f(n)|)\right)\times$$
$$\times\left(|f'(n)|\left((\Omega^{-1}).\sum_{0\le l\le \Omega-1} (f'(n).\overline{a}')^l\right) + (1-|f'(n)|)\right),$$

which can be written as a linear form of products of the functions $|f(n)|$, $|f'(n)|$, $f(n)^k$, $(f'(n))^l$, $0\le k\le \Omega-1$, $0\le l\le \Omega-1$, and we remark that such products are in F.

This can be generalized immediately to any finite intersection of elements of the family $\{S(f,a); f\in F, a\in \mathcal{F}\cup\{0\}\}$. So, the characteristic function I_W of a finite intersection W of elements of the family $\{S(f,a); f\in F, a\in\mathcal{F}\cup\{0\}\}$ can be written as a linear form of elements of F with complex coefficients. As a consequence, it has a mean value, since all the elements of F have a mean value, and this clearly defines a finitely additive measure on $B(\mathcal{F})$. Moreover, this implies also that I_W belongs to the complex algebra generated by F.

Now, to prove that the complex algebra generated by the characteristic functions of the family $\{S(f,a); f\in F, a\in\mathcal{F}\cup\{0\}\}$ is identical to the complex algebra generated by F, it is sufficient to prove that if f is an element of F, it can be expressed as a complex linear form of characteristic functions of elements of $B(\mathcal{F})$. But this is evident, since we have

$$f(n) = \sum_{a \in \mathcal{F}} a.I_{S(f,0)}(n) + I_{S(f,0)}(n).$$

3.3.3. The algebra $A(F)$

We denote by $A(F)$ (resp. $K(F)$) the complex algebra generated by F (resp. K).

Remark 2. *$K(F)$ is an ideal of $A(F)$.*

Proof. If a is in $K(F)$, then we have

$$a = \sum_{finite} \lambda_i f_i,$$

where $m(|f_i|) = 0$ for all i. An element h in $A(F)$ can be written as

$$h = \sum_{finite} \mu_i h_i.$$

The product $a.h$ is a finite linear form of products $f_i h_j$, and since

$$|f_i h_j| \leq 1.|f_i|,$$

we get that $a.h$ is in $K(F)$.

Now, we prove the following result.

Theorem 18. 1) *The function $\langle a, b \rangle$ defined on $A(F)^2$ by the relation $\langle a, b \rangle = m(a.\bar{b})$ is a bilinear form.*

2) *$a \longmapsto \sqrt{m(|a|^2)}$ is a norm on $A(F) - K(F)$.*

3) *The family of arithmetical functions $\{s.c_q; s \in F_c/F'_c,\ q \in N^*\}$ is dense in $A(F)$ for the topology induced by the bilinear form $\langle ., . \rangle$.*

Proof. 1) The first assertion is a simple consequence of the fact that $A(F)$ is an algebra in which any element has a mean value.

2) We assume that there exists some element h in $A(F) - K(F)$ such that $m(|h|^2) = 0$. We have

$$h(n) = \sum_{finite} \lambda_f f(n),$$

where $f(n)$ is in $F-K$. But we know that there is a decomposition of F given by $F = K \cup \left(\underset{s\in U}{\cup} s.F' \right)$, where U is a set of representatives of the quotient group F_c/F'_c. This gives us that each f can be written as $f = s_f f',\ s_f \in U,\ f' \in F'$, and so we have

$$h(n) = \sum_{finite} \lambda_f f(n) = \sum_{finite} \lambda_f s_f(n) f'(n).$$

Now, since $m(|h|^2) = 0$, we get that

$$m\left(\left| \sum_{finite} \lambda_f s_f(n) f'(n) \right|^2 \right) = 0,$$

and this can be written as

$$\sum_{finite} \lambda_f \overline{\lambda_g} m(s_f(n)\overline{s_g(n)} f'(n)\overline{g'(n)}) = 0.$$

Due to the definition of U, we have $m(s_f(n)\overline{s_g(n)} f'(n)\overline{g'(n)}) = 0$ if $s_f \neq s_g$. This remark leads to rewrite the equation with the following method: first, select the functions s which appear in the formula giving h. Then, we write the functions f' which are associated to s. This allows to give to h a different form which is

$$h(n) = \sum_{s\in finite} s(n). \left(\sum_{finite} \lambda_{s,f'} f'(n) \right).$$

Now, due to the remark above, we get that

$$\begin{aligned} 0 = m(|h|^2) = m\left(\left| \sum_{s\in finite} s(n). \left(\sum_{finite} \lambda_{s,f'} f'(n) \right) \right|^2 \right) &= \\ = \sum_{s\in finite} m\left(\left| s(n). \left(\sum_{finite} \lambda_{s,f'} f'(n) \right) \right|^2 \right) &= \\ = \sum_{s\in finite} m\left(\left| \left(\sum_{finite} \lambda_{s,f'} f'(n) \right) \right|^2 \right). \end{aligned}$$

It remains to prove that if

$$m\left(\left|\left(\sum_{finite}\lambda_{s,f'}f'(n)\right)\right|^2\right)=0,$$

then

$$\left(\sum_{finite}\lambda_{s,f'}f'(n)\right)=0$$

for all n in N^*.

