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GRÖBNER BASES FOR PERMUTATIONS AND ORIENTED TREES

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This paper is dedicated to professor Imre Kátai on the occasion of his 65th birthday

Abstract. Let $\mathbb{F}$ be a field. We describe Gröbner bases for the ideals of polynomials vanishing on the sets X_n and Y_m. Here $X_n = X(\alpha_1, \ldots, \alpha_n)$ is the set of all permutations of some $\alpha_1, \ldots, \alpha_n \in \mathbb{F}$. Y_m is the set of characteristic vectors of the oriented trees on an m-element vertex set.

1. Introduction

Let $\mathbb{F}$ be a field and $n \geq 1$ an integer. For a subset X of the affine space $\mathbb{F}^n$ one may consider the ideal $I(X)$ of polynomial functions $f \in S = \mathbb{F}[x_1, \ldots, x_n]$ vanishing on X. Many interesting (combinatorial) properties of X can be formulated in terms of the polynomial functions $X \to \mathbb{F}$. This approach leads to the study of $I(X)$, and in particular to the study of Gröbner bases, standard monomials and the Hilbert function of $S/I(X)$ (see Subsection 1.1 for definitions). In [11], [133] we described Gröbner bases and related data for the complete uniform families, i.e., when X consists of all 0,1-vectors in $\mathbb{F}^n$ which, for a fixed k, have precisely k ones as coordinate values. Applications are given in [3], [11], [13] and [14].

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In this note we consider another two types of interesting finite subsets of $\mathbb{F}^n$. Let $\alpha_1, \ldots, \alpha_n$ be n different elements of $\mathbb{F}$ and put

$$X_n := X_n(\alpha_1, \ldots, \alpha_n) := \{(\alpha_{\pi(1)}, \ldots, \alpha_{\pi(n)}) : \ \pi \in S_n\}.$$

X_n is the set of all permutations of the α_i, viewed as a subset of $\mathbb{F}^n$.

For general terminology on directed graphs (path, circuit, cycle, etc.) we refer to Lovász [17]. An *oriented tree* with vertex set V is a weakly connected directed graph T on V with $|V|-1$ edges such that there is a $v \in V$, the root of T, which is reachable by a directed path from every $w \in V$. An *oriented forest* is a digraph whose weak components are oriented trees.

Let m be a positive integer and $\mathcal{T}_m$ be the set of all oriented trees with vertex set $[m] := \{1, 2, \ldots, m\}$. It is known that $|\mathcal{T}_m| = m^{m-1}$, see for example Section 2.3.4.4 in [15], or § 4 in [17]. We represent the trees $T \in \mathcal{T}_m$ by their characteristic vectors $v(T) \in F^n$, where $n = m(m-1)$. The coordinate functions $x_{(i,j)}$ in F^n are indexed with directed edges (i,j), $i \neq j \in [m]$. The (i,j)-component $v(T)_{(i,j)}$ is 1 if (i,j) is an edge of T and $v(T)_{(i,j)} = 0$ otherwise. We put

$$Y_m := \{v(T) : \ T \in \mathcal{T}_m\} \subseteq \mathbb{F}^n.$$

In Theorems 2.2 and 3.2 we describe Gröbner bases for the ideals $I(X_n)$ and $I(Y_m)$ above. Before formulating the precise statements, we overview the facts from the theory of Gröbner bases we need later on.

Suppose that we have a set of variables x_ℓ indexed by elements ℓ of a set J. In the sequel J will be either $[n]$, or the set of edges of the complete digraph KD_m on $[m]$. For a subset $H \subseteq J$ we denote by x_H the monomial $\prod_{\ell \in H} x_\ell$, in particular, $x_\emptyset = 1$.

1.1. Gröbner bases and standard monomials

A total ordering $\prec$ on the monomials $x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}$ from variables $x_1, x_2, \ldots, x_n$ is a *term order*, if 1 is the minimal element of $\prec$, and $uw \prec vw$ holds for any monomials u, v, w with $u \prec v$. There are many term orders, important examples being the lexicographic order $\prec_l$ and the deglex order $\prec_{dl}$. We have

$$x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} \prec_l x_1^{j_1} x_2^{j_2} \cdots x_n^{j_n}$$

iff $i_k < j_k$ holds for the smallest index k such that $i_k \neq j_k$. As for deglex, we have $u \prec_{dl} v$ iff either $\deg u < \deg v$, or $\deg u = \deg v$, and $u \prec_l v$.

