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PESSIMISTIC AND OPTIMISTIC INTERVAL SOLUTIONS OF PERTURBED LINEAR SYSTEMS

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Abstract. In this paper we estimate the error in the solution of a linear system with imprecise input data. We give the parameters by their possibilistic distribution and compute a family of possibilistic distributions of the solution. We will show that among the distributions belonging to this family there are minimal (optimistic) and maximal (pessimistic) elements by the inclusion. The results will be applied to investigate the effect of the microgeometric imperfections in the statically indetermined internal forces of a rod system acting on the points of virtual intersections.

1. Motivation of the research

A well-known method for modelling a rod system is the matrix force method. The fundamental equation, which describes the rod structure by this method is

$$\mathbf{B}^T\mathbf{R}\mathbf{B}\mathbf{x} + \mathbf{B}^T\mathbf{R}\mathbf{a} = \mathbf{0}, \tag{1}$$

where

B – matrix of $I\!R^{m\times n}$, $m \le n$ with rank $\mathbf{B} = m$. It describes the displacements of the statically determined basic system under unit loads on the points of the virtual intersection;

R – nonsingular blockdiagonal flexibility matrix of $I\!R^{m\times m}$;

a – vector of $I\!R^m$, it describes the displacements of the basic system under external loads;

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$\mathbf{x}$ – vector of $I\!R^n$ for the statically undetermined internal forces (moments) acting on the points of virtual intersections;

$\mathbf{L}$ – internal forces of the undetermined structure, where $\mathbf{L} = \mathbf{B}\mathbf{x} + \mathbf{a}$;

m – number of rod (beam) sections $\times k$, where

$$k = \begin{cases} 1 & \text{for constant internal forces,} \\ 2 & \text{for linear forces (bending moments),} \\ 3 & \text{for piecewise second degree internal force (moment) function,} \end{cases}$$

n – degree of redundancies.

With the given assumptions the matrix of the system (1) is nonsingular, therefore to find its solution with high accuracy there are lots of methods.

However, the problem of finding the solution of a linear systems, in general, belongs to the family of the so called ill posed problems. It means that a little perturbation in the input data of the problem can cause large deviations in the solution. To know the measure of these deviations is very important in the practice. For example, in the case of rod systems the flexibility matrix has the structure

$$\mathbf{R} = \mathrm{diag}\langle \mathrm{r}_1\mathbf{M}_1, \ldots, \mathrm{r}_\mathrm{k}\mathbf{M}_\mathrm{k}\rangle,$$

where

- $r_i = \frac{\ell_i}{6IE}$ for the bending moments under concentrated load, while $r_i = \frac{\ell_i}{30IE}$ for bending moments under distributed load of constant intensity per section, where
  - $\sim$ ℓ_i is the i-th section length;
  - $\sim$ IE is the stiffness for bending;
- $\mathbf{M}_i$, $(i = 1, \ldots, k)$ are symmetric positive definite matrices and

$$\sum_{i=1}^{k} \dim \mathbf{M}_i = m.$$

For the values of r_i, $(i = 1, \ldots, k)$ nominal values are known, the practical values usually differ from these ones because of the microgeometrical imperfections. To know the effect of these microgeometrical imperfections is very important, since it may contribute to damage of the structure.

However, the classical error analysis usually gives a very draft, often unbelievable for the experts estimation. The main reason of these situations is that the classical error estimation uses only the worst values from the possible ones for the parameters. It would be possible to give a probabilistic estimation taking into consideration the probability distribution of the randomly given imprecise data. But to work with the multidimensional distributions is too labor-intensive.

2. Formalization of the discussed problem

In this paper we will investigate a general linear system

$$\mathbf{F}\mathbf{x} = \mathbf{f}, \tag{2}$$

where

$\mathbf{F}$ – nonsingular matrix of $I\!R^{n\times n}$ with the elements f_{ij}, $(i = 1,\dots,n,\ j = 1,\dots,n)$;

$\mathbf{f}$ – vector of $I\!R^{n}$ with the elements f_{0i}, $(i = 1,\dots,n,\ j = 1,\dots,n)$;

$\mathbf{x}$ – solution vector of $I\!R^{n}$ with the elements x_i, $(i = 1,\dots,n)$.