To do that, we remark that given y a positive integer, if f' is in F', for all $l\leq y$, denoting by N_y the product $\prod\limits_{p\leq y}p$, we have the identity

$$\lim_{x\to+\infty}\left(\sum_{n\leq x,\ (n,N_y)=1}1\right)^{-1}\times\sum_{n\leq x,\ (n,N_y)=1}f'(ln)=$$

$$=\lim_{x\to+\infty}\left(x.\prod_{p\leq y}(1-p^{-1})\right)^{-1}\times\sum_{n\leq x,\ (n,N_y)=1}f'(ln)=$$

$$=f'(l).\prod_{p\geq y}\left((1-p^{-1})\sum_{0\leq k}f'(p^k)p^{-k}\right),$$

and we recall that

$$\lim_{y\to+\infty}\prod_{p\geq y}\left((1-p^{-1})\sum_{0\leq k}f'(p^k)p^{-k}\right)=1.$$

By the Cauchy inequality, we get that

$$\left|\sum_{n\leq x,\ (n,N_y)=1}\left(\sum_{finite}\lambda_{s,f'}f'(ln)\right)\right|^2\leq$$

$$\leq\left(\sum_{n\leq x,\ (n,N_y)=1}\left|\sum_{finite}\lambda_{s,f'}f'(ln)\right|^2\right).\left(\sum_{n\leq x,\ (n,N_y)=1}1\right)\leq$$

$$\leq\left(\sum_{n\leq x}\left|\sum_{finite}\lambda_{s,f'}f'(n)\right|^2\right).\left(\sum_{n\leq x,\ (n,N_y)=1}1\right),$$

and so we have

$$\left| \lim_{x\to+\infty} \left(x.\prod_{p\le y}(1-p^{-1}) \right)^{-1} \times \left(\sum_{n\le x,\ (n,N_y)=1} \left(\sum_{finite} \lambda_{s,f'} f'(n) \right)\right) \right|^2 \le$$

$$\le \lim_{x\to+\infty} \left(\left(x.\prod_{p\le y}(1-p^{-1}) \right)^{-1} . \left(\sum_{n\le x} \left| \sum_{finite} \lambda_{s,f'} f'(n) \right|^2 \right)\right) \times$$

$$\times \left(\left(x.\prod_{p\le y}(1-p^{-1}) \right)^{-1} . \left(\sum_{n\le x,\ (n,N_y)=1} 1 \right)\right) \le$$

$$\le \left(\prod_{p\le y}(1-p^{-1}) \right)^{-1} .m \left(\left| \sum_{finite} \lambda_{s,f'} f'(n) \right|^2 \right) = 0,$$

since

$$m \left(\left| \sum_{finite} \lambda_{s,f'} f'(n) \right|^2 \right) = 0.$$

Now, by the remarks above, we know that

$$\lim_{x\to+\infty} (1/x) \sum_{n\le x,\ (n,N_y)=1} \left(\sum_{finite} \lambda_{s,f'} f'(ln) \right) =$$

$$= \sum_{finite} \lambda_{s,f'} f'(l) \prod_{p\ge y} \left((1-p^{-1}) \sum_{0\le k} f'(p^k)p^{-k} \right),$$

and from

$$\lim_{y\to+\infty} \prod_{p\ge y} \left((1-p^{-1}) \sum_{0\le k} f'(p^k)p^{-1} \right) = 1,$$

we get that

$$\sum_{finite} \lambda_{s,f'} f'(l) = 0,$$

and so, the only h in $A(F) - K(F)$ such that $m(|h|^2) = 0$ is 0.

3) It is sufficient to prove the following result.

Lemma 19. *The elements of the system c_q are in $A(F')$, the algebra generated by F'.*

Completing the proof of the assertion 3) then turns to be a simple consequence of the well known fact that any element of F' is in B^2_{inv} [4], and of the decomposition theorem for the space F.

It is known [4] that

$$c_q(n) = \sum_{d|q, d|n} d\mu(q/d),$$

where μ is the Möbius function.

We write it as

$$c_q(n) = \sum_{d|q} d\mu(q/d) I_d(n),$$

where $I_d(n) = 1$ if $d|n, = 0$ if $d \nmid n$.

Now, we remark that

$$I_d(n) = \prod_{p^\alpha \| d, \alpha > 0} I_{p^\alpha}(n).$$

But the function $J_{p^\alpha}(n)$, $\alpha > 0$, defined by $J_{p^\alpha}(n) = 1 - I_{p^\alpha}(n)$ is in F' since it takes the values 1 or 0 and $m(J_{p^\alpha}) = 1 - m(I_{p^\alpha}) = 1 - p^{-\alpha} > 0$, and it is easy to check that we have

$$J_{p^\alpha}(nn') = J_{p^\alpha}(n) J_{p^\alpha}(n')$$

when $(n, n') = 1$.

As a consequence, since $I_{p^\alpha} = 1 - J_{p^\alpha}$ is in $A(F') - K(F)$, I_d is also in $A(F') - K(F)$ and the same for c_q.

Remark 3. *Assertion 2) in the above theorem implies that any element f in $A(F)$ can be written in a unique way $f = h + g$, where $h \in K(F)$, $g \in$ $\in A(F) - K(F)$. As a consequence, since $K(F)$ is an ideal, we get that for any k in N^*, we have $f^k = h_k + g^k$, where $h_k \in K(F)$ and g^k is the k-th power of g.*

As a corollary of the above theorem, we have

Corollary 20. *The system $\{\varphi(q)^{-1/2}(s.c_q);\ s \in F_c/F'_c,\ q \in N^*\}$, where φ is the Euler function, is an orthonormal system in $A(F)$.*

Proof. This can be obtained by a direct computation.