The *leading monomial* $\mathrm{lm}(f)$ of a nonzero polynomial f from the ring $S = \mathbb{F}[x_1, x_2, \dots, x_n]$ is the largest (with respect to $\prec$) monomial which occurs with nonzero coefficient in the standard form of f.

Let I be an ideal of S. A finite subset $G \subseteq I$ is a *Gröbner basis* of I if for every $f \in I$ there exists a $g \in G$ such that $\mathrm{lm}(g)$ divides $\mathrm{lm}(f)$. A term order is well-founded, implying that G generates I, i.e. $\mathcal{G}$ is a basis of I. A fundamental fact is (cf. [10, Chapter 1, Corollary 3.12] or [1, Corollary 1.6.5, Theorem 1.9.1]) that every nonzero ideal I of S has a Gröbner basis with respect to any term order $\prec$.

A monomial $w \in S$ is a *standard monomial* for I if it is not a leading monomial of any $f \in I$. Let $\mathrm{sm}(\prec, I, \mathbb{F})$ stand for the set of all standard monomials of I with respect to the term-order $\prec$ over $\mathbb{F}$. It is known (see [10, Chapter 1, Section 4]) that for a nonzero ideal I the set $\mathrm{sm}(\prec, I, \mathbb{F})$ is a basis of the $\mathbb{F}$-vector space S/I. More precisely every $g \in S$ can be written uniquely as $g = h + f$ where $f \in I$ and h is a unique $\mathbb{F}$-linear combination of monomials from $\mathrm{sm}(\prec, I, \mathbb{F})$.

Now if $X \subseteq \mathbb{F}^n$ is a finite set, then an easy interpolation argument gives that every function from X to $\mathbb{F}$ is a polynomial function. The latter two facts imply that

$$(1) \qquad |\mathrm{Sm}(\prec, I(X), \mathbb{F})| = |X|.$$

A Gröbner basis $\{f_1, \dots, f_m\}$ of I is *reduced* if the coefficient of $\mathrm{lm}(f_i)$ is 1, and no nonzero monomial in f_i is divisible by any $\mathrm{lm}(f_j)$, $j \neq i$. By a theorem of Buchberger ([1, Theorem 1.8.7]) a nonzero ideal has a unique reduced Gröbner basis.

The *initial ideal* $\mathrm{in}(I)$ of I is the ideal in S generated by the monomials $\{\mathrm{lm}(f) : f \in I\}$.

Next we introduce reduction, a notion closely related to Gröbner bases. Let $\mathcal{G}$ be a set of polynomials in $\mathbb{F}[x_1, \dots, x_n]$ and let $f \in \mathbb{F}[x_1, \dots, x_n]$ be a fixed polynomial. Let $\prec$ be an arbitrary term-order. We can reduce f by the set $\mathcal{G}$ with respect to $\prec$. This gives a new polynomial $h \in \mathbb{F}[x_1, \dots, x_n]$.

Here *reduction* means that we possibly repeatedly replace monomials in f by smaller ones (with respect to $\prec$) as follows: if w is a monomial occurring in f and $\mathrm{lm}(g)$ divides w for some $g \in \mathcal{G}$ (i.e. $w = \mathrm{lm}(g)u$ for some monomial u), then we replace w in f with $u(\mathrm{lm}(g) - g)$. Clearly the monomials in $u(\mathrm{lm}(g) - g)$ are $\prec$-smaller than w. If $\mathcal{G}$ is a Gröbner basis then any $f \in S$ can be reduced into a (unique) $\mathbb{F}$-linear combination of standard monomials.

Let I be an ideal of $S = \mathbb{F}[x_1, \dots, x_n]$. The *Hilbert function* of the algebra S/I is the sequence $h_{S/I}(0), h_{S/I}(1), \dots$. Here $h_{S/I}(m)$ is the dimension over $\mathbb{F}$

of the factor space $\mathbb{F}[x_1,\dots,x_n]_{\le m}/(I\cap\mathbb{F}[x_1,\dots,x_n]_{\le m})$ (see [5, Section 9.3]). It is easy to see that $h_{S/I}(m)$ is the number of standard monomials of degree at most m, where the ordering $\prec$ is deglex.

