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DIOPHANTINE EQUATIONS WITH QUADRATIC FORMS

G. Belozorov and P. Varbanets (Odessa, Ukraine)

Dedicated to Professor Imre Kátai on his 65th birthday

1. Introduction

Let $\varphi_1(u,v)$ and $\varphi_2(u,v)$ be two quadratic forms. We denote $I_h(\varphi_1,\varphi_2,N)$ the number represetations of a natural k in the form

$$k = \varphi_1(u_1,v_1) - \varphi_2(u_2,v_2)$$

under the condition $|\varphi_1(u_1,v_1)| \le N$, $u_1,v_1 \in \mathbb{Z}$.

In the works [1]-[5] the diophantine equation

$$(1) \qquad u_1v_1 - u_2v_2 = 1, \quad u_1,v_1 \in \mathbb{N}, \quad u_1v_1 \le N$$

is investigated. This problem is equivalent to the problem on the estimation of the sum

$$\sum_{n\le N} \tau(n)\tau(n+1),$$

where $\tau(n)$ is the divisor function.

Y.Motohashi [6] proved the asymptotic formula for $1 \le h \le x^{20/27}$

$$\sum_{n\le N} \tau(n)\tau(n+1) = N\sum_{i=0}^{2}(\log N)^i \sum_{j=0}^{2} c_{ij} \sum_{d|k} \frac{(\log d)^j}{d} + O_\varepsilon\left(N^{2/3+\varepsilon}\right).$$

In 1981 I.Chalk [7] studied the distribution of solutions of the diophantine equation

$$(2) \qquad a(u_1^2+u_2^2) - b(u_3^2+u_4^2) = k, \quad u_1^2+u_2^2 \le N, \quad a,b \in \mathbb{N}.$$

The number $A(N,a,b,k)$ of solutions of the equation (2) is defined by the sum

$$A(N,a,b,k) \sum_{\substack{k/n<n\leq N\\ an\equiv k \pmod b}} r(n) r\left(\frac{an-k}{b}\right), \qquad r(n)=\sum_{n=u^2+v^2} 1.$$

In the special case $a=b=k=1$ G.Bantle [8] obtained an asymptotic formula for $A(N,a,b,k)$ with the remainder term $R(N)=O(N^{2/3+\varepsilon})$.

Unfortunately, we cannot apply the methods of works [5], [6], [8] in the general case.

In our paper we study the mean square of the remainder term in the asymptotic formula for the number of solutions of the diophantine equation $A_\varphi(N,a,b,k)$

$$a\varphi(u_1,v_1)-b(u_2^2+v_2^2)=k, \qquad 0<\varphi(u_1,v_1)\leq N,$$

where $\varphi(u_1,v_1)$ is a positively definite quadratic form

$$a_1u_1^2+2b_1u_1v_1+c_1v_1^2, \quad a_1c_1-b_1^2=D>0, \quad (a_1,2)=1.$$

2. Premilinary results

Let $\rho(\varphi;l,q)$ denote the number of solutions of the congruence

$$\varphi(u,v)\equiv l \pmod q.$$

Lemma 1. *Let*

$$\rho(\varphi;l,p^\alpha)=$$

$$=\begin{cases} p^\alpha-p^{\alpha-1}\left(\frac{-D}{p}\right)^{\gamma+1}+p^\alpha\left(1-\frac{1}{p}\right)\sum\limits_{\beta=\alpha-\gamma}^{\alpha-1}\left(\frac{-D}{p}\right)^{\alpha-\beta} & \text{if } (D,p)=1 \\ & \text{and } \gamma<\alpha, \\ p^\alpha+p^\alpha\left(1-\frac{1}{p}\right)\sum\limits_{\beta=0}^{\alpha-1}\left(\frac{-D}{p}\right)^{\alpha-\beta} & \text{if } (D,p)=1 \\ & \text{and } \gamma=\alpha, \\ p^\alpha+\left(\frac{-a_1l_1}{p}\right)p^{\alpha-\gamma-1}\left(\frac{D_1}{p}\right)^\gamma & \text{if } D=D_1, \\ & l=l_1p^\gamma, \\ & \gamma<\alpha, \\ p^\alpha & \text{if } \gamma=\alpha, \end{cases}$$

where (n/p) denotes the Legendre symbol.

This lemma can be proved if we use the equality

$$\rho(\varphi; l, q) = \frac{1}{q} \sum_{h,l_1,l_2=0}^{q-1} e^{2\pi i \frac{h(\varphi(l_1,l_2)-1)}{q}}$$

and the properties of the Gauss' sums.