3.3.4. λ-closures of $A(F)$

Some notations and definitions. Let λ be a real number greater or equal to 1. For an algebra H of arithmetical functions, we shall denote by $\overline{H}^{\lambda}$ the set of the sequences $h(n),\ n \in N^*$, satisfying the condition: for any $\varepsilon > 0$, there exists some f in H such that

$$\limsup_{x \to +\infty} (1/x) \sum_{n \le x} |f(n) - h(n)|^{\lambda} \le \varepsilon.$$

This space will be called the λ-closure of H.

Similarly, we shall denote by D^{λ} the set of the sequences $h(n),\ n \in N^*$, satisfying the condition

$$\limsup_{x \to +\infty} (1/x) \sum_{n \le x} |h(n)|^{\lambda} = 0.$$

Remark 4. To simplify the notations, the exponent λ will be dropped in the formulas when there will be no need to specify it.

We recall that a distribution function σ is a non-decreasing function $R \to \to [0, 1]$ such that

$$\lim_{t \to -\infty} \sigma(t) = 0, \qquad \lim_{t \to +\infty} \sigma(t) = 1.$$

We say that a sequence $a(n)$ has a limit distribution σ_a if there exists a distribution function σ such that for any continuity point t of σ the relation

$$\lim_{x \to +\infty} (1/x) \sum_{n \le x,\ a(n) < t} 1 = \sigma(t)$$

holds.

Some results and properties of $\overline{A}^{\lambda}(F)$. We shall now give some properties of these spaces as consequences of the results given above on $A(F)$.

First, we recall that a function $f : N^* \to C$ is a $(\mathcal{L}^{\lambda})$-uniformly summable function if for all $\varepsilon > 0$, there exists a finite linear form

$$f_{\varepsilon} = \sum_{b \in B} \alpha_b I_b,$$

where α_b is in C, and $I_b(n)$ is the characteristic function of b in $\{S(f, a);\ f \in \in F,\ a \in \mathcal{F} \cup \{0\}\}$, such that

$$\limsup_{x \to +\infty} (1/x) \sum_{n \le x} |f_{\varepsilon}(n) - f(n)|^{\lambda} \le \varepsilon.$$

([1] p. 204).

The relationship with the spaces of $(\mathcal{L}^\lambda)$-uniformly summable functions is given by the following result.

Theorem 21. *The space of $(\mathcal{L}^\lambda)$–uniformly summable functions built on the Boole algebra $B(\mathcal{F})$ generated by the family $\{S(f,a); f \in F, a \in \mathcal{F} \cup \{0\}\}$ equipped with the finitely additive measure m is identical to $\overline{A}^\lambda(F)$.*

Proof. Assume that $f : N^* \to C$ is a $(\mathcal{L}^\lambda)$*–uniformly summable function.* For any $\varepsilon > 0$, there exists a finite linear form

$$f_\varepsilon = \sum_{b \in B} \alpha_b I_b,$$

where α_b is in C, and $I_b(n)$ is the characteristic function of b in $\{S(f,a);\ f \in \in F,\ a \in \mathcal{F} \cup \{0\}\}$, such that

$$\limsup_{x \to +\infty} (1/x) \sum_{n \le x} |f_\varepsilon(n) - f(n)|^\lambda \le \varepsilon.$$

Such a f_ε is in $B(\mathcal{F})$, and since we have identically $A(F) = B(\mathcal{F})$, this gives the conclusion. We have also the following result.

Theorem 22. *$\overline{A}^\lambda(F)$ is identical to the λ-closure of the space $\bigoplus_{s \in U} s.B^\lambda_{inv}$, where $\bigoplus_{s \in U} s.B^\lambda_{inv}$ is the complex vector space of finite linear forms $\sum \alpha_{s,f} s.f$, with $\alpha_{s,f}$ in C, s is in U, and f is in B^λ_{inv}.*

Proof. This is a simple adaptation in the present case of the transitivity principle for closure of spaces, i.e. if $K \subset L \subset M$, then $\overline{K} \subset \overline{L} \subset \overline{M}$.

We recall that the λ-closure of the vector space generated by the system c_q of the Ramanujan sums is B^λ_{inv}, the c_q are in $A(F')$ and any element of F' is in B^λ_{inv} [4]. So, we have $B^\lambda_{inv} = \overline{A}^\lambda(F')$. This gives us that for s in U, $s.B^\lambda_{inv} = s.\overline{A}^\lambda(F')$. But it is clear that $s\overline{A}^\lambda(F') = \overline{A}^\lambda(s.F')$, and so, we get that the λ-completion of the set of the finite linear forms $\sum \alpha_{s,f} s.f$, where $\alpha_{s,f}$ are complex numbers, s is in a finite and fixed subset S of U, and f is in F', is $\bigoplus_{s \in S} s.B^\lambda_{inv}$. And this gives us that $\overline{A}^\lambda(F)$ is identical to the λ-closure of the space $\bigoplus_{s \in U} s.B^\lambda_{inv}$.