In the case when $I=I(X)$ for some $X\subseteq F^n$, the number $h_X(m):=$ $:=h_{S/I}(m)$ is the dimension of the space of functions from X to $\mathbb{F}$ which are polynomials of degree at most m.

2. Permutations

We recall the definition of the complete symmetric polynomials. Let i be a nonnegative integer and write

$$h_i(x_1,\dots,x_n)=\sum_{a_1+\dots+a_n=i} x_1^{a_1}x_2^{a_2}\cdots x_n^{a_n}.$$

Thus, $h_i\in\mathbb{F}[x_1,\dots,x_n]$ is the sum of all monomials of total degree i. For $0\le i\le n$ we write σ_i for the i-th elementary symmetric polynomial:

$$\sigma_i(x_1,\dots,x_n)=\sum_{S\subset[n],\ |S|=i} x_S.$$

$\sigma_i\in\mathbb{F}[x_1,\dots,x_n]$ is the sum of all square free monomials of degree i in the variables $x_1,\dots,x_n$.

Let $\alpha_1,\dots,\alpha_n$ be n different elements of $\mathbb{F}$, and $X_n=X_n(\alpha_1,\dots,\alpha_n)\subseteq\mathbb{F}^n$ be the set of permutations of $\alpha_1,\dots,\alpha_n$.

For $1\le k\le n$ we introduce the polynomials $f_k\in S$ as follows:

$$f_k=\sum_{i=0}^{k}(-1)^i h_{k-i}(x_k,x_{k+1},\dots,x_n)\sigma_i(\alpha_1,\dots,\alpha_n).$$

We remark, that $f_k\in\mathbb{F}[x_k,x_{k+1},\dots,x_n]$. Moreover, $\deg f_k=k$ and the leading monomial of f_k is x_k^k with respect to any term order $\prec$ for which $x_1\succ x_2\succ\ldots\succ x_n$.

Proposition 2.1. *Let $v\in X_n$. Then $f_k(v)=0$ for $1\le k\le n$.*

Proof. The statement is immediate from the following known (see, e.g. [9, p. 314]) identities. Let $1 \le k \le n$. Then

$$\sum_{i=0}^{k}(-1)^i h_{k-i}(x_k,\dots,x_n)\sigma_i(x_1,\dots,x_n) = 0. \tag{2}$$

For the convenience of the reader we sketch a proof of (2). For a fixed k one verifies first that

$$\sigma_i(x_1,\dots,x_n) = \sum_{S\subseteq[k-1]} x_S\sigma_{i-|S|}(x_k,\dots,x_n), \tag{3}$$

where we understand $\sigma_j = 0$ for $j < 0$.

We need also the fundamental relation connecting complete symmetric polynomials to the elementary ones, see [20, Theorem 4.3.7] or [18, p.14]. If t, m are positive integers then, with the convention $\sigma_i = 0$ for $i > m$, we have

$$\sum_{i=0}^{t}(-1)^i h_{t-i}(w_1,\dots,w_m)\sigma_i(w_1,\dots,w_m) = 0. \tag{4}$$

Now using (3), we obtain

$$\sum_{i=0}^{k}(-1)^i h_{k-i}(x_k,\dots,x_n)\sigma_i(x_1,\dots,x_n) =$$

$$= \sum_{S\subseteq[k-1]} x_S \sum_{j=|S|}^{k}(-1)^i h_{k-i}(x_k,\dots,x_n)\sigma_{i-|S|}(x_k,\dots,x_n).$$

To establish (2), it suffices to verify that the coefficient of x_S is 0 for every $S \subseteq [k-1]$. For this we can apply (4) with $t = k - |S| > 1$, and $m = n - k + 1$.

We can state now the main result of this section. A related weaker statement is given in [9, Proposition 5, Chapter 7].