Lemma 2. *Let $q = 2^\alpha$, $(l, q) = 2^\gamma$, $l = l_0 2^\gamma$. Then for $(D, 2) = 1$*

$$\rho(\varphi; l, 2^\alpha) =$$

$$= \begin{cases} 2^{\alpha+1} & \text{if } 2l_1 - a_1(1+D) \equiv 0 \ (mod\ 8), \quad \gamma \le \alpha - 2, \\ 2^\alpha & \text{if } 2l_1 - a_1(1+D) \equiv 0 \ (mod\ 8), \quad \gamma \ge \alpha - 1, \\ 0 & \text{in other cases.} \end{cases}$$

For $D = 2D_1$, $(D_1, 2) = 1$

$$\rho(\varphi; l, 2^\alpha) =$$

$$= \begin{cases} 2^{\gamma+3} & \text{if } l_1 + 2a_1 + a_1 D \equiv 0 \ (8) \ \text{or} \quad l_1 + a_1 D_1 = 0 \ (8) \\ & \text{or } l_1 + a_1 + 2a_1 D_1 \equiv 0 \ (8) \ \text{or } l_1 + a_1 \equiv 0 \ (8), \\ 2^{\gamma+2} & \text{if } l_1 + 2a_1 + a_1 D \equiv 4 \ (8) \ \text{or } l_1 + a_1 D_1 = 4 \ (8) \\ & \text{or } l_1 + a_1 + 2a_1 D_1 \equiv 4 \ (8) \ \text{or } l_1 + a_1 \equiv 0 \ (8), \\ 0 & \text{in other cases.} \end{cases}$$

Let $r_\varphi(n)$ denote the number of representations of n in form $n = \varphi(u, v)$, $u, v \in \mathbb{Z}$.

Lemma 3. *Uniformly for $q \ll N^{1-\varepsilon}$*

$$B(\varphi; l, q, N) = \sum_{\substack{n \equiv l(q) \\ n \le N}} r_\varphi(n) = \frac{\pi N}{\sqrt{D} q^2} \rho(\varphi; l, q) +$$

$$+ \frac{\sqrt{DN}}{q} \sum_{n \le M} \frac{\Phi(n, l, q)}{\sqrt{n}} I_1\left(\frac{2\pi}{q}\sqrt{DnN}\right) + O(N^\varepsilon) + O\left(\left(\frac{D}{M}\right)^{1/2} N^{1/2+\varepsilon}\right),$$

where

$$\Phi(n, l, q) = \sum_{\substack{n_1, n_2 \in \mathbb{Z} \\ D\psi(n_1, n_2) = n}} \sum_{\substack{l_1, l_2 \pmod q \\ \phi(l_1, l_2) \equiv l(q)}} e^{2\pi i} \frac{n_1 l_1 + n_2 l_2}{q},$$

ψ is the inverse form to φ, $I_1(z)$ is the Bessel function.

This assertion can be obtained by applying Perron's formula, the functional equation for Epstein zeta-function and the estimation of trigonometric sum

$$\sum_{\substack{l_1,l_2 \pmod q\\ \varphi(l_1,l_2)\equiv l\ (q)}} e^{2\pi i\frac{n_1l_1+n_2l_2}{q}} \ll q^{1/2}(n_1,n_2,q)^{1/2}\tau(q).$$

3. Main results

We assume that $(a,b)=(a,k)=(b,k)=1$ and $(a_1,2)=1$. We shall build an asymptotic formula for $A_\varphi(A,a,b,k)$. From the definitions of $r(n)$ and $r_\varphi(n)$ we have

$$A_\varphi(N,a,b,k)= \tag{3}$$

$$=\sum_{\substack{k/a\le n\le N\\ an\equiv k\ (b)}} r_\varphi(n)r\left(\frac{an-k}{b}\right)=4\sum_{k/a\le n\le N} r_\varphi(n)\sum_{d|\frac{an-k}{b}}\chi_4(d)=$$

$$=4\sum_{\substack{k/a\le n\le N\\ an\equiv k\ (b)}} r_\varphi(n)\sum_{\substack{d|\frac{an-k}{b}\\ d\le\sqrt{N_1}}}\chi_4(d)+4\sum_{\substack{k/a\le n\le N\\ an\equiv k\ (b)}} r_\varphi(n)\sum_{\substack{d|\frac{an-k}{b}\\ d>\sqrt{N_1}}}\chi_4(d)=$$

$$=4\Sigma_1+4\Sigma_2,$$

where $N_1=\dfrac{aN-k}{b}$, χ_4 is nonprincipal character mod 4.