We will assume, that there are given a matrix $\mathbf{E}$ of $I\!R^{n\times n}$ with the elements $e_{ij} \geq 0$, $(i = 1,\dots,n,\ j = 1,\dots,n)$ and a vector $\mathbf{e}_0$ of $I\!R^{n}$ with the elements $e_{0i} \geq 0$, $(i = 1,\dots,n)$ such that the relative error in the parameters of the perturbed system

$$\bar{\mathbf{F}}\mathbf{x} = \bar{\mathbf{f}} \tag{3}$$

fulfills the following inequality:

$$\frac{|f_{ij} - \bar{f}_{ij}|}{|f_{ij}|} \leq \varepsilon e_{ij}, \qquad (i = 1,\dots,n,\ j = 0,1,\dots,n).$$

The following questions will be discussed:

- How we can determine the exact set of perturbed solutions;
- Define the maximal subset of perturbed solution set, where all coordinates of the solution preserve their sign in the perturbed solution set;
- How we can determine interval solution;
- How we can obtain a possibilistic distribution of the solution and its marginal fuzzy values.

The results will be illustrated by an example, obtained from the model of the rod system problem.

3. Preliminaries from the fuzzy mathematics

We will say that $\tilde{a}(\mathbf{x})$ is a *normal fuzzy value* on $I\!R^n$ if for any $\mathbf{x} \in \mathbf{R^n}$ $\mathbf{x} \in [0,1]$ holds and $\exists \mathbf{x}_0$ such that $\tilde{a}(\mathbf{x}_0) = 1$. The function $\mu : I\!R^n \to I\!R$

is usually called the membership function of the fuzzy value. The set of the normal fuzzy values on $I\!R^n$ will be denoted by $\mathcal{F}(I\!R^n)$ and its elements will be denoted by either Greek letters or Latin letters with tilde.

The fuzzy value μ is said to be *convex* if its membership function is upper semi-continuous, and the λ-level sets

$$[\tilde{a}]^\lambda = \begin{cases} \{\mathbf{x} \in I\!R^n : \tilde{a}(\mathbf{x}) \geq \lambda & \text{if } \lambda \in (0,1], \\ \overline{\{\mathbf{x} \in I\!R^n : \tilde{a}(\mathbf{x}) > \lambda\}} & \text{if } \lambda = 0 \end{cases} \tag{4}$$

are convex for any $\lambda \in [0,1]$, where overline denotes the closure operator. The level set with $\lambda = 0$ is the *support* of the fuzzy value.

The normal convex fuzzy value on $I\!R$ with compact support is called *fuzzy number*. The set of fuzzy numbers will be denoted by $\mathcal{FN}$.

$\tilde{a}_i(x_i) \in \mathcal{F}(I\!R^n)$ is the i-th *marginal value* of $\tilde{a}(\mathbf{x}) \in \mathcal{F}(I\!R^n)$ if

$$\tilde{a}_i(x_i) = \max\{\tilde{a}(\mathbf{y}) : \mathbf{y} = (y_1, \ldots, y_n),\ y_i = x_i\}.$$

$\pi(\mathbf{x}) \in \mathcal{F}(I\!R^n)$ is the *joint possibility distribution* of the fuzzy numbers $a_i(x_i) \in \mathcal{FN}$, $(i = 1, \ldots, n)$, if the marginal values of $\pi(\mathbf{x})$ are $a_i(x_i)$, $(i = = 1, \ldots, n)$.

It is obvious that a given convex fuzzy value on $I\!R^n$ uniquely defines its marginal values, but it is not true inversely. A lot of fuzzy values on $I\!R^n$ has the same marginal values, consequently from a system of fuzzy numbers as marginal values of its potential joint possibility distribution we can generate different family of joint possibility distributions. One of these constructions is given in the following theorem:

Proposition 1. *Let us assume, that the following conditions hold:*

i) $a_i(x_i) \in \mathcal{FN}\ \ i = 1, \ldots, n$*;*

ii) $g(t)$ *is a continuous monotone decreasing function on* $[0,1]$ *with the boundary values* $g(0) = 1,\ g(1) = 0$ *and with pseudoinverse*

$$g^{(-1)}(x) = \begin{cases} g^{-1}(x) & \textit{if } x \in [0,1)], \\ 0 & \textit{if } x \geq 1. \end{cases}$$

iii) $\pi_p(\mathbf{x}) \in \mathcal{F}(I\!R^n)$ *is given by the formula*

$$\pi_p(\mathbf{x}) = g^{(-1)}\left(\left(\sum_{i=1}^n g^p(\tilde{a}_i(x_i))\right)^{1/p}\right)$$

if $1 \le p < \infty$ *and* $\pi_\infty(\mathbf{x}) \in \mathcal{F}(I\!R^n)$ *is given by the formula*

$$\pi_\infty(\mathbf{x}) = \min_{i=1,\dots,n} \{\tilde{a}_1(x_1), \dots, \tilde{a}_n(x_n)\}.$$

Then $\pi_p(\mathbf{x})$ *is a joint possibilistic distribution with the exponent* $p \in [1, \infty]$ *and with the marginal fuzzy values* $\tilde{a}_i(x_i)$.

