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ON APPROXIMATE SANDWICH AND DECOMPOSITION THEOREMS

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Dedicated to the 65th birthday of
Zoltán Daróczy and Imre Kátai

Abstract. At the 8th International Conference on Functional Equations and Inequalities in Złockie, Poland, 2001, the first author posed a problem concerning approximately subadditive and superadditive functions. Here a solution of that problem is given in the form of an approximate sandwich theorem. An analogous approximate Stone-type decomposition theorem is also obtained and the connection of these two results is explained.

0. Introduction

Separation, sandwich and extension theorems generalizing the Hahn-Banach separation theorem to various settings have important and interesting applications in several fields of mathematics. A recent survey on these developments is due to Buskes [2].

The case when the underlying structure is an abelian semigroup was first considered by Kaufman [8] and Kranz [12]. In order to formulate their result, we recall (and redefine) the notion of sub- and superadditive functions defined on semigroups.

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First we introduce the *upper and lower addition* on the extended real line $\overline{\mathbb{R}} := [-\infty, \infty]$ in the following way:

$$x \dot{+} y = \begin{cases} x + y & \text{if} \quad x, y \in \mathbb{R}, \\ \infty & \text{if} \quad \infty \in \{x, y\}, \\ -\infty & \text{if} \quad \infty \notin \{x, y\} \text{ and } -\infty \in \{x, y\}, \end{cases}$$

and

$$x \underset{\cdot}{+} y = \begin{cases} x + y & \text{if} \quad x, y \in \mathbb{R}, \\ -\infty & \text{if} \quad -\infty \in \{x, y\}, \\ \infty & \text{if} \quad -\infty \notin \{x, y\} \text{ and } \infty \in \{x, y\}. \end{cases}$$

The idea of these extensions is due to König [11, p.10-11] who observed that these are the only associative and commutative binary operations on $\overline{\mathbb{R}}$ that produce the usual addition in all doubtless cases. That is, the only case when the sums $x \dot{+} y$ and $x \underset{\cdot}{+} y$ are different is when $\{x, y\} = \{\infty, -\infty\}$ since

$$\infty \dot{+} (-\infty) = \infty \qquad \text{and} \qquad \infty \underset{\cdot}{+} (-\infty) = -\infty.$$

When one of the summands x, y is a (nonextended) real number then $x \dot{+} y = x \underset{\cdot}{+} y$, hence we shall simply write $x + y$ instead of them. As usual, $x - y$ denotes $x + (-y)$. We shall also denote by $\underset{\cdot}{\sum}$ and $\dot{\sum}$ the summation with respect to the lower and upper addition, respectively.

The use of these operations will make the formulation of the sandwich and separation theorems more symmetric.

Denote by $(S, +)$ an abelian semigroup in the sequel and let $f : S \to \overline{\mathbb{R}}$. Then f is called *subadditive* if, for all x, y in S,

$$f(x + y) \le f(x) \dot{+} f(y)$$

The function f is said to be *superadditive* if

$$f(x) \underset{\cdot}{+} f(y) \le f(x + y)$$

for all x, y in S, and f is called *additive* if it is both sub- and superadditive. Clearly, a function f is superadditive if and only if $(-f)$ is subadditive.

The basic sandwich theorem due to Krantz [12] (cf. also [1] and [10]) can now be formulated as follows.

Theorem A. *Let $p : S \to \overline{\mathbb{R}}$ be a subadditive and $q : S \to \overline{\mathbb{R}}$ be a superadditive function such that $q \le p$ on S. Then there exists an additive function $a : S \to \overline{\mathbb{R}}$ separating q from p, that is, satisfying the inequality $q \le a \le p$.*

We note that this result is slightly more general than that of [12] since in the original result the functions p and q did not take the value ∞, hence the use of upper and lower addition was also avoided. Generalizations of Theorem A are due to Fuchssteiner [4], Rodé [19], Volkmann and Weigel [22], König [9] and interesting applications can be found in the book of Fuchssteiner and Lusky [5].

At the 8th International Conference on Functional Equations and Inequalities in Złockie, Poland, 2001, the first author posed the following problem that asks for an approximate version of Theorem A (see [17]).

First we define the notions of approximate sub- and superadditivity allowing extended real valued functions. Let ε be a nonnegative real number. A function $f : S \to \overline{\mathbb{R}}$ is called *ε-subadditive* if, for all x, y in S,

$$f(x+y) \le f(x) \dot{+} f(y) + \varepsilon.$$

If f satisfies

$$f(x) \underset{\cdot}{+} f(y) - \varepsilon \le f(x+y)$$

for all x, y in S, then it is said to be *ε-superadditive*, furthermore, f is called *ε-additive* if it is both ε-sub- and ε-superadditive. Clearly, f is ε-subadditive (resp. ε-superadditive) if and only if $f + \varepsilon$ (resp. $f - \varepsilon$) is subadditive (resp. superadditive). When f takes real values only, then it is ε-additive if and only if

$$|f(x+y) - f(x) - f(y)| \le \varepsilon \qquad (x, y \in S),$$

which leads to the stability problem of the Cauchy functional equation raised by Ulam [21] in 1940 and answered first by Hyers [7]. Surveys offering further details of this theory are, for instance, due to Forti [3] and Székelyhidi [20].