Theorem 2.2. *Let $\mathbb{F}$ be a field and let $\prec$ be an arbitrary term order on the monomials of $\mathbb{F}[x_1,\dots,x_n]$ such that $x_n \prec \dots \prec x_1$. Then the reduced Gröbner basis of $I(X_n)$ is*

$$\{f_i : \ 1 \le i \le n\}.$$

Moreover the set of standard monomials is

$$\mathrm{Sm}(\prec, I(X_n), \mathbb{F}) = \{x_1^{\alpha_1}\dots x_n^{\alpha_n} : \ 0 \le \alpha_i \le i-1, \ \text{for } 1 \le i \le n\}. \tag{5}$$

Proof. Let $\mathcal{M}$ denote the set of monomials on the right hand side of (5). The leading monomial of f_k is x_k^k, hence if a monomial w is not in $\mathcal{M}$ then w is clearly a leading term for $I(X_n)$. We infer that the standard monomials are a subset of $\mathcal{M}$. The reverse inclusion follows at once from $|\mathcal{M}| = n! = |X_n|$ and (1). Now (5) implies that the monomials x_k^k, $(1 \leq k \leq n)$ generate the initial ideal for $I(X_n)$, therefore $\{f_1, \dots f_n\}$ is a Gröbner basis for $I(X_n)$.

Reducedness is immediate: on one hand, there are no divisibilities among the x_k^k. On the other hand, except for the leading term, all monomials in f_k are standard monomials.

In [2] E. Artin proved that $\mathcal{M}$ is a basis of the quotient ring

$$\mathbb{F}[x_1, \dots, x_n]/(\sigma_1, \dots, \sigma_n).$$

Our result can be considered as a refinement of Artin's theorem. We call the elements of $\mathcal{M}$ *Artin monomials*.

There is a useful and simple bijection between permutations and Artin monomials, more precisely their exponent vectors. This is the Hall map [16, Section 5.1.1]. To a permutation π of $\{1, \dots, n\}$ the Hall map associates the sequence of integers $b_n, b_{n-1}, \dots, b_1$, where b_j is the number elements $k \in [n]$ such that $k > j$ and k appears in π to the left of j. Clearly we have $b_i \leq n - i$ for $i = 1, \dots, n$, hence $x_1^{b_n} x_2^{b_2} \cdots x_n^{b_1} \in \mathcal{M}$. It is not hard to show that this map is invertible. Monomials of degree k correspond under the Hall map to permutations with exactly k inversions. These latter objects have been studied intensively. Writing simply $h(m)$ for the Hilbert function $h_{S/I(X_n)}(m)$, we have

$$h(m) - h(m-1) = I_m(n), \quad m = 1, 2, \dots, \binom{n}{2},$$

where $I_m(n)$ is the number of permutations of n symbols with m inversions. In [16, Section 5.1.1.] there are some explicit formulae for $I_m(n)$, $m \leq n$. Asymptotic estimates are given in [7] and [19].

The Fundamental Theorem on Symmetric Polynomials asserts that every symmetric polynomial $f \in \mathbb{F}[y_1, \dots, y_n]$ admits a unique expression of the form

$$f = \sum_{p \geq 0} a_p \sigma_1^{p_1} \sigma_2^{p_2} \cdots \sigma_n^{p_n},$$

where $p = (p_1, p_2, \dots, p_n)$, $a_p \in \mathbb{F}$, and the σ_i are the elementary symmetric polynomials in the y_i. In [12] Garsia obtained a beautiful generalization. Here we present a simple proof. Let $\mathcal{N}$ be the set of Artin monomials in the y_i (we substitute y_i in the place of x_i for every $w \in \mathcal{M}$).

Corollary 2.3. *Every polynomial* $f \in \mathbb{F}[y_1, \ldots, y_n]$ *has a unique expansion of the form*

$$f(y_1, \ldots, y_n) = \sum_{w \in \mathcal{N}} \sum_{p \geq 0} a_{w,p} w \sigma_1^{p_1} \sigma_2^{p_2} \cdots \sigma_n^{p_n},$$

where $a_{w,p} \in \mathbb{F}$.