Proof. In the case $1 \le p < \infty$ we have that

$$\sup_{\mathbf{y} \in I\!R^{n-1}} g^{(-1)} \left(\left(\sum_{i \ne j} g^p(\tilde{a}_i(y_i)) + g^p(\tilde{a}_j(x_j)) \right)^{1/p} \right) =$$

$$= g^{(-1)} \left(\left(\inf_{\mathbf{y} \in I\!R^{n-1}} \left(\sum_{i \ne j} g^p(\tilde{a}(y_i)) \right) + g^p(\tilde{a}_j(x_j)) \right)^{1/p} \right).$$

In the case $p = \infty$ we obtain that

$$\sup_{\mathbf{y} \in I\!R^{n-1}} \min(\min_{i \ne j} \tilde{a}_i(y_j), \tilde{a}_j(x_j) = \max(\sup_{\mathbf{y} \in I\!R^{n-1}} \min_{i \ne j} \tilde{a}_i(y_i), \tilde{a}_j(x_j)).$$

Since $\tilde{a}_j(y_j)$ is a normal fuzzy value, then there exists a point $\bar{y}_i \in I\!R$ such that $\tilde{a}(\bar{y}_i) = 1$, therefore

$$\inf_{\mathbf{y} \in I\!R^{n-1}} \left(\sum_{i \ne j} g^p(\tilde{a}_i(y_i)) \right) = 0 \quad \text{and} \quad \sup_{\mathbf{y} \in I\!R^{n-1}} \min_{i \ne j} \tilde{a}_i(y_i) = 1,$$

therefore both statements of the theorem fulfill.

Remarks.

i) The indices in the notation for the elements of this family of possibilistic distributions is consequent since with a simple computation we can observe that $\lim_{p \to \infty} \pi_p(\mathbf{x}) = \pi_\infty(\mathbf{x})$.

ii) The definition of the possibilistic distributions $\pi_p(\mathbf{x})$ can be reformulated saying that it connects the fuzzy numbers $\tilde{a}_i(x_i)$, $(i = 1, \dots, n)$ with an Archimedean t-norm given by the additive generator function $g^p(t)$.

iii) From the monotonicity of t-norms follows that

$$\pi_p(\mathbf{x}) \le \pi_q(\mathbf{x}) \ \ \forall \ \ 1 \le q \le p \le \infty.$$

The *Zadeh's extension principle* defines the mapping of a fuzzy set. If $G : I\!R^n \to I\!R$ and μ is a possibility distribution on $I\!R^n$ then

$$\tilde{G}(\mu)(y) = \sup_{y=G(\mathbf{x})} \mu(\mathbf{x}).$$

It is obvious, if $\mu_i(x_i)$, $(i = 1, \ldots, n)$ are fuzzy numbers then their different joint possibility distributions define different mappings.

Let $G_a : I\!R^n \to I\!R$ be a parametrical function on $I\!R^n$ depending on the parameter $\mathbf{a} \in I\!R^s$. Then the fuzzification of this function by the possibilistic distribution $\mu \in \mathcal{F}(I\!R^s)$ on the parameter is given by $\tilde{G}_\mu : I\!R^n \to I\!R^n \times \mathcal{F}(I\!R)$, it can be obtained by the assignation

$$\mathbf{x} \mapsto \sup_{\mathbf{a}:y=G_a(\mathbf{x})} \mu(\mathbf{a}) = G_\mu[x](y)$$

4. Optimistic and pessimistic possibilistic distribution of the solution of the linear system

For the system (2) let us use the fuzzification technique, described in [3].

Let a class of the fuzzy numbers $\mathcal{F}_g$ contain the fuzzy numbers with membership functions

$$\tilde{\mu}(u) = \begin{cases} 1, & \text{if } u = u^* \\ g^{-1}\Big(\dfrac{|u-u^*|}{h}\Big), & \text{if } |u-u^*| \le h, \\ 0, & \text{otherwise} \end{cases} \tag{5}$$

for all $u^*, h \in I\!R$, where the function g is taken from the Theorem 1. Shortly these fuzzy numbers will be described by their two parameters (u^*, h).

