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REMARKS ON PRIME–INDEPENDENT MULTIPLICATIVE FUNCTIONS

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Dedicated to Professor Imre Kátai
on his sixty fifth birthday

1. Introduction

There is a large literature on multiplicative arithmetic functions $f(n)$ which are prime-independent. Hence it is difficult to state anything that is really new. However, I hope that at least some of the following is perhaps new.

We are concerned here with the set S of multiplicative arithmetic functions f such that for every prime p and every positive integer k we have $f(p^k) =$ an integer, independent of p. There are plenty of examples of such functions. Some examples are: the Möbius function $\mu(n)$, the numbers of divisors of n denoted by $d(n)$, the Liouville function $(-1)^{\Omega(n)}$, where $\Omega(n)$ denotes the total number of prime divisors of n – multiplicity being taken into account, the unitary divisor function, $d^*(n)$ $(= 2^{\omega(n)}$, $\omega(n) =$ the number of distinct prime divisors of n, with $\omega(1) = 0)$, the unitary Möbius function $\mu^*(n) = (-1)^{\omega(n)}$, more generally, $\mu_k(n) = (-k)^{\omega(n)}$, if n is square-free, and $= 0$ otherwise, with $\mu_k(1) = 1$, $\mu^*_{(k)}(n) = k^{\omega(n)}$, $d^*_{(k)}(n) = k^{\omega(n)} =$ the number of representations of $1 < n$ as

$$\sum_{\substack{m_1 m_2 \dots m_k = n \\ (m_i, m_j)=1,\, i \neq j}} m_1 m_2 \dots m_k.$$

A further example is the exponential divisor function $\tau^{(e)}(n)$ with $\tau^*(p^a) = d(a)$, p prime. We may also mention, the extensions of $d(n)$ and $\tau^{(e)}_k(n)$

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obtained by using k-fold divisor products:

$$d_r(n) = \sum_{m_1 m_2 \ldots m_r = n}$$

and

$$\tau_r^{(e)}(n) = \prod p_1^{d_r(a_1)} \ldots p_k^{d_r(a_r)},$$

where $n = p_1^{a_1} \ldots p_k^{a_k}$.

Note that we define $f(1) = 1$ for every multiplicative arithmetic function $f(n)$. A further set of examples of functions that belong to the set S is provided by the following easy

Theorem 1. *If $f(n)$ and $g(n) \in S$, then their Dirichlet product $(f \cdot g)(n)$ and Dirichlet inverse f^{-1} also belongs to S, where we recall the definition*

$$(f \cdot g)(n) = \sum_{ab=n} f(a)g(b).$$

Proof. We only have to recall the well known result that $(f \cdot g)(n)$ is multiplicative since f and g are, and for each prime p and $k = 1, 2, \ldots,$ we have

$$(f \cdot g)(p^k) = \sum_{a=0}^{k} f(p^a)g(p^{k-a}).$$

Since every term in the summation on the right is independent of p, so is the left member.

That f^{-1} – the Dirichlet inverse of f – belongs to the set S follows from the relation that for each $k \geq 1$,

$$f^{-1}(p^k)f(1) + f^{-1}(p^{k-1})f(p) + \cdots + f^{-1}(1)f(p^k) = 0$$

which shows recursively that $f^{-1}(p), f^{-1}(p^2), \ldots$ are all independent of p and are integral since $f \in S$.

2. A representation theorem

In what follows, p denotes an arbitrary prime: $\sum\limits_{n}$, $\prod\limits_{p}$ respectively products over all natural numbers n and all primes, and the summations $\sum\limits_{n}$, $\sum\limits_{p}$ have analogous meanings. All our series and products are formal and all questions of convergence are therefore ignored. $\zeta(s)$ is the Riemann zeta function. One of our main results is the following. I did not find this (as well as much of the material that follows in Sections 2 and 3) formally stated in the literature.

Theorem 2. *A function $f(n)$ belongs to the set S if and only if its generating Dirichlet function $\sum f(n)n^{-s}$ has the representation*

$$\sum_n f(n)n^{-s} = \prod_{n=1}^{\infty} \big(\zeta(ns)\big)^{b(n)}, \tag{2.1}$$

where $b(n)$ are integers (positive, negative or zero).