At first we study Σ_1. By Lemma 3 we obtain

$$\Sigma_1=\sum_{d\le\sqrt{N_1}}\chi_4(d)\sum_{\substack{an\equiv k\ (bd)\\ n\le N}} r_\varphi(n)-\sum_{d\le\sqrt{N_1}}\chi_4(d)\sum_{\substack{an\equiv k\ (bd)\\ n\le k/a}} r_\varphi(n)=$$

$$=\sum_{\substack{d\le\sqrt{N_1}\\ (a,d)=1}}\chi_4(d)\left\{\frac{\pi N}{\sqrt{D}(bd)^2}\rho(a\varphi;k,bd)+\right.$$

$$\left.+\frac{\sqrt{DN}}{bd}\sum_{n\le M}\frac{\Phi(n,k,bd)}{\sqrt{n}}I_1\left(\frac{2\pi}{bd}\sqrt{DnN}\right)\right\}-$$

$$-\sum_{\substack{d\le\sqrt{N_1}\\ (a,d)=1}}\chi_4(d)\left\{\frac{\pi k}{a\sqrt{D}(bd)^2}\rho(a\varphi;k,bd)+\right.$$

$$+\frac{\sqrt{D\frac{k}{a}}}{bd}\sum_{n\le M}\frac{\Phi(n,k,bd)}{\sqrt{n}}I_1\left(\frac{2\pi}{bd}\sqrt{\frac{Dnk}{a}}\right)\Bigg\}+$$

$$+O\left(\left(\frac{D}{M}\right)^{1/2}N^{1/2}\left(\frac{aN-k}{b}\right)^{1/2}\right)+O\left(D^{1/2}N^{1/2+\varepsilon}\right)=$$

$$=\Sigma_{11}-\Sigma_{12}+O\left(\left(\frac{aD}{bM}\right)^{1/2}N\right)+O\left(N^{1/2+\varepsilon}D^{1/2}\right).$$

Investigate the sum Σ_{11}. We have

(4)
$$\Sigma_{11}=\frac{\pi N}{b\sqrt{D}}\sum_{\substack{d\le\sqrt{N_1}\\(a,d)=1}}\frac{\chi_4(d)\rho(a\varphi;k,bd)}{bd^2}+$$

$$+\frac{\sqrt{DN}}{b^2}\sum_{\substack{d\le\sqrt{N_1}\\(a,d)=1}}\frac{\chi_4(d)}{d}\sum_{n\le M}\frac{\Phi(n,k,bd)}{\sqrt{n}}I_1\left(\frac{2\pi\sqrt{DnN}}{bd}\right).$$

Let us consider the function

$$F(s)=\sum_{d=1}^{\infty}\frac{\chi_4(d)\rho(a\varphi;k,bd)}{bd^{s+2}}$$

for Re $s>1$. From Lemmas 1, 2 we deduce

$$F(s)=\frac{L(s+1,\chi_4)}{L(s+2,\chi_4^0)}\cdot g(s),$$

where χ_4^0 is the principal character *mod* 4 and the function $g(s)$ is defined by

$$g(s)=\prod_{p|abk}\left(1-\frac{\chi_4(p)}{p^{s+1}}\right)\left(1-\frac{\chi_4(p)}{p^{s+2}}\right)^{-1}\times$$

$$\times\prod_{p|k}\left(1+\frac{\chi_4(p)\rho(a\varphi;k,p)}{p^{s+2}}\right)\prod_{l}\left(\frac{\chi_4(l)\rho(a\varphi;k,bl)}{bl^{s+2}}\right),$$

where $l=p_1^{\beta_1}\dots p_m^{\beta_m}$, $\beta_j=0,1,\dots;\ j=1,\dots,m$, if $b=p_1^{\alpha_1}\dots p_m^{\alpha_m}$ is the canonical decomposition of b.

Notice that

(5)
$$(\log\log(abk))^{-3}\ll g(0)\ll(\log\log(abk))^2.$$

Hence, by Perron's formula we obtain

$$\Sigma_{11} = \frac{\pi}{b\sqrt{D}} \frac{L(1,\chi_4)}{L(2,\chi_4)} g(0)N + \tag{6}$$

$$+\frac{\sqrt{DN}}{b} \sum_{\substack{d\le\sqrt{N_1}\\(a,d)=1}} \frac{\chi_4(d)}{d} \sum_{n\le M} \frac{\Phi(n,k,bd)}{\sqrt{n}} I_1\left(\frac{2\pi\sqrt{DnN}}{bd}\right) + O(N^{1/2}b^{-1}D^{-1/2}).$$