Existence of a limit distribution

Theorem 23. *Any real-valued element f of $\overline{A}^{\lambda}(F)$ has a limit distribution.*

Proof. We shall give a more general proof of this result.

Let H be a complex algebra of arithmetical functions such that if f is any element of H,

$$\lim_{x\to+\infty}(1/x)\sum_{1\le n\le x} f(n)$$

exists.

We shall denote by $\overline{H}$ the set of arithmetical functions h such that for any $\varepsilon>0$ there exists an element f in H such that

$$\limsup_{x\to+\infty}(1/x)\sum_{1\le n\le x}|h(n)-f(n)|<\varepsilon.$$

$\overline{H}$ has the following property.

Theorem 24. *For any real element h of $\overline{H}$, the sequence $h(n)$ has a distribution law σ_h.*

Proof. 1) We shall denote by $\nu_l, l\in N^*$, the sequence of probability laws defined on $\overline{H}$ by

$$\nu_l(h)=(1/l)\sum_{1\le n\le l} h(n).$$

Now, we shall prove that any real element of H has a distribution law.

Let f be a real-valued element of H. Since H is an algebra, for all m in N, f^m is in H. So, for all m in N,

$$\lim_{x\to+\infty}(1/x)\sum_{1\le n\le x} f(n)^m$$

exists.

This means that for all m in N, $\lim\limits_{l\to+\infty}\nu_l(f^m)$ exists, and as a consequence, by the "moments theorem", f has a distribution law.

2) Now, let h be a given real element of $\overline{H}$. We shall prove that

$$\lim_{x\to+\infty}(1/x)\sum_{1\le n\le x}\exp ith(n)$$

exists for all real t and is a continuous function, which will give us the conclusion by a classical result of P. Lévy.

For any k in N^*, there exists some real-valued f_k in H such that

$$\limsup_{x\to+\infty}(1/x)\sum_{1\leq n\leq x}|f_k(n)-h(n)|\leq(1/k^2).$$

We remark that

$$\limsup_{x\to+\infty}(1/x)\sum_{1\leq n\leq x,\ |f_k(n)-h(n)|>1/k}(1/k)\leq$$

$$\leq\limsup_{x\to+\infty}(1/x)\sum_{1\leq n\leq x,\ f_k(n)-h(n)|>1/k}|f_k(n)-h(n)|\leq$$

$$\leq\limsup_{x\to+\infty}(1/x)\sum_{1\leq n\leq x}|f_k(n)-h(n)|\leq(1/k^2).$$

Hence we get that

$$(1/k)\limsup_{x\to+\infty}(1/x)\sum_{1\leq n\leq x,\ |f_k(n)-h(n)|>1/k}1\leq(1/k^2),$$

i.e.

$$\limsup_{x\to+\infty}(1/x)\sum_{1\leq n\leq x,\ |f_k(n)-h(n)|>1/k}1\leq 1/k.$$

Now, we remark that, if t is any real number and k is large enough,

$$(1/x)\sum_{1\leq n\leq x}|\exp itf_k(n)-\exp ith(n)|=$$

$$=(1/x)\sum_{1\leq n\leq x}|\exp it(f_k(n)-h(n))-1|=$$

$$=\left((1/x)\sum_{1\leq n\leq x,\ |f_k(n)-h(n)|>1}|\exp it(f_k(n)-h(n))-1|\right)+$$

$$+\left((1/x)\sum_{1\leq n\leq x,\ |f_k(n)-h(n)|\leq 1/k}|\exp it(f_k(n)-h(n))-1|\right)\leq$$

$$\leq\left((1/x)\sum_{1\leq n\leq x,\ |f_k(n)-h(n)|>1/k}2\right)+$$

$$+\left((1/x)\sum_{1\leq n\leq x,\ |f_k(n)-h(n)|\leq 1/k}|t|.|f_k(n)-h(n)|\right),$$

and this gives that

$$\limsup_{x\to+\infty}(1/x)\sum_{1\le n\le x}|\exp itf_k(n)-\exp ith(n)|\le$$
$$\le\left(2\limsup_{x\to+\infty}(1/x)\sum_{1\le n\le x,|f_k(n)-h(n)|>1/k}1\right)+$$
$$+|t|.\limsup_{x\to+\infty}(1/x)\sum_{1\le n\le x}|f_k(n)-h(n)|\le(2/k)+(|t|/k^2).$$

This gives us that

$$\limsup_{x\to+\infty}\left|\left((1/x)\sum_{1\le n\le x}(\exp itf_k(n))-\left((1/x)\sum_{1\le n\le x}\exp ith(n)\right)\right)\right|\le$$
$$\le\limsup_{x\to+\infty}(1/x)\sum_{1\le n\le x}|\exp itf_k(n)-\exp ith(n)|\le(2/k)+(|t|/k^2).$$

Now, since f_k is in H, it has a distribution law and so,

$$\lim_{x\to+\infty}(1/x)\sum_{1\le n\le x}\exp itf_k(n)$$

exists and is continuous.

Hence we get that on any closed segment of the real line

$$\lim_{x\to+\infty}(1/x)\sum_{1\le n\le x}\exp ith(n)$$

exists and as a uniform limit of a sequence of continuous functions, it is continuous.