Proof. To prove the 'if' part, let $f(n)$ satisfy (2.1). Then

$$\sum_n f(n)n^{-s} = \prod_p \prod_n (1-p^{-ns})^{-b(n)}. \tag{2.2}$$

On using the binomial expansions for the terms in the inner product on the right side of (2.2), we have

$$\sum_n f(n)n^{-s} = \prod_p \left(\sum_{k=0}^{\infty} c(p^k)p^{-ks}\right), \tag{2.3}$$

where $c(p^k)$ is an integer whose value is independent of the value of the prime p. We now define $c(n)$ to be the multiplicative function whose values for prime powers p^k of the argument n are as determined in the right side of (2.3). Then (2.3) gives

$$\sum_n f(n)n^{-s} = \sum_n c(n)n^{-s}. \tag{2.4}$$

In view of the uniqueness of representation of a function by a Dirichlet series, we see from (2.4) that $f(n) = c(n)$. This proves that $f \in S$ and completes the proof of the 'if' part.

To prove the 'only if' part we borrow an idea in a paper of Carlitz [1] and remark that a given formal power series $1 + a_1x + a_2x^2 + \dots$ can be expressed in the form

$$1 + a_1x + a_2x^2 + \dots = (1-x)^{b(1)}(1-x^2)^{b(2)}(1-x^3)^{b(3)}\dots, \tag{2.5}$$

where $b(1),\ b(2),\ b(3),\dots$ are constants that can be determined recursively. Moreover, $b(1),\ b(2),\ b(3),\dots$ are integers if and only if $a_1, a_2, \dots$ are integers.

Now let $f(n) \in S$. Then its formal Dirichlet series has the Euler factorization

$$\sum_n f(n)n^{-s} = \prod_p \Big(1 + \sum_n f(p^n)p^{-ns}\Big).$$

We recall that $f(p^n)$ is independent of p and define

$$f(p^n) = a_n \quad (n = 1, 2, \ldots);\ f(1) = 1.$$

Using the representation in (2.5) we have

$$\begin{aligned}\sum_n f(n)n^{-s} &= \prod_p \Big(1 + \sum_n a_n p^{-ns}\Big) = \\ &= \prod_p \prod_n \big(1 - p^{-ns}\big)^{b(n)} = \\ &= \prod_n \big(\zeta(ns)\big)^{b(n)}.\end{aligned}$$

Since a_n, and hence $b(n)$, are integers, the proof is complete.

For a given $f(n) \in S$, to represent its Dirichlet series in the form (2.1), one can determine $b(1),\ b(2), \ldots$ recursively from the relation (2.5), where $a_n = f(p^n)$. But there are easier methods of doing this for special forms of $\sum f(n)n^{-s}$. One such case is the following.

Theorem 3. *Let A be an absolute constant. Then*

$$\prod_p (1 - Ap^{-s})^{-1} = \prod_n \big(\zeta(ns)\big)^{b(n)},$$

where

$$b(n) = \frac{1}{n} \sum_{d|n} A^d \mu(n/d) \quad (n = 1, 2, \ldots).$$

Proof. Let

$$1 - Ax = \prod_n (1 - x^n)^{b(n)}.$$

Taking logarithms of both sides and expanding in series and equating the coefficients of x^n we get

$$\frac{A^n}{n} = \sum_{d|n} b(d)(d/n).$$

Using the Möbius inversion formula, we get

$$b(n) = \frac{1}{n} \sum_{d|n} A^d \mu(n/d).$$

The theorem now follows easily.

(2.6) Remark. It is easily seen that Theorem 3 can be extended as follows:

$$\prod_p \prod_j (1 - A_j p^{-s}) = \prod_n \left(\zeta(ns)\right)^{b(n)},$$

where

$$b(n) = \frac{1}{n} \sum_{d|n} \left(\sum_j A_j^d \right) \mu(n/d),$$

j varying over a finite or infinite sequence of natural numbers.

3. Some applications and examples

(i) Let $\mu_k(n)$ be the function defined in Section 1. An application of Theorem 3 shows that

$$\sum_n \mu_k(n) n^{-s} = \prod_n \left(\zeta(ns)\right)^{b_k(n)}, \tag{3.1}$$

where

$$b_k(n) = \frac{1}{n} \sum_{d|n} k^d \mu(n/d). \tag{3.2}$$

(ii) Next, we refer to the definitions of $d^*(n)$ and apply Theorem 3 to obtain

$$\begin{aligned} 1 + \sum_n \left(d^*(p^n)\right)^k p^{-ns} &= 1 + 2^k x + 2^k x^2 + \dots \quad (x = p^{-s}) \\ &= (1-x)^{-1}\bigl(1 - (1-2^k)x\bigr). \end{aligned}$$

Hence

$$\sum \left(d^*(n)\right)^k n^{-s} = \zeta(s) \prod_n \left(\zeta(ns)\right)^{b_{1-2}k(n)}, \tag{3.3}$$

where $b_k(n)$ is given by (3.2). The corresponding formula for $d(n)$ was given by Ramanujan [7] for $k = 2$ and by Grotze [4] for general k.