Similarly one can see that

$$\Sigma_{12} = \frac{\pi}{b\sqrt{D}} \frac{L(1,\chi_4)}{L(2,\chi_4^0)} g(0)\frac{k}{a} + \tag{7}$$

$$+\sqrt{\frac{Dk}{ab^2}} \sum_{\substack{d\le\sqrt{N_1}\\(a,d)=1}} \frac{\chi_4(d)}{d} \sum_{n\le M} \frac{\Phi(n,k,bd)}{\sqrt{n}} I_1\left(\frac{2\pi\sqrt{Dnk}}{bd\sqrt{a}}\right) + O(N^{1/2}b^{-1}D^{-1/2}).$$

Now we investigate the sum Σ_2. Observe that

$$\Sigma_2 = \sum_{\substack{k/a\le n\le N\\ an\equiv k\ (b)}} r_\varphi(n) \sum_{\substack{dt=\frac{an-k}{b}\\ d>\sqrt{N_1}}} \chi_4(d) =$$

$$= \sum_{t\le\sqrt{N}} \left\{ \sum_{\substack{N_2t+\frac{k}{a}<n\le N\\ an=k\ (bt)\\ \frac{an-k}{bt}\equiv 1\ (4)}} r_\varphi(n) - \sum_{\substack{N_2t+\frac{k}{a}<n\le N\\ an\equiv k\ (bt)\\ \frac{an-k}{bt}\equiv -1\ (4)}} r_\varphi(n) \right\} =$$

$$= \sum_{\substack{t\le\sqrt{N}\\(t,a)=1}} \left\{ \sum_{\substack{N_2t+\frac{k}{a}<n\le N\\ an\equiv k+bt\ (4bt)}} r_\varphi(n) - \sum_{\substack{N_2t+\frac{k}{a}<n\le N\\ an\equiv k-bt\ (4bt)}} r_\varphi(n) \right\} =$$

$$= \sum_{\substack{t\le\sqrt{N}\\(t,a)=1}} \left\{ \sum_{n\le N}{}^{+} - \sum_{n\le N}{}^{-} \right\} - \sum_{\substack{t\le\sqrt{N}\\(t,a)=1}} \left\{ \sum_{n\le N_2t+k/a}{}^{+} - \sum_{n\le N_2t+k/a}{}^{-} \right\} =$$

$$= \Sigma_{21} - \Sigma_{22}.$$

Here $N_2 = N_1^{1/2}\dfrac{b}{a}$ and the signs $'+'$ and $'-'$ correspond to the conditions of summing $an \equiv k + bt$ or $an \equiv k - bt$.

We study Σ_{21}. By Lemma 3 we infer

$$\Sigma_{21} = \sum_{\substack{t\le\sqrt{N_1}\\(t,a)=1}} \frac{\pi N}{D(4bt)^2}(\rho(a\varphi;k+bt,4bt)-\rho(a\varphi;k-bt,4bt))+$$
$$+\sum_{\substack{t\le\sqrt{N_1}\\(t,a)=1}} \sqrt{DN\cdot 4bt}\times$$
$$\times\sum_{n\le M}\frac{\Phi(n,k+bt,4bt)-\Phi(n,k-4bt,4bt)}{\sqrt{n}}I_1\left(\frac{2\pi\sqrt{DnN}}{4bt}\right)+$$
$$+O\left(\left(\frac{D}{M}\right)^{1/2}\left(\frac{a}{b}\right)^{1/2}N\right)+O(D^{1/2}N^{1/2+\varepsilon}).$$

Let for Re $s>1$

$$F_+(s)=\sum_{\substack{t=1\\(t,a)=1}}^{\infty}\frac{\rho(a\varphi;k+bt,4bt)}{t^{s+2}},\quad F_-(s)=\sum_{\substack{t=1\\(t,a)=1}}^{\infty}\frac{\rho(a\varphi;k-bt,4bt)}{t^{s+2}}.$$