Assume that the parameters of the linear system (2) are given from the $\mathcal{F}_g$ in the form

$$\tilde{f}_{ij} = (f_{ij}, d_{ij}) \in \mathcal{F}_g,\ (i = 1, \ldots, n,\ j = 0, 1, \ldots, n),$$

where f_{ij}, $(i = 1, \ldots, n)$ are the nominal values of the parameters of (2) and

$$d_{ij} = e_{ij}|f_{ij}|.$$

Let $\pi_i^p(\mathbf{f})$ be the joint possibility distribution of fuzzy values $\tilde{f}_{ij}$, $(j = 0, \ldots, n)$ obtained by the Theorem 1 with $1 \le p \le \infty$. As it was shown in [2], using the the Zadeh's extension principle, we obtain for the left hand side of the i-th equality in the system $\mathbf{Fx} - \mathbf{f}_0 = 0$ that

$$\tilde{\ell}_i(\mathbf{x}, y) = g^{(-1)}\left(\frac{\left|y - \sum_{j=1}^{n} f_{ij}x_j - f_{0i}\right|}{D_i^q(\mathbf{x})}\right),$$

where

$$(6) \qquad D_i^q(\mathbf{x}) = \begin{cases} \sum\limits_{j=1}^{n} (d_{ij}^q |x_j|^q + d_{0i}^q)^{1/q} & \text{if } 1 \le q < \infty,\ q = p/(p-1), \\ \max(\max\limits_{j=1,\ldots,n} d_{ij}|x_j|, d_{i0}) & \text{if } q = \infty. \end{cases}$$

Using the fuzzification of the equality relation introduced in [2] we obtaine that

$$\sigma_i^p(\mathbf{x}) = g^{(-1)}\left(\frac{\left|\sum_{j=1}^{n} f_{ij}x_j - f_{0i}\right|}{D_i^q(\mathbf{x})}\right)$$

is the possibilistic value of the satisfaction of the i-th equation at the point $\mathbf{x}$, and $\sigma^{pr}(\mathbf{x}) = \bigwedge\limits_{i=1}^{n} \sigma_i^r(\mathbf{x})$, where $\bigwedge$ is an intersection operator given by an Archimedean t-norm with the generator function $g^p(t)$ if $1 \le r < \infty$ and $\bigwedge = \min$ if $r = \infty$.

If $\mathbf{x}^*$ is the solution of the exact system (2), then $\sigma_i(\mathbf{x}^*) = 1$ for every $i \in \{1, \ldots, m\}$.

The perturbed solution set on the grade $\lambda \ge 0$ is the solution of the inequality system

$$(7). \qquad \mathcal{N}_p(\lambda) = \left|\sum_{j=1}^{n} f_{ij}x_j - f_{0i}\right| \le g(\lambda)D_i^q(\mathbf{x}), \qquad (i = 1, \ldots, n)$$

It is obvious that $[\sigma^{pr}]^\lambda \supset [\sigma^{pr}]^\mu$ and $\mathcal{N}_p(\lambda) \supset \mathcal{N}_p(\mu)$ if $\lambda < \mu$.

The following statements show the dependence on the different parameters of the level sets and the perturbed solution sets.

Proposition 2. *If* $1 \le p \le \infty$ *and* $q = \frac{p}{p-1}$, *then*

$$[\sigma^{p1}]^\lambda \subset [\sigma^{pq}]^\lambda \subset [\sigma^{p\infty}]^\lambda = \mathcal{N}_p(\lambda) \;\; \forall\lambda \in [0,1].$$

Proof. The inclusions follow from the monotonicity in p of the t-norms. The last equality follows from the definitions of the given sets.

Proposition 3. $\mathcal{N}_1(\lambda) \subset \mathcal{N}_p(\lambda) \subset \mathcal{N}_q(\lambda) \subset \mathcal{N}_\infty(\lambda)$ *for all* $\lambda \in [0,1]$, *where* $1 \le p \le 2$ *and* $q = poverp - 1$.

Proof. Since $D_i^q(\mathbf{x})$ is the weighted q-norm of the vector $\mathbf{z} = (\mathbf{x}, 1)$, from the ordering of different norms in the n-dimensional Euclidean space follows the statement.

The error-estimation given by the exponent $p = 1$ or $p = \infty$ will be called *optimistic* and *pessimistic* error-estimation, respectively and in the case $1 < < p < \infty$ we will speak about p-*intermediate* error estimation.