(iii) From the definition of $\mu^*(n)$ given in (1.3), we have

$$\begin{aligned}\sum_n \mu^*(n) n^{-s} &= \prod_p (1 - p^{-s} - p^{-2s} - \ldots) = \\ &= \prod_p (1 - 2p^{-s})(1 - p^{-s})^{-1}.\end{aligned}$$

From (2.6) we see that

$$\sum_n \mu^*(n) n^{-s} = \zeta^3(s) \sum_{n=2}^{\infty} \left(\zeta(ns)\right)^{b_2(n)},$$

where $b_2(n)$ is as given in (3.2).

(3.4) Let $\delta^{(e)}(n)$ be the multiplicative function for prime powers p^k defined as follows:

$$\delta^{(e)}(p^k) = \begin{cases} 1 & \text{if } k \text{ is a triangular number,} \\ & \text{i.e. is of the form } m(m+1)/2, \\ & \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\sum \delta^{(e)}(n) n^{-s} = \prod_{n=1}^{\infty} \frac{\zeta(2n-1)s}{\zeta(2ns)}.$$

This follows from the famous identity of Gauss [MacMahon, 6, p. 24]:

$$\frac{(1-x^2)(1-x^4)(1-x^6)\ldots}{(1-x)(1-x^3)(1-x^5)\ldots} = 1 + x + x^3 + x^6 + \ldots + x^{n(n-1)/2} + \ldots \tag{3.5}$$

(3.6) Let $\beta(n)$ be the number of abelian groups of order n. Then, as is well known,

$$\sum \beta(n) n^{-s} = \prod_n \zeta(ns).$$

This follows from the fact that $\beta(n)$ is multiplicative and $\beta(p^n) = p(n)$, where $p(n)$ is the number of partitions of n so that

$$1 + \sum p(n)x^n = \prod_n (1 - x^n)^{-1}.$$

(3.7) Let $\gamma(n)$ be the multiplicative function for which

$$\gamma(p^k) = p_2(k), \text{ the number of planar partitions of } n.$$

Then

$$\sum \gamma(n)n^{-s} = \prod_n \left(\zeta(ns)\right)^n.$$

This follows from MacMahon's result [6, p. 175] that

$$1 + \sum_n p_2(n) = \prod_n (1 - x^n)^{-n}.$$

The interested reader would find many such examples that could be constructed from identities in Combinatorial Analysis.

Thus on combining one of Jacobi's famous identities [6]:

$$(1 + x + x^3 + \ldots)^4 = \sigma(1) + \sigma(3)x + \sigma(5)x^2 + \ldots + \sigma(2n+1)x^n + \ldots$$

(where $\sigma(n)$ denotes the sum of the divisors of n) with (3.5) and recalling the definition of $\delta^{(e)}(n)$ given in (3.4), we obtain

$$\prod_{n=1}^{\infty} \left(\frac{\zeta(2n-1)}{\zeta(2n)}\right)^4 = \sum_n \sigma'(n)n^{-s}$$

and

$$\sum_{d_1 d_2 d_3 d_4 = n} \delta^{(e)}(d_1)\delta^{(e)}(d_2)\delta^{(e)}(d_3)\delta^{(e)}(d_4) = \sigma'(n),$$

where $\sigma'(n)$ is the multiplicative function defined by $\sigma'(n) = \sigma(2a+1)$.

4. Segal's theorem and consequences

For prime-independent multiplicative real-valued – but not necessarily integer – valued-functions, it is possible to obtain a theorem somewhat analogous to Theorem 2. It can also be derived using the following theorem of Segal [8].

Theorem (Segal). *Suppose $h(n)$ is an additive arithmetic function such that $h(p^k)$ (p prime) depends only on k. Then there is a function $f(k)$ such that in some half-plane*

$$\sum_{n=1}^{\infty} h(n)n^{-s} = \zeta(s) \sum_{k=1}^{\infty} f(k) \log \zeta(ks) \quad (s = \sigma + it),$$

provided the left-hand side series converges in some half plane (and so absolutely in some half place). Actually the $f(k)$ in the theorem is given by

$$f(k) = \sum_{d|k} \frac{\mu(d)}{d} \left(h(p^{r/d}) - h^{p^{(r/d)-1}}\right),$$

μ being the Möbius function, and the prime p being arbitrary. Note that h is rational valued, so is f.