Let p^{α_b} and p^{α_k} denote that $p^{\alpha_b}\|b$, $p^{\alpha_k}\|k$. Then

$$F_\pm(s)=\prod_{(p,2abk)=1}\left(1+\frac{\rho(a\varphi;1,p)}{p^{s+2}}+\frac{\rho(a\varphi;1,p^2)}{p^{2(s+2)}}+\dots\right)\times$$
$$\times\prod_{p|b}\left(\rho(a\varphi;1,p^{\alpha_b})+\frac{\rho(a\varphi;1,p^{\alpha_b+1})}{p^{s+2}}+\dots\right)\times$$
$$\times\prod_{p|k}\left(1+\frac{\rho(a\varphi;0,p)}{p^{s+2}}+\frac{\rho(a\varphi;0,p^2)}{p^{2(s+2)}}+\dots+\right.$$
$$\left.+\frac{\rho(a\varphi;0,p^{\alpha_k})}{p^{\alpha_k(s+2)}}+\frac{\rho(a\varphi;k,p^{\alpha_k+1})}{p^{(\alpha_k+1)(s+2)}}+\dots\right)\times$$
$$\times\left(\rho(a\varphi;k\pm b,2^2)+\frac{\rho(a\varphi;k+2b,2^3)}{2^{s+2}}+\frac{\rho(a\varphi;k+4b,2^4)}{2^{2(s+2)}}+\dots\right)=$$
$$=\frac{\zeta(s+1)}{L(s+2,\chi_4)}G_\pm(s),$$

where $G_\pm(s)$ is a regular function for Re $s>-1$.

Observe that

$$G_+(s)=G_0(s)g_+(s),\qquad G_-(s)=G_0(s)g_-(s),$$

where

$$g_{\pm}(s) = \rho(a\varphi; k \pm b, 2^2) + \frac{\rho(a\varphi; k \pm 2b, 2^3)}{2^{s+2}} + \frac{\rho(a\varphi; k \pm 4b, 2^4)}{2^{2(s+2)}} + \dots$$

Hence,

$$F_+(s) - F_-(s) = \frac{\zeta(s+1)}{L(s+2,\chi_4)} G_0(s)(g_+(s) - g_-(s)).$$

It is obvious that for $a \equiv 0 \pmod 2$ or $a \equiv 2 \pmod 4$

$$\rho(a\varphi; k + 2^m b, 2^{m+2}) = \rho(a\varphi; k - 2^m b, 2^{m+2}).$$

For $a \equiv 0 \pmod 4$ we have

(8) $\rho(a\varphi; 0, 4) = 16, \quad \rho(a\varphi; 2, 4) = 0, \quad \rho(a\varphi; k \pm 2^m b, 2^{m+2}) = 0 \quad \text{if} \;\; m \geq 1.$

So we have

$$F_+(s) = F_-(s) \quad \text{if} \;\; a \equiv 0 \pmod 4.$$

For $a \equiv 0 \pmod 4$ and $k \equiv b \pmod 4$

$$S = \sum_{\substack{t \leq \sqrt{N_1} \\ (a,t)=1}} \frac{\rho(a\varphi; k + bt, 4bt)}{t^2} - \sum_{\substack{t \leq \sqrt{N_1} \\ (a,t)=1}} \frac{\rho(a\varphi; k - bt, 4bt)}{t^2} =$$

$$= \sum_{\substack{t \leq \sqrt{N_1} \\ (a,t)=1}} \left[\frac{\rho(a\varphi; k + bt, bt)\rho(a\varphi; k + bt, 4)}{t^2} - \frac{\rho(a\varphi; k - bt, bt)\rho(a\varphi; k - bt, 4)}{t^2} \right] =$$

$$= \left\{ \sum_{\substack{t \leq \sqrt{N_1} \\ (t,a)=1 \\ t \equiv 1 \ (4)}}{}^{+} + \sum_{\substack{t \leq \sqrt{N_1} \\ (t,a)=1 \\ t \equiv -1 \ (4)}}{}^{+} \right\} - \left\{ \sum_{\substack{t \leq \sqrt{N_1} \\ (t,a)=1 \\ t \equiv 1 \ (4)}}{}^{-} + \sum_{\substack{t \leq \sqrt{N_1} \\ (t,a)=1 \\ t \equiv -1 \ (4)}}{}^{-} \right\}.$$

By (8) we obtain for $a \equiv 0 \ (4), \; k \equiv b \ (4)$

$$S = -16 \sum_{\substack{t \leq \sqrt{N_1} \\ (a,t)=1}} \frac{\chi_4(t)\rho(a\varphi; k, bt)}{t^2}. \tag{9}$$

Similarly, if $a \equiv 0\ (4),\quad k \equiv -b\ (4)$

$$S = 16 \sum_{\substack{t \le \sqrt{N_1} \\ (a,t)=1}} \frac{\chi_4(t)\rho(a\varphi; k, bt)}{t^2}. \tag{10}$$