Proof. Let $\{y_1, \ldots, y_n\}$ be variables, and consider the set of permutations $X_n = X_n(y_1, \ldots, y_n)$ in $\mathbb{K}^n$, where $\mathbb{K}$ is the function field $\mathbb{F}(y_1, \ldots, y_n)$. The polynomial $f(x_1, \ldots, x_n)$ can be considered as an element of $\mathbb{K}[x_1, \ldots, x_n]$. We apply the preceding Theorem with $\mathbb{K}$ in the place of $\mathbb{F}$ and $\alpha_i = y_i$. The reduction of $f(x_1, \ldots, x_n)$ with respect to $f_1, \ldots, f_n$ shows the existence of a unique expansion of the form

$$f(x_1, \ldots, x_n) = \sum_{w \in \mathcal{M}} w g_w, \tag{6}$$

where $g_w \in \mathbb{F}[y_1, \ldots, y_n]$ are symmetric polynomials in the y_i. This holds because the leading coefficient of an f_i is 1, and the non leading terms of f_i are of the form $w g_w$ as above. The two sides of (6) are equal as functions on X_n. Now the substitutions $x_i = y_i$ and the Fundamental Theorem on Symmetric Polynomials gives the claim.

Remark. The Corollary, together with the proof we presented here, offers an algorithmic version of the fact that $\mathbb{F}[y_1, \ldots, y_n]$ is a free module of rank $n!$ over the ring of symmetric polynomials. The reduction procedure gives an expression of f in terms of the Artin basis.

3. Oriented trees

Let m be a positive integer. Recall that Y_m is the set of characteristic vectors of the oriented trees on $[m]$. We have $Y_m \subset \mathbb{F}^n$, where $n = m(m-1)$. The coordinate functions on $\mathbb{F}^n$ are indexed with the edges of the complete directed graph KD_m with vertex set $[m]$. We consider the ideal $I(Y_m)$ in the polynomial ring $S = \mathbb{F}[x_{(i,j)} : \ 1 \leq i, j \leq m, \ i \neq j]$. We work with the lexicographic order, where the ordering of the variables is as follows:

$$x_{(2,1)} \succ x_{(3,1)} \succ \ldots \succ x_{(m,1)} \succ x_{(1,2)} \succ \ldots \succ x_{(1,m)} \succ x_{(3,2)} \succ \ldots, \tag{7}$$

i.e. the edges entering 1 are the largest, then follow the edges leaving 1, and we proceed similarly for $2, 3, \ldots, m$.

In this section i, j, k denote three different integers from $[m]$. We introduce four sets of polynomials.

$$\mathcal{A} = \{x_{(i,j)}^2 - x_{(i,j)} :\ j > 1\},$$
$$\mathcal{B} = \{x_{(i,j)} x_{(i,k)} :\ j, k > 1\},$$

$$\mathcal{C} = \{x_C :\ C \text{ is a directed cycle in } KD_m, \text{ which avoids vertex } 1\}.$$

We define the polynomials $g_i \in S$, $i > 1$ as follows

$$g_i := \left(-1 + \sum_{j \neq i} x_{(i,j)}\right)\left(-1 + \sum_P x_P\right) - \sum_C x_C, \tag{8}$$

where P ranges over the directed paths in KD_m from 1 to i and C runs through the subgraphs KD_m which consist of a directed path Q from 1 to i and an edge (i, j), where j is a node on Q. Please note that the leading term of g_i is $x_{(i,1)}$. Also, simplification of (8) shows that the non leading terms of g_i are of the shape $\alpha_F x_F$, where $\alpha_F \in \mathbb{F}$ and F is an oriented forest on $[m]$, without edges entering 1. We put

$$\mathcal{D} = \{g_i :\ i > 1\},$$

and set

$$\mathcal{G} := \mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \cup \mathcal{D}.$$

Proposition 3.1. *We have $\mathcal{G} \subseteq I(Y_m)$.*

Proof. The polynomials from $\mathcal{A}$ vanish on all 0,1-vectors. We have $f(v(T)) = 0$ for $f \in \mathcal{B}$ and $T \in \mathcal{T}_m$ because the out-degree of a vertex in T is at most 1. The polynomials from $\mathcal{C}$ vanish on $v(T)$ because T does not contain directed cycles.

Finally $g_i(v(T)) = 0$ for $i > 1$ because either the out-degree of vertex i in T is 1, or else i is the root of T, and hence T contains a directed path from 1 to i.

Let $\mathcal{F}_m$ be the set of all oriented forests F on $[m]$ which do not contain edges entering vertex 1. We note that $|\mathcal{T}_m| = |\mathcal{F}_m|$. Indeed, from a tree $T \in \mathcal{T}_m$ we obtain a forest $F \in \mathcal{F}_m$ by just deleting the edges entering vertex 1. This map is invertible: from F we recover T by adding edges $(i, 1)$, where i is the root of a component C of F for which $1 \notin C$.