Corollary 25. *1) For all $\lambda\ge 1$ and f in $A(F)$, $m(f)$ and $m(|f|^\lambda)$ exists. 2) If h is in $A^\lambda(F)$, $m(h)$ and $m(|h|^\lambda)$ exists.*

Proof. 1) Since f is in $A(F)$, f takes only a finite number of values and can be written as

$$f=\sum_{finite}v.I_{v,f},$$

where $I_{v,f}(n)$ is the characteristic function of the n in N^* such that $f(n) = v$. Our theorem gives that $m(I_{v,f})$ exists, hence $m(f)$ also, and since

$$|f|^\lambda = \sum_{finite} |v|^\lambda . I_{v,f},$$

we have the conclusion.

2) is a simple consequence of 1) and of Hölder inequality.

A consequence of the existence of the limit distribution. We can give the following complement to the above result.

Theorem 26. *Let h be a real-valued element of $\overline{A}^\lambda(F)$ and σ_h its distribution function. Then, if t is a continuity point of σ_h, the sequence $I_{t,h}(n)$ defined by $I_{t,h}(n) = 1$ if $h(n) < t$, $= 0$ if $h(n) \geq t$, belongs to $\overline{A}(F)$.*

Proof. 1) First, we prove the following result.

Lemma 27. *The theorem is true for f a given real-valued element of $A(F)$, and more precisely, in this case, $I_{t,f}$ is in $A(F)$.*

Proof. $f(n)$ can be written as

$$f(n) = \sum_{finite} a_{s,f_s} s(n) f_s(n),$$

where s is in U and f_s is in F'. This implies that $V(F)$, the set of the values of f, is finite since $s.f_s$ takes a finite number of values. More precisely, we have

$$V(f) \subset \left\{ \sum_{finite} a_{s,f_s} X_{s,f_s};\ X_{s,f_s} \in \mathcal{F} \cup \{0\} \right\}.$$

Hence we get that if v is a value of f, then there exists a finite family $\mathcal{S}$ of distinct systems $S(v)$ of values $X_{s,f_s,v}$ of X_{s,f_s} such that

$$\sum_{finite} a_{s,f_s} X_{s,f_s,v} = v.$$

This gives us that

$$\{n; f(n) = v\} = \bigcup_{S(v)\in\mathcal{S}} \left(\bigcap_{X_{s,f_s}\in\mathcal{S}(v)} \{n; s(n) f_s(n) = X_{s,f_s}\} \right),$$

and this is a partition. As a consequence, we get that

$$I_{f=v}(n) = \sum_{S(v)\in\mathcal{S}} \left(\prod_{X_{s,f_s}\in\mathcal{S}(v)} I_{s.f_s=X_{s,f_s}}(n) \right),$$

where $I_{f=v}(n) = 1$ if $f(n) = v$, 0 if not, and similarly, $I_{s.f_s=X_{s,f_s}}(n) = 1$ if $s(n)f_s(n) = X_{s,f_s}$, 0 if not.

But we know that $I_{s.f_s=X_{s,f_s}}(n)$ is in $A(F)$, and so, $I_{f=v}(n)$ is also in $A(F)$, and this gives us the result.

2) To finish the proof, it will be sufficient to show that the function $I_{t,h}(n)$ defined by $I_{t,h}(n) = 1$ if $h(n) < t, = 0$ if $h(n) \geq t$, belongs to $\overline{A}(F)$.

Let $\varepsilon > 0$ and f be an element of $A(F)$ such that $\overline{m}(|f-h|) \leq \varepsilon$. Denote by $I_{t,f}(n)$ the function defined by $I_{t,f}(n) = 1$ if $f(n) < t, = 0$ if $f(n) \geq t$.

We have

$$\begin{aligned}
\overline{m}(|I_{t,h} - I_{t,f}|) &= \limsup_{x\to+\infty}(1/x)\sum_{n\leq x} |I_{t,h}(n) - I_{t,h}(n)| \leq \\
&\leq \limsup_{x\to+\infty}(1/x) \sum_{n\leq x,\ I_{t,h}(n)\neq I_{t,f}(n)} 1 \leq \\
&\leq \limsup_{x\to+\infty}(1/x) \sum_{n\leq x,\ I_{t,h}(n)=1,\ I_{t,f}(n)=0} 1 + \\
&+ \limsup_{x\to+\infty}(1/x) \sum_{n\leq x,\ I_{t,h}(n)=0,\ I_{t,f}(n)=1} 1.
\end{aligned}$$

Now, the set of the n such that $I_{t,h}(n) = 0,\ I_{t,f}(n) = 1$ is contained in

$$\{n\in N^*;\ f(n)\leq t,\ t+\varepsilon' \leq h(n)\} \cup \{n\in N^*;\ f(n)\leq t,\ t\leq h(n)\leq t+\varepsilon'\},$$

where ε' is chosen to satisfy the condition

$$\lim_{\varepsilon\to 0}\varepsilon' = \lim_{\varepsilon\to 0}\varepsilon/\varepsilon' = 0.$$

Since

$$\{n\in N^*;\ f(n)\leq t,\ t\leq h(n)\leq t+\varepsilon'\} \subset \{n\in N^*;,\ t\leq h(n)\leq t+\varepsilon'\},$$

the upper density of $\{n \in N^*;\ f(n) \leq t,\ t \leq h(n) \leq t+\varepsilon'\}$ is bounded by $\sigma_h(t+\varepsilon') - \sigma_h(t)$.