Hence,

$$\Sigma_{21} = \begin{cases} R_1(M,N) & \text{if } a \not\equiv 0 \pmod 4, \\ \Sigma_{11} & \text{if } a \equiv 0 \pmod 4 \text{ and } k \equiv -b \pmod 4, \\ -\Sigma_{11} & \text{if } a \equiv 0 \pmod 4 \text{ and } k \equiv b \pmod 4, \end{cases} \tag{11}$$

where

$$R_1(M,N) =$$

$$\sum_{\substack{t \le \sqrt{N_1} \\ (a,y)=1}} \frac{\sqrt{DN}}{4bt} \sum_{n \le M} \frac{\Phi(n, k+bt, 4bt) - \Phi(n, k-bt, 4bt)}{\sqrt{n}} I_1\left(\frac{2\pi}{4bt}\sqrt{DnN}\right) +$$

$$+ O\left(\left(\frac{aD}{bM}\right)^{1/2} N\right) + O(D^{1/2}N^{1/2+\varepsilon}).$$

Now consider Σ_{22}. Similarly as sum Σ_{21} we have

$$\Sigma_{22} = \frac{\pi\left(N_2 + \frac{k}{a}\right)}{4\sqrt{D}b^2} \left\{ \sum_{\substack{t \le \sqrt{N_1} \\ (a,t)=1}} \frac{\rho(a\varphi; k+bt, 4bt) - \rho(a\varphi; k-bt, 4bt)}{t^2} \right\} +$$

$$+ \frac{\sqrt{D}}{4b} \sum_{t \le \sqrt{N_1}} \frac{\sqrt{N_2 t + \frac{k}{a}}}{t} \sum_{n \le M} \frac{\Phi(n, k+bt, 4bt) - \Phi(n, k-bt, 4bt)}{\sqrt{n}} \times$$

$$\times I_1\left(\frac{2\pi}{4bt}\sqrt{Dn\left(N_2 t + \frac{k}{a}\right)}\right) + O\left(\left(\frac{aD}{bM}\right)^{1/2} N\right) + O(N^{1/2+\varepsilon}D^{1/2}).$$

Thus after analogous calculations we obtain

$$\Sigma_{22} = \begin{cases} R_2(M,N) & \text{if } a \not\equiv 0 \pmod 4, \\ \Sigma_{11} & \text{if } a \equiv 0 \pmod 4 \text{ and } k \equiv -b \pmod 4, \\ -\Sigma_{11} & \text{if } a \equiv 0 \pmod 4 \text{ and } k \equiv b \pmod 4, \end{cases} \tag{12}$$

where

$$R_2(M,N) = \frac{\sqrt{D}}{4b} \sum_{\substack{t\le\sqrt{N_1}\\(a,t)=1}} \frac{\sqrt{N_2t+\frac{k}{a}}}{t} \sum_{n\le M} \frac{\Phi(n,k+bt,4bt)-\Phi(n,k-bt,4bt)}{\sqrt{n}} \times$$

$$\times I_1\left(\frac{2\pi}{4bt}\sqrt{Dn\left(N_2t+\frac{k}{a}\right)}\right) + O\left(\left(\frac{aD}{M}\right)^{1/2} N\right) + O(N^{1/2+\varepsilon}).$$

Hence, we proved

Theorem 1. *Let $\varphi(u,v) = a_1u^2+2b_1uv+c_1v^2$ be primitive positive definite quadratic form, $D = a_1c_1 - b_1^2 > 0$, D is square-free number, $(a_1,2) = 1$. Let $A_\varphi(N,a,b,k)$ be the number of the solutions of diophantine equation*

$$a\varphi(u_1,v_1) - b(u_1^2+v_1^2) = k, \quad a,b,k\in\mathbb{N}, \quad (a,b)=(a,k)=(b,k)=1$$

under the condition $\varphi(u_1,v_1)\le N$. Then the asymptotic formula

$$A_\varphi(N,a,b,k) = E(a,b,k)\cdot\left(N-\frac{k}{a}\right) + S_1(N) - S_2(N) + S_3(N) +$$

$$+O\left(N^{1/2}\left(\frac{k}{a}\right)^{\varepsilon}\right) + O\left(\frac{kD}{ab}\log N\right) + \left(\left(\frac{aD}{bM}\right)^{1/2} N\right) + O(N^{1/2+\varepsilon})$$

holds, where

$$E(a,b,k) = \begin{cases} E_2(a,b,k) & \text{if } a\not\equiv 0 \ (mod\ 4),\\ 2E_2(a,b,k) & \text{if } a\equiv 0 \ (mod\ 4),\ k\equiv -b \ (mod\ 4),\\ 0 & \text{if } a\equiv 0 \ (mod\ 4),\ k\equiv b \ (mod\ 4),\end{cases}$$