Problem. Let ε, δ be nonnegative numbers and let $p, q : S \to \mathbb{R}$ be such that p is ε-subadditive, q is δ-superadditive and $q \le p$. Then $p + \varepsilon$ and $q - \delta$ are sub- and superadditive functions and $q - \delta \le p + \varepsilon$, therefore, by Theorem A, there exists an additive function $a : S \to \mathbb{R}$ such that

$$q - \delta \le a \le p + \varepsilon \tag{1}$$

holds. Define, for any x in S,

$$\varphi(x) = \begin{cases} q(x) & \text{if} \quad a(x) < q(x), \\ a(x) & \text{if} \quad q(x) \le a(x) \le p(x), \\ p(x) & \text{if} \quad p(x) < a(x). \end{cases}$$

Then, obviously $q \le \varphi \le p$, and by (1), we have $a - \varepsilon \le \varphi \le a + \delta$. Thus, for all x, y in S,

$$\varphi(x+y) \le a(x+y) + \delta = a(x) + a(y) + \delta \le \varphi(x) + \varphi(y) + (2\varepsilon + \delta).$$

Similarly, for all x, y in S, we have

$$\varphi(x+y) \ge \varphi(x) + \varphi(y) - (2\delta + \varepsilon).$$

Hence φ is $(2\varepsilon + \delta)$-subadditive and $(2\delta + \varepsilon)$-superadditive. The question formulated in [17] is *whether there exists an ε-subadditive and δ-superadditive function $f : S \to \mathbb{R}$ such that $q \le f \le p$.*

The main result of the first section will give an affirmative answer to this question allowing also functions with extended real values.

1. Approximate sandwich theorem

The following lemma easily follows by induction.

Lemma 1. *Let ε be a nonnegative number and $f : S \to \overline{\mathbb{R}}$ be an ε-subadditive (resp. ε-superadditive) function. Then, for all n in $\mathbb{N}$, $x_1, \dots, x_n$ in S,*

$$f\left(\sum_{i=1}^{n} x_i\right) \le \sum_{i=1}^{n} f(x_i) + (n-1)\varepsilon \quad \left(\textit{resp.} \quad \sum_{i=1}^{n} f(x_i) - (n-1)\varepsilon \le f\left(\sum_{i=1}^{n} x_i\right)\right).$$

If the semigroup $(S, +)$ contains a neutral element, then it is unique and it will be denoted by 0. It may happen that there is no neutral element in S. In this case, let 0 denote an arbitrary element and define the addition in $S \cup \{0\}$ by

$$x + y = \begin{cases} x + y & \text{if } x, y \in S, \\ x & \text{if } x \in S,\ y = 0, \\ y & \text{if } x = 0,\ y \in S, \\ 0 & \text{if } x = y = 0. \end{cases}$$

Then $(S \cup \{0\}, +)$ will trivially be an abelian semigroup. Therefore, in the sequel, we can always use the addition in with this extension regardless whether S does or does not contain a neutral element. If n is a nonnegative integer then, for any x in S, the element nx is defined as 0 if $n = 0$, otherwise $nx = x+\cdots+x$, where the number of terms on the right hand side is exactly n.

The next result gives an affirmative answer to the problem described in the introduction.

Theorem 1. *Let ε, δ be nonnegative real numbers, and let $p : S \to \overline{\mathbb{R}}$ be an ε-subadditive and $q : S \to \overline{\mathbb{R}}$ be a δ-superadditive function with $q \le p$. Then there exists an ε-subadditive and δ-superadditive function $\varphi : S \to \overline{\mathbb{R}}$ satisfying the separation inequality $q \le \varphi \le p$.*

In the particular case $\varepsilon = \delta = 0$, the above result obviously specializes to Theorem A, thus it can be considered as a generalization.

Proof. By Zorn's lemma we can find a minimal ε-subadditive function p_0 with respect to the pointwise ordering, such that $q \le p_0 \le p$. Similarly, we can find a maximal ε-subadditive function p_0 with respect to the same ordering, such that $q \le q_0 \le p_0 \le p$. We will show that $q_0 = p_0$, hence $\varphi = q_0 = p_0$ will be the desired separating function.