Remark. Write $H(n) = \exp h(n)$. Suppose $h(n)$ is prime-independent, additive and rational valued. Then $H(n)$ is prime-independent and multiplicative, but not necessarily rational integer-valued. The following theorem indicates when this happens.

Theorem. *Let $h(n)$ be a prime-independent, additive and rational valued. Set $H(n) = \exp h(n)$. Then $H(n)$ is prime-independent and multiplicative and rational integer valued if and only if for all $k \geq 1$, we have*

$$\sum_{rs=k} rh(p^r)\mu(s) \equiv 0 \pmod{k}$$

where μ is the Möbius function.

This can be proved directly or immediately deduced from the following

Theorem (Carlitz [1]). *Let $a_1, a_2, \ldots$ be a sequence of rational numbers and set*

$$\exp\left(\sum_{m=1}^{\infty} a_m x^m\right) = \sum_{n=1}^{\infty} c_n x^n$$

the power series being formal. A necessary and sufficient condition that all the coefficients c_n be rational integers is that for all $k \geq 1$

$$\sum_{rs=k} ra_r\mu(s) \equiv 0 \pmod{k}.$$

We shall not go into the details of such a representation. Segal also proved the following asymptotic estimate:

Theorem II (Segal). *If $h(n)$ is additive and $h(p^k)$ depends only on k and $h(p^m) \equiv O(2^{m/2})$, then*

$$\sum_{n\leq x} h(n) = h(p)x\log\log x + Ax + O(x/\log x),$$

where A is a constant.

5. Concluding remarks

(5.1) The function $\delta^{(e)}(n)$ defined in (3.4) does not seem to have been studied in the literature so far. We can show that, c being an absolute constant

$$\sum_{n\leq x} \delta^{(e)}(n) = cx + O\left(x^{1/2}e^{-(\log^{3/5}x)(\log\log x)^{-1/5}}\right).$$

Note that these $\delta^{(e)}$ numbers, which we also call exponential delta numbers, that is those n for which $\delta^{(e)}(n) = 1$, numbers include square-free numbers and their cubes or sixth powers, etc. Their distribution among residue class (mod k) for any given k and the number of representations of n as a sum of k exponential delta numbers are among the many problems that the author hopes to study in a subsequent paper.

(5.2) Let $\Delta^{(e)}(n)$ be defined as the additive function representing the number of exponents a_i in the canonical form of $n = p_1^{a_1}\dots p_r^{a_r}$ that are triangular numbers. Segal's Theorem II shows

$$\sum_{n\leq x} \Delta^{(e)}(n) = x\log\log x + Ax + O(x/\log x),$$

where A is a constant.

(5.3) If we apply Segal's theorem to $\log \tau^{(e)}(n)$, we get

$$\sum_{n \le x} \log \tau^{(e)}(n) = Ax + O(x/\log x), \quad x \to \infty,$$

where A is a constant.

In contrast, Fabrykowski and the author proved in [3] that

$$\sum_{n \le x} \tau^{(e)}(n) = A_1 x + O(x^{1/2} \log x), \quad x \to \infty.$$

This was improved by Wu [10] to

$$\sum_{n \le x} \tau^{(e)}(n) = A_1 x + A_2 \sqrt{x} + O(x^{2/9} \log x),$$

where A_1 and A_2 are constants.

Very recently, Kátai and the author proved [5], among other results, that if, as usual, $\omega(n)$ (respectively $\Omega(n)$) denote the number of distinct prime factors of n (respectively, total number of prime factors of n, multiple factors counted multiply), then, as $x \to \infty$, we have

$$\frac{1}{x} \sum_{n \le x} \omega\big(\tau^{(e)}(n)\big) = A + O\left(\frac{\log\log x}{\sqrt{x}}\right)$$

and

$$\frac{1}{x} \sum_{n \le x} \Omega\big(\tau^{(e)}(n)\big) = B + O\left(\frac{\log\log x}{\sqrt{x}}\right),$$

where A and B are absolute constants.

The first of these results slightly improves an earlier result of Smati and Wu [9]. The second result is believed to be new.

Note. I just noticed after finishing this paper that J. Knopfmacher and J.N. Ridley, in their paper, Prime-independent arithmetical functions, Ann. Mat. Pure Applied **101** (1974), 153-169, have proved in their Theorem 2.1 a result somewhat similar to my Theorem 2, using a different terminology. But their entire approach is different from mine and there is really nothing in common between their subsequent results and mine. In particular, they have not even referred to Segal's theorem that I used here in my paper. Further, my results in Section 3, 4 and 5 do not appear at all in their paper.

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