$$E_2(a,b,k) = \frac{4\pi}{\sqrt{Db}}\cdot\frac{L(1,\chi_4)}{L(2,\chi_4^0)}g(0),$$

$$S_1(N) = \frac{\sqrt{DN}}{b}\sum_{n\le M}\frac{1}{\sqrt{n}}\sum_{\substack{d\le\sqrt{N_1}\\(a,d)=1}}\frac{\Phi(n,k,bt)}{d}\chi_4(d)I_1\left(\frac{2\pi\sqrt{DnN}}{bd}\right),$$

$$S_2(N) =$$

$$= \frac{\sqrt{DN}}{4b} \sum_{n \le M} \frac{1}{\sqrt{n}} \sum_{\substack{t \le \sqrt{N_1} \\ (a,t)=1}} \frac{\Phi(n, k+bt, 4bt) - \Phi(n, k-bt, 4bt)}{t} I_1\left(\frac{2\pi\sqrt{DnN}}{4bt}\right),$$

$$S_3(N) =$$

$$\frac{\sqrt[4]{D^2 N}}{4(ab)^{1/2}} \sum_{n \le M} \frac{1}{\sqrt{n}} \sum_{\substack{t \le \sqrt{N_1} \\ (a,t)=1}} \frac{\Phi(n, k+bt, 4bt) - \Phi(n, k-bt, 4bt)}{\sqrt{t}} I_1\left(\frac{2\pi\sqrt{DnN_2}}{b\sqrt{t}}\right),$$

$$N_1 = \frac{aN-k}{b}, \quad N_2 = \frac{b}{a} N_1^{1/2}.$$

Corollary. *For every* $\varepsilon > 0$, $(ab)^{3+\varepsilon} \ll N$, $k = o(aN)$ *and* $N \to \infty$ *the asymptotic formula*

$$A_\varphi(N, a, b, k) = E(a, b, k) \cdot \left(N - \frac{k}{a}\right) + O\left(\left(\frac{Da}{b}\right)^{1/2} N^{5/6+\varepsilon}\right)$$

holds.

Now we denote

$$\Delta(x) = A_\varphi(x, a, b, k) - E(a, b, k) \cdot \left(x - \frac{k}{a}\right)$$

and consider

$$\frac{1}{x} \int_{k/a}^{X} \Delta^2(x) dx.$$

Theorem 2. *Let* $b \equiv k \equiv 1$ *(mod 2) and* $a > b$. *Then*

$$\frac{1}{x} \int_{1}^{X} \Delta^2(x) dx \ll \frac{aD}{b^{5/3}} X^{4/3+\varepsilon} + \left(\frac{Dk}{ab}\right)^2 \log^2 X.$$

Proof. By Theorem 1 and the Cauchy inequality we obtain

$$\int_{k/a}^{X} \Delta^2(x) dx \ll$$

$$\ll \sum_{i=1}^{3} \int\limits_{k/a}^{X} |S_i(x)|^2 dx + O\left(\left(\frac{Dk}{ab}\right)^2 X \log^2 X\right) + O\left(\frac{aD}{bM} X^3\right) + O(X^{2+\varepsilon} D).$$

The integrals $\int\limits_{k/a}^{X} |S_i(x)|^2 dx$ can be considered on the identical scheme. Thus we obtain the estimate only of the first integral

$$\int\limits_{k/a}^{X} |S_1(x)|^2 dx =$$

$$= \frac{D}{b^2} \sum_{n_1, n_2 \le M} \frac{1}{\sqrt{n_1 n_2}} \int\limits_{k/a}^{X} \sum_{d_1, d_2 \le \sqrt{M} \frac{ax-k}{b}} \frac{\Phi(n_1, k, bd_1)\Phi(n_2, k, bd_2)}{d_1 d_2} \chi_4(d_1 d_2) \times$$

$$\times I_1\left(\frac{2\pi\sqrt{Dn_2 x}}{bd_2}\right) I_1\left(\frac{2\pi\sqrt{Dn_1 x}}{bd_1}\right) dx.$$

We account that for $z \to +\infty$

$$I_1(z) = \sqrt{\frac{2}{\pi}} \frac{\cos\left(z - \frac{3\pi}{4}\right)}{\sqrt{z}} + O\left(\frac{1}{z^{3/2}}\right).$$

Apply a refinement of Hilbert's inequality given by Montgomery and Vaughan [9].