Now, we have the inclusion

$$\{n \in N^*;\ f(n) \le t,\ t+\varepsilon' \le h(n)\} \subset \{n \in N^*;\ \varepsilon' \le |f(n)-h(n)|\},$$

and since $\overline{m}(|f-h|) \le \varepsilon$, we get that the upper density of $\{n \in N^*;\ \varepsilon' \le |f(n)- -h(n)|\}$ is bounded by ε/ε'.

This gives us that

$$\limsup_{x\to+\infty}(1/x) \sum_{n\le x, I_{t,h}(n)=0,\ I_{t,f}(n)=1} 1 \le \sigma_h(t+\varepsilon') - \sigma_h(t) + \varepsilon/\varepsilon'.$$

In a similar way, we remark that the set of the n such that $I_{t,h}(n)=1,\ I_{t,f}(n)= =0$ is contained in

$$\{n \in N^*;\ t-\varepsilon' \le h(n) \le t\} \cup \{n \in N^*;\ f(n) \ge t,\ h(n) \le t-\varepsilon'\},$$

and the upper density of this union is bounded by $\sigma_h(t) - \sigma_h(t-\varepsilon') + \varepsilon/\varepsilon'$.

This gives us that

$$\limsup_{x\to+\infty}(1/x) \sum_{n\le x,\ I_{t,h}(n)=1,\ I_{t,f}(n)=0} 1 \le \sigma_h(t) - \sigma_h(t-\varepsilon') + \varepsilon/\varepsilon'.$$

Hence we have

$$\begin{aligned}\overline{m}(|I_{t,h}-I_{t,f}|) &\le (\sigma_h(t+\varepsilon') - \sigma_h(t) + \varepsilon/\varepsilon') + (\sigma_h(t) - \sigma_h(t-\varepsilon') + \varepsilon/\varepsilon') = \\ &= \sigma_h(t+\varepsilon') - \sigma_h(t-\varepsilon') + 2\varepsilon/\varepsilon',\end{aligned}$$

and so, t being a continuity point of σ and due to the choice of ε', we have

$$\lim_{\varepsilon\to 0} \overline{m}(|I_{t,h}-I_{t,f}|) = 0,$$

and since $I_{t,f}$ is in $A(F)$, this implies that $I_{t,h}$ is in $\overline{A}(f)$.

The next result is related to the Fourier series of an element of $\overline{A}(F)$. We have

Theorem 28. *An element h fo $\overline{A}^\lambda(F)$ admits a generalized Fourier series*

$$h(n) \sim \sum \hat{h}_{s,q} s(n) c_q(n)$$

and this series characterizes the class of h in $\overline{A}^\lambda(F)/D^\lambda$.

Proof. I) We prove the existence of the Fourier series of an element of $\overline{A}^{\lambda}(F)$.

Let f be in $\overline{A}^{\lambda}(F)$. Then, given any k in N^*, there exists an element a_k of $\bigoplus_{s\in U} s.B^{\lambda}_{inv}$, (the vector space of finite linear forms $\sum \alpha_{s,f} s.f$, where $\alpha_{s,f}$ are complex numbers, s is in U, and f is in F') such that $\left(\overline{m}(|f-a_k|^{\lambda})\right)^{1/\lambda} <$ $< k^{-1}$. The existence of the Fourier coefficients for f is a simple consequence of this approximation, for the sequence $\{a_k\}_{k\in N^*}$ is a Cauchy sequence and so, $m(a_k.\overline{s}c_q)$ is a Cauchy sequence and so it has a limit which is also equal to $m(f.\overline{s}c_q)$, expression that we shall denote by $\hat{f}_{s,c}$.

II) It remains to prove that the Fourier series characterize the classes in $\overline{A}^{\lambda}(F)/D^{\lambda}$.

It is sufficient to prove that if h is in $\overline{A}^{\lambda}(F)$ and have a Fourier series identically equal to zero, h is in D^{λ}.

To simplify the notations, the proof will be given only for $\lambda = 1$. The same patterns will work immediately for any $\lambda \geq 1$, thanks to the Hölder and Minkovski inequalities.

a) First, we prove

Lemma 29. *Let H be a complex algebra of arithmetical functions such that if g is in H, $\overline{g}$ is also in H.*

1) If f is in H and $\|f\|_{\infty} = \sup|f(n)|$ is finite, then the function $n \to$ $\to |f(n)|$ is a uniform limit of elements of H.

2) If h is in $\overline{H}$, $|h|$ is also in $\overline{H}$.

Proof. 1) Let f be an element of H such that $\|f\|_{\infty}$ is finite. We remark that f and $\overline{f}$ are in H and so, $|f|^2$ is in H. Denoting by C the upper bound of $|f(n)|$, we remark that the modulus of the function g defined by $g(n) = f(n)/C$ is in $[-1,+1]$. We have $|g| = (1-(1-|g|^2))^{1/2}$, and since $1-|g|^2$ is in $[0,1]$ and the Taylor series of the function $z \to (1-z)^{1/2}$ is uniformly convergent on $[-1,+1]$, $|g|$ is a uniform limit of polynomials in $|g|^2$, which all are in H since $|g|^2$ is in H, and the uniformity of the limit gives immediately the conclusion.