The Figure 2 shows the perturbed solution sets for the system

$$(8) \qquad \begin{aligned} 4x + 1y &= 2000 \\ x + 10y &= 4000 \end{aligned}$$

in the cases $p = 1, 2, \infty$.

5. Optimistic and pessimistic interval solutions

To find the possible perturbation interval for every coordinate of the solution let us consider the optimization problems

$$(9) \qquad \min x_i \qquad \text{and} \qquad \max x_i, \qquad (i = 1, \dots, m)$$

subject to

$$(10). \qquad \mathbf{x} \in \mathcal{N}_p(0).$$

These problems define the lower and upper bounds of the perturbed solutions of (3), that is these values define an n-dimensional interval which comprises all the solutions of the perturbed system. Figure 1 shows this interval solution for the system (8), too.

The problems (9)-(10) are well defined if $\mathcal{N}_p(0)$ is bounded.

Proposition 4. *$\mathcal{N}_p(\lambda)$ is bounded for all $1 \leq p \leq \infty$ if $\| \mathbf{D} \|_q \| F^{-1} \|_q <$ < 1, where $\mathbf{D}$ is a matrix with the elements d_{ij} and $\| \mathbf{A} \|_q = \sum_{i=1=1}^{n} a_{ij}^q {}^{1/q}$.*

Proof. It is enough to prove, that $\mathcal{N}_p(0)$ is bounded.

Let us assume that there exist $\mathbf{x} \in \mathcal{N}_p(0)$ and $\mathbf{s} \in I\!R^n$ such that $\mathbf{x} + \alpha\mathbf{s} \in$ $\in \mathcal{N}_p(0)$ for all $\alpha > 0$. It means that

$$\left| \sum_{j=1=1}^{n} f_{ij}(x_j + \alpha s_j) \right| \leq D_i^q(\mathbf{x} + \alpha\mathbf{s}),$$

where $D_i^q(\mathbf{x})$ is defined in (6). Dividing by α the last inequalities and taking the limit $\alpha \to \infty$ we obtain that

$$\left| \sum_{i=1=1}^{n} f_{ij} s_j \right| \leq \| \mathbf{D}_i \mathbf{s} \|_q \leq \| \mathbf{D} \|_q \| \mathbf{s} \|_q, \tag{11}$$

where $\mathbf{D}_i$ is a diagonal matrix, the j-th diagonal element of which is d_{ij}. From (11) follows that

$$\| \mathbf{Fs} \|_q \leq \| \mathbf{D} \|_q \| \mathbf{s} \|_q \, . \tag{12}$$

Let $\mathbf{z} = \mathbf{Fs}$. Then (12) turns into the inequality

$$1 \leq \| \mathbf{D} \| \frac{\| \mathbf{F}^{-1}\mathbf{z} \|_q}{\| \mathbf{z} \|_q} \leq \| \mathbf{D} \|_q \| \mathbf{F}_{-1} \|_q,$$

which contradicts to the assumption of the theorem.

To obtain the numerical solution of the problem (9)-(10) is not trivial. It can be explained by the fact that the set $\mathcal{N}(0)$ is, in general, not convex. The convexity can be guaranteed only if the coordinates of the solutions do not change sign under the possible perturbation. In this case $\mathcal{N}(0)$ is a polieder contained in that ortant in which the nominal solution is present. This permits the symbols for absolute value to be removed from the definition of $\mathcal{N}_p(0)$.

So, we have to solve the problem

$$\min/\max \;\; x_i \qquad (i = 1, \ldots, m) \tag{13}$$

subject to

$$\left| \sum_{j=1}^{n} f_{ij} x_j - f_{0i} \right| \leq D_i^q(\mathbf{x}), \qquad (i = 1, \ldots, n). \tag{14}$$

$$x_j \mathrm{sg} x_j^* \geq 0, \qquad (j = 1, \ldots, n), \tag{15}$$

where $\mathbf{x}^*$ is the solution of (2).

If any of the problems(13)-(15) has a solution vector in which at least one coordinate is equal to zero, than this coordinate can change sign. Let I denote the set of indeces of these coordinates. Then changing the constraining ortant (15) by one of the adjacent ortant on the side of the i-th axis, where $i \in I$ and resolve the problem (13(-(15) we can find the solutions of (11)-(12) after no more than 2^n iterations.