Theorem 3.2. *Let $\mathbb{F}$ be a field and let $\prec$ be the lex order on the monomials of S, as specified in (7). Then $\mathcal{G}$ is the reduced Gröbner basis of $I(Y_m)$. Moreover the set of standard monomials is*

$$\mathrm{Sm}(\prec, I(Y_m), \mathbb{F}) = \{x_F :\ F \in \mathcal{F}_m\}. \tag{9}$$

Proof. We prove first (9). In view of $|\mathcal{T}_m| = |\mathcal{F}_m|$ and (1) it suffices to show $\subseteq$. Let w be a monomial which is not divisible by the leading monomial of any $f \in \mathcal{G}$. The leading term of g_i is $x_{(i,1)}$, hence w does not contain any of the these variables. The polynomials in $\mathcal{A}$ ensure now that w is not divisible by the square of any variable, hence $w = x_G$ for some subgraph G of KD_m. The monomials in $\mathcal{B}$ do not divide w, hence the out-degree of any vertex of G is at most 1. Likewise, the monomials in $\mathcal{C}$ ascertain that G does not contain directed cycles. It is a simple and well known (see, e.g. Exercise 2.3.4.2.7 in [15]) fact that such a G must be an oriented forest. As G has no edges entering 1, we conclude that $G \in \mathcal{F}_m$.

The argument above gives also that the leading terms of $\mathcal{G}$ generate the initial ideal of $I(Y_m)$, hence $\mathcal{G}$ is a Gröbner basis of $I(Y_m)$.

Concerning reducedness, the set of the leading monomials of $\mathcal{G}$ is

$$\{x^2_{(i,j)} : \ j > 1\} \cup \mathcal{B} \cup \mathcal{C} \cup \{x_{(i,1)} : \ i > 1\},$$

and there are no nontrivial divisibilities among these monomials. The other (non leading) terms of an $f \in \mathcal{G}$ are all standard monomials.

In this case we have a nice formula for the number of standard monomials of degree i. We shall use a result of Clarke [8]: for $1 \le k \le m-1$ let $L(m,k)$ denote the number of undirected spanning trees on the vertex set $[m]$ where the degree of 1 is k. Clarke proved that

$$L(m,k) = \binom{m-2}{k-1}(m-1)^{m-k-1}.$$

We observe that for $m \ge 2$, $L(m,k)$ is the number of oriented forests on $\{2,\dots,m\}$ with $m-1-k$ edges. Indeed from such an oriented forest we obtain a spanning tree on $[m]$ by joining the roots of the trees to 1 and then forgetting the orientation of edges. This map is clearly invertible.

Proposition 3.3. *We have*

$$|\{F \in \mathcal{F}_m : \ F \text{ has exactly } i \text{ edges}\}| = \binom{m-1}{i}(m-1)^i$$

for $0 \le i \le m-1$.

Proof. The formula is obviously correct for $m = 1, 2$ and in general for $i = 0$. The case $i = m-1$ is also easy. Then F spans an oriented tree on $\{2,\dots,m\}$ and has an edge leaving 1. The number of such graphs is $(m-$ $-1)^{(m-2)}(m-1) = (m-1)^{(m-1)}$.

Assume now that $1 < i < m-1$. Let $F \in \mathcal{F}_m$ and $|F| = i$. Then F has 0 or 1 edges starting at 1. The subgraph of F spanned by $\{2, \dots, m\}$ is an oriented tree on $m-1$ points with i edges in the former case and with $i-1$ edges in the latter case.

The number of possibilities therefore is

$$L(m, m-1-i) + L(m, m-i)(m-1) =$$

$$= \binom{m-2}{m-2-i}(m-1)^i + \binom{m-2}{m-i-1}(m-1)^{i-1}(m-1) =$$

$$= \left(\binom{m-2}{i} + \binom{m-2}{i-1}\right)(m-1)^i = \binom{m-1}{i}(m-1)^i.$$

The proof is complete.

4. A concluding remark

It would be interesting to give Gröbner bases for $I(Y_m)$ with respect to some degree compatible order $\prec$, such as deglex. This would likely be helpful to determine the Hilbert function of $S/I(Y_m)$.

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