Let $\{a_n\}$ and $\{b_n\}$ be two sequences of real numbers, $b_i \ne b_j$ for $i \ne j$. Then

$$\left| \sum_{\substack{r,s=1 \\ r \ne s}}^{M} \frac{a_r a_s}{(rs)^{1/4}(\sqrt{r} - \sqrt{s})} e^{i(b_2 - b_3)} \right| \ll \sum_{r=1}^{M} a_r^2.$$

Thus the main contribution in the estimation of $\int\limits_{k/a}^{X} |S_i(x)|^2 dx$ is given by the integral

$$I = \frac{\sqrt{D}}{b} \sum_{n_1, n_2 \le M} \frac{1}{(n_1 n_2)^{3/4}} \int\limits_{k/a}^{X} \sqrt{x} \sum_{\substack{d_1, d_2 \le \sqrt{\frac{ax-k}{b}} \\ n_1 d_2^2 = n_2 d_1^2 \\ (d_1 d_2, a) = 1}} \frac{|\Phi(n_1, k, d_1)\Phi(n_2, k, d_2)|}{(d_1 d_2)^{1/2}} dx.$$

By the estimate $|\Phi(n,k,d)| = O(d^{1/2}(n,d)^{1/2}\tau(d))$ we obtain

$$I \ll \frac{\sqrt{D}}{b} \sum_{n_1,n_2\le M} \frac{1}{(n_1n_2)^{3/4}} \int\limits_{k/a}^{X} \sqrt{x} \sum_{\substack{d_1,d_2\le\sqrt{\frac{ax-b}{b}}\\ n_1d_2^2=n_2d_1^2}} \sqrt{(n_1,d_1)(n_2d_2)}\tau(d_1)\tau(d_2)dx.$$

Observe, that the collection of n_1, n_2, d_1 define at most one value of d_2 such that $n_1d_2^2 = n_2d_1^2$.

Now we have

$$\sum_{n_1,n_2} \frac{1}{(n_1n_2)^{3/4}} \sum_{\substack{d_1,d_2\\ n_1d_2^2=n_2d_1^2}} \sqrt{(n_1,d_1)(n_2,d_2)}\tau(d_1)\tau(d_2) \ll$$

$$\ll \sum_{n_1,n_2} \sum_{\substack{t_1|n_1\\ t_2|n_2}} \frac{1}{(n_1n_2)^{3/4}} \sum_{\substack{1\le d_i\le 1/t_i\sqrt{\frac{ax}{b}}\\ i=1,2}} \tau(d_1t_1)\tau(d_2t_2)(t_1,t_2)^{1/2} \ll$$

$$\ll \left(\frac{ax}{b}\right)^{\varepsilon} \sum_{n_1,n_2\le M} \frac{\tau(n_1)\tau(n_2)}{(n_1n_2)^{3/4}} \cdot \left(\frac{ax}{b}\right)^{1/2} \ll M^{1/2} \left(\frac{ax}{b}\right)^{1/2+2\varepsilon} \log^2 M.$$

Hence,

$$I \ll \frac{\sqrt{D}}{b} \left(\frac{a}{b}\right)^{1+2\varepsilon} M^{1/2} X^2 \log^2 M,$$

$$\frac{1}{X} \int\limits_{k/a}^{X} |S_1(x)|^2 dx \ll \frac{\sqrt{D}a^{1+2\varepsilon}}{b^2} M^{1/2} X \log^2 M.$$

Similar estimates we obtain for $\frac{1}{X}\int\limits_{k/a}^{X} |S_i(x)|^2 dx$ $(i = 2, 3)$. And therefore

$$\int\limits_{k/a}^{X} \left(A_\varphi(x,a,b,k) - E(a,b,k)\cdot\left(x - \frac{k}{a}\right)\right)^2 dx \ll$$

$$\ll \frac{aD}{bM} X^3 + \left(\frac{kD}{ab}\right)^2 X \log^2 M + X^{2+\varepsilon} D + \frac{\sqrt{D}n^{1+2\varepsilon}}{b^2} M^{1/2} X \log^2 M.$$

We take $M = (bX)^{2/3}$ and then we obtain our theorem.

Corollary. *Let $k \ll N^{2/3}a^{2/3}b^{1/6}D^{-1/2}$. Then for almost all $N \leq X$ and any $\varepsilon > 0$ the asymptotic formula*

$$A_\varphi(N,a,b,k) - E(a,b,k) \cdot \left(N - \frac{k}{a}\right) + O_\varepsilon\left(N^{2/3+\varepsilon}\frac{a^{1/2}D^{1/2}}{b^{5/6}}\right)$$

holds.

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G. Belozorov and P. Varbanets
Dept. of Computer Algebra and Discrete Mathematics
Odessa I. Mechnikov National University
Dvoryanska 2
65000 Odessa, Ukraine