6. Application for a rod systems

Let the examined rod system given on the Figure 1, which is described by the parameters

$$\mathbf{B}^T = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{R} = \begin{bmatrix} 2 & 1 & & & & \\ 1 & 2 & & & & \\ & & 2 & 1 & & \\ & & 1 & 2 & & \\ & & & & 4 & 2 \\ & & & & 2 & 4 \end{bmatrix}, \quad \mathbf{a} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2000 \\ 0 \\ 0 \end{bmatrix}.$$

If the nominal values of the parameters are

$$r_1 = 1, \ r_2 = 1, \ r_3 = 2,$$

then the exact system is given by (8), and its nominal solution will be

$$x_1^* = -347.826, \ x_2^* = -608.697,$$

furthermore, for the stress vector belonging to the nominal forces we obtain

$$\mathbf{L}^T = [\underbrace{0, -347.8}_{1.}, \underbrace{-347.8, 1391.3}_{2.}, \underbrace{-608.7, 0}_{3.}] \qquad \text{number of sections} .$$

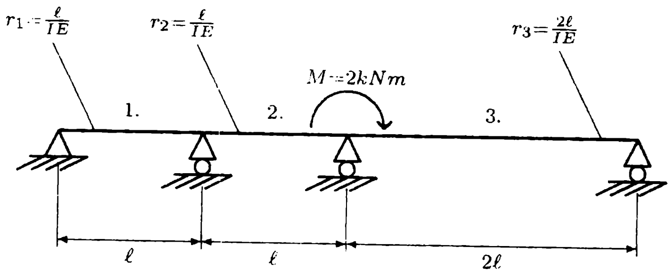

Figure 1.

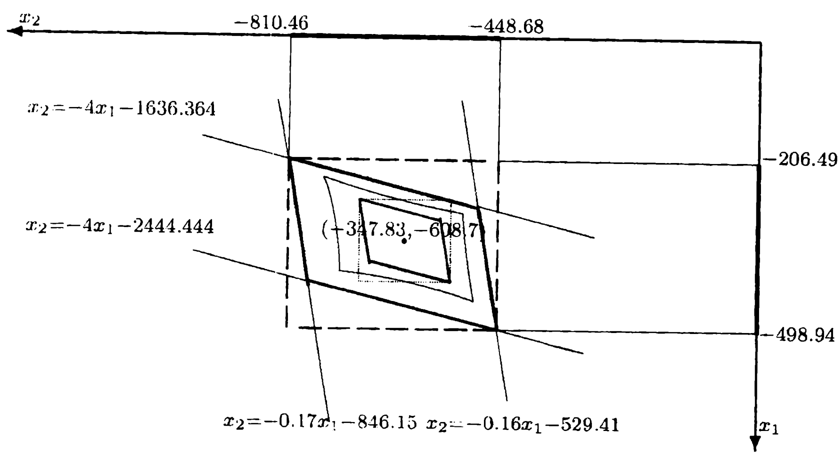

Figure 2.

If the triangular fuzzy numbers

$$\tilde{r}_i = (r_i, \varepsilon e_i r_i) \in \mathcal{F}_g,$$

where $\varepsilon = 0.1$, $e_1 = e_2 = 1$, $e_3 = 1.5$, are used to model the imperfect flexibility matrix elements, then the fuzzy coefficients are taken also from $\mathcal{F}_g$. The perturbed constraints set for $p = 1, 2, \infty$ and the possibilistic intervals of the solution vector for $p = 1, \infty$ are seen on the Figure 2 with $\varepsilon = 0.1$.

Finally, the possibilistic intervals of the final load can be given by the inclusion

$$\mathbf{Bx} + \mathbf{a} \in \underset{\textit{pessimistic}}{\begin{bmatrix} [0, 0] \\ [-498.94, -206.49] \\ [-498.94, -206.49] \\ [118.54, 1551.32] \\ [-810.46, -448.68] \\ [0, 0] \end{bmatrix}} \text{ or } \underset{\textit{optimistic}}{\begin{bmatrix} [0, 0] \\ [-417.42, -278.37] \\ [-417.42, -278.37] \\ [1313.49, 1469.71] \\ [-686.51, -530.29] \\ [0, 0] \end{bmatrix}}$$

References

[1] **Dubois D. and Prade H.,** *Fuzzy sets and systems. Theory and applications*, Academic Press, New York, 1980.

[2] **Kovács M.,** A stable embedding of ill-posed linear systems into fuzzy systems. *Fuzzy Sets and Systems*, **45** (3) (1992), 305-312.

[3] **Nándori E. and Kovács M.,** Possibilistic estimations of imperfections in structural analysis, *Acta Technica Acad. Sci. Hung.,* **107** (3-4) (1995-96), 369-394.

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