2) This is an immediate consequence of 1) and the inequality that

$$\overline{m}(\||f|-|h|\|) \leq \overline{m}(|f-h|).$$

b) Now, we prove that given h in $\overline{A}(F)$, we have

$$\lim_{L\to+\infty} \limsup_{x\to+\infty}(1/x) \sum_{n\leq x,\ |f(n)|\geq L} |h(n)| = 0.$$

This is the original definition of a uniformly summable function [1]. For sake of completeness, I give the proof of this statement.

Let h be in $\overline{A}(F)$. Then $|h|$ is in $\overline{A}(F)$ and for all k in N^*, there exists a non-negative a_k in $A(F)$ such that $\overline{m}(\||h| - a_k|) \leq 1/k$. But $\|a_k\|_\infty$ is bounded, say by L_k. This gives us that if $|f(n)| \geq 2L_k$, then $2(|f(n)| - a_k(n)) \geq |f(n)|$. Hence we get that

$$\sum_{n \leq x,\ |f(n)| \geq 2L_k} |f(n)| \leq 2 \sum_{n \leq x} |f(n) - a_k(n)| \leq 2/k,$$

and so, for any $L \geq 2L_k$, we have

$$\limsup_{x \to +\infty} \sum_{n \leq x,\ |f(n)| \geq L} |f(n)| \leq 2k,$$

and this ends the proof.

c) We can prove now that if the Fourier series of h is identical to 0, then h is in D.

i) Assume that h is not in D. Then there exists some constant $c > 0$ such that $\overline{m}(|h|) \geq 2c > 0$. Now, for $\eta > 0$, we have

$$0 < 2c \leq \overline{m}(|h|) \leq \overline{m}(|h| \leq \eta) + \overline{m}(|h| \geq \eta).$$

Since h has a distribution function, we can find some sequence of $\eta_k > 0$ such that η_k and $-\eta_k$ are continuity points of σ_h and $\overline{m}(|h| \leq \eta_k) \leq 1/k$.

This choice implies that $2c - (1/k) \leq \overline{m}(|h| \geq \eta_k)$, and as a consequence, at least one of the following inequalities holds: either $\overline{m}(h \geq \eta_k) \geq c - k^{-1}$ or $\overline{m}(-h \geq \eta_k) \geq c - k^{-1}$.

We shall assume that it is the first one.

We denote by h_k the function defined by $h_k(n) = h(n)$ if $h(n) \geq \eta_k, = 0$ if not, $I_k(n)$ the characteristic function of the n in N^* such that $h_k(n) > 0$.

Remark that $h_k(n) = h(n).I_k(n)$.

ii) We prove that for any k in N^*, there exists a non-negative element f_k of $A(F)$ such that $\|f_k\|_\infty \leq 1$ and $\overline{m}(|I_k - f_k|) \leq (1/k)$.

Proof. By density of $A(F)$, there exists some f'_k in $A(F)$ such that

$$\overline{m}(|I_k - f'_k|) \leq (1/k)$$

for I_k is in $\overline{A}(F)$. since f'_k is in $A(F)$, we can write f'_k as

$$f'_k = \sum_{finite} v.I_{v,f'_k},$$

where all the $I_{v,f'_k}(n)$, the characteristic functions of the n in N^* such that $f'_k(n) = v$, are in $A(F)$.

Since

$$|f'_k| = \sum_{finite} |v|.I_{v,f'_k}$$

is in $A(F)$ and $|1 - |f'_k||$ is also in $A(F)$, the function $f_k = \inf(1, |f'_k|)$ is also in $A(F)$ since we have

$$f_k = 1 - (1/2)\left((1 - |f'_k|) + |1 - |f'_k||\right).$$

This function f_k satisfies the conditions given in the conclusion since we have

$$m(|I_k - f_k|) \le m\left(|I_k - |f'_k||\right) \le m\left(|I_k - f'_k|\right).$$

iii) To finish the proof, we recall that $c - k^{-1} \le \overline{m}(h_k)$, and this can be written as $c - k^{-1} \le \overline{m}(h.I_k)$.

If J is any real parameter, the functions h_{J+} (resp. h_{J-}) are defined by $h_{J+}(n) = |h(n)|$ if $|h(n)| > J$, 0 if not (resp. $h_{J-}(n) = |h(n)|$ if $|h(n)| \le J$, 0 if not).

We have

$$(1/x)\sum_{n\le x} h(n).I_k(n) = (1/x)\sum_{n\le x} h(n).I_k(n) - f_k(n) + (1/x)\sum_{n\le x} h(n).f_k(n),$$

and so

$$\limsup_{x\to+\infty}(1/x)\sum_{n\le x} h(n).I_k(n) \le$$

$$\le \limsup_{x\to+\infty}\left|(1/x)\sum_{n\le x} h(n).(I_k(n) - f_k(n))\right| + \limsup_{x\to+\infty}\left|(1/x)\sum_{n\le x} h(n).f_k(n)\right|.$$

iii-1) We prove that the second term in the right member of this inequality is equal to zero.

We remark that since f_k is in $A(F)$, it can be written as

$$f_k = \sum_{finite} a_{s,f'} s.f',$$

where s is in U and f' in F'. Now, for $y > 0$ and f' in F', we define P_y by

$$P_y(n) = \prod_{p^\alpha \| n,\ p^\alpha \le y} f'(p^\alpha).$$

Since it is a periodical multiplicative function, P_y is in the algebra generated by $c_q,\ q \in N^*$ (see [4]). Moreover, it is known that

$$\lim_{y\to+\infty} m(|f' - P_y|) = 0$$

(see [4]). Now, we have $m(hf_k) = m(h(f_k - P_y)) + m(hP_y)$. By hypothesis, $m(hP_y) = 0$. So, we have to prove that $m(h(f_k - P_y)) = 0$. We have

$$m(h(f_k - P_y)) = m(h_{J-}(f_k - P_y)) + m(h_{J+}(f_k - P_y)).$$

But $|h_{J-}(f_k - P_y)| \le J|f_k - P_y|$, and so, $|m(h_{J-}(f_k - P_y))| \le Jm(|f_k - P_y|)$. And $|m(h_{J+}(f_k - P_y))| \le 2.m(|h_{j+}|)$. So, since we can choose y such that $m(|f_k - P_y|) = o(J),\ J \to +\infty$, we have the conclusion.

iii-2) We finish the proof. We have

$$\overline{m}(|h.(I_k - f_k)|) = \limsup_{x\to+\infty}(1/x)\sum_{n\le x}|h(n).(I_k(n) - f_k(n))|.$$

We write $|h(n).(I_k(n) - f_k(n))|$ as $(h_{J-}(n) + h_{J+}(n)).|(I_k(n) - f_k(n))|$, where J is a real parameter.

Now, we have

$$|h_{J+}(n).(I_k(n) - f_k(n))| \le (1 + \|f_k\|_\infty)h_{J+}(n)$$

and as a consequence of the uniform summability of h and of the inequality $\|f_k\|_\infty \le 1$, we get that

$$\overline{m}(|h_{J+}.(I_k - f_k|)) = o(1), \quad J \to +\infty.$$

Now, we remark that $|h_{J-}(n).(I_k(n) - f_k(n))| \le J.|I_k(n) - f_k(n)|$, and so we have

$$\overline{m}(|h_{J-}.(I_k - f_k|)) \le J.\overline{m}(|(I_k - f_k|)) \le J/k.$$

This leads us to the inequality

$$c - k^{-1} \leq \limsup_{x \to +\infty} (1/x) \sum_{n \leq x} h(n).I_k(n) \leq J/k + o(1), \quad J \to +\infty.$$

Putting $J = \sqrt{k}$, this gives us that $c - k^{-1} \leq 1/\sqrt{k} + o(1),\ k \to +\infty$, and so, since $c > 0$, we have a contradiction. Hence we deduce that if an element h of $\overline{A}(F)$ has a Fourier series identically equal to zero, it belongs to D.

Complement: the special case of $\overline{A}^2(F)$. In the case of $\overline{A}^2(F)$, we have the following precise result.

Theorem 30. *$\overline{A}^2(F)/D^2$ is isomorphic to the Hilbert space sum of the*

$$\bigoplus_{s \in U} s.B^2_{inv'} \quad s \in U.$$

Proof. Let f be in $\overline{A}^2(F)$. Then, given any k in N^*, there exists an element a_k of $\bigoplus_{s \in U} s.B^2_{inv}$, (the vector space of finite linear forms $\sum \alpha_{s,f} s.f$, where $\alpha_{s,f}$ are complex numbers, s is in U, and f is in B^2_{inv}) such that $\left(\overline{m}(|f - a_k|^2)\right)^{1/2} < k^{-1}$. We remark that the sequence $\{a_k\}_{k \in N^*}$ is a Cauchy sequence and that to each of the a_k, one can associate a finite set S_k of elements $s_{j(k)}$ of U. So, the family $\{a_k\}_{k \in N^*}$ belongs to $\mathcal{H}_a$, the countable sum of the Hilbert spaces $s_j B^2_{inv}$, $s_j \in S_a = \bigcup_{k \in N^*} S_k$ which is a Hilbert space of sequences (by the Marcinkievic theorem) with its norm induced by $m(|\ldots|^2)^{1/2}$. As a consequence, the limit, say a, of the a_k exists in $\mathcal{H}_a$ and is characterized by its Fourier series

$$a \sim \sum_{s \in S_a, q \in N^*} \hat{a}_{s,q} s(n) c_q(n),$$

where the coefficients satisfy the relation

$$\sum_{s \in S_a,\ q \in N^*} |\hat{a}_{s,q}|^2 \varphi(q) < +\infty.$$

It is now straightforward that a and f are in the same class of $\overline{A}^2(F)/D^2$ and this ends the proof of the theorem.

References

[1] **Indlekofer K.-H.,** On some spaces of arithmetical functions II., *Analysis Math.,* **18** (1992), 203-221.

[2] **Indlekofer K.-H.,** Number theory - Probabilistic, heuristic and computational approaches, *Computers and Mathematics with Applications,* **43** (2002), 1035-1061.

[3] **Ruzsa I.Z.,** General multiplicative functions, *Acta Arith.,* **32** (1977), 313-347.

[4] **Schwarz W. and Spilker J.,** *Arithmetical functions,* London Mathematical Society Lecture Notes Series **184**, Cambridge University Press, 1994.

[5] **Weil A.,** *L' intégration dans les groupes topologiques et ses applications,* Actualités scientifiques et industrielles **1145**, Hermann, Paris, 1941.

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