Assume that there exists a point u in S such that $q_0(u) < p_0(u)$. Choose a real c with $q_0(u) < c < p_0(u)$. Then define the function $\overline{p} : S \to \mathbb{R}$ for any t in S by the formula

$$\overline{p}(t) := \inf \left\{ kc + \dot{\sum_{i=1}^{n}} p_0(x_i) + (k+n-1)\varepsilon \;\middle|\; \begin{array}{c} k, n \ge 0,\ k+n \ge 1,\ x_1, \ldots, x_n \in S, \\ ku + x_1 + \cdots + x_n = t \end{array} \right\},$$

Then, by taking $k = 0$, $n = 1$ and $x_1 = t$, we get $\overline{p}(t) \le p_0(t)$, and with $k = 1$, $n = 0$, we obtain $\overline{p}(u) \le c < p_0(u)$. Now we are going to show that $\overline{p}$ is also ε-subadditive. For, let t, s in S be arbitrary. If either $\overline{p}(t)$ or $\overline{p}(s)$ equals ∞ then

$$\overline{p}(t+s) \le \overline{p}(t) \dotplus \overline{p}(s) + \varepsilon \tag{2}$$

is obvious. Thus, we may assume that $\overline{p}(t) < \infty$ and $\overline{p}(s) < \infty$. Choose $\alpha, \beta \in \mathbb{R}$ arbitrarily such that

$$\overline{p}(t) < \alpha \qquad \text{and} \qquad \overline{p}(s) < \beta.$$

Then there exist nonnegative integers $k, n \ge 0$ with $k + n \ge 1$ and $x_1, \ldots, x_n$ in S with $ku + x_1 + \cdots + x_n = t$, furthermore nonnegative integers $\ell, m \ge 0$ with $\ell + m \ge 1$ and $y_1, \ldots, y_m$ in S with $\ell u + y_1 + \cdots + y_m = s$ such that

$$kc + \dot{\sum_{i=1}^{n}} p_0(x_i) + (k+n-1)\varepsilon < \alpha \quad \text{and} \quad \ell c + \dot{\sum_{j=1}^{m}} p_0(y_j) + (\ell+m-1)\varepsilon < \beta. \tag{3}$$

On the other hand,

$$(k+\ell)u + x_1 + \cdots + x_n + y_1 + \cdots + y_m = t + s,$$

hence, by (3) and the definition of $\overline{p}(s+t)$,

$$\overline{p}(s+t) \le (k+\ell)c + \dot{\sum_{i=1}^{n}} p_0(x_i) \,\dot{+} \dot{\sum_{j=1}^{m}} p_0(y_j) + (k+\ell+n+m-1)\varepsilon < \alpha + \beta + \varepsilon.$$

Computing the limits $\alpha \downarrow \overline{p}(t)$ and $\beta \downarrow \overline{p}(s)$, we get that (2) is valid also in the general case.

Since $\overline{p} \le p_0$ and $\overline{p}(u) < p_0(u)$ hence, by the minimality of p_0 we have that $q_0 \not\le \overline{p}$, that is there exists t such that $\overline{p}(t) < q_0(t)$. Thus, there are nonnegative integers k, n with $k+n \ge 1$ and elements $x_1, \ldots, x_n$ in S such that

$$kc + \dot{\sum_{i=1}^{n}} p_0(x_i) + (k+n-1)\varepsilon < q_0\Bigl(ku + \sum_{i=1}^{n} x_i\Bigr). \tag{4}$$

Here $k = 0$ is impossible, because then, by using Lemma 1, we get

$$p_0\Bigl(\sum_{i=1}^{n} x_i\Bigr) \le \dot{\sum_{i=1}^{n}} p_0(x_i) + (n-1)\varepsilon < q_0\Bigl(\sum_{i=1}^{n} x_i\Bigr),$$

which is an obvious contradiction.

Interchanging the roles of p_0, q_0 and ε, δ a similar argument shows, that there exists nonnegative integers ℓ, m with $\ell + m \ge 1$ and elements $y_1, y_2, \ldots, y_m$ in S such that

$$p_0\Bigl(\ell u + \sum_{j=1}^{m} y_j\Bigr) < \ell c + \underset{\cdot}{\sum_{j=1}^{m}} q_0(y_j) - (\ell+m-1)\delta. \tag{5}$$

Again we have here $\ell \ge 1$. First observe that the left hand sides of (4) and (5) cannot be equal to ∞ and the right hand sides cannot take the value $-\infty$. Thus, their upper and lower sums are the same. Multiplying (4) by ℓ, (5) by k, adding up these two inequalities, and then adding $-k\ell c$ to both sides, we get

$$\begin{aligned} &\ell \dot{\sum_{i=1}^{n}} p_0(x_i) \,\dot{+}\, kp_0\Bigl(lu + \sum_{j=1}^{m} y_j\Bigr) + \ell(k+n-1)\varepsilon < \\ &\qquad < k \underset{\cdot}{\sum_{j=1}^{m}} q_0(y_j) \underset{\cdot}{+} \ell q_0\Bigl(ku + \sum_{i=1}^{n} x_i\Bigr) - k(\ell+m-1)\delta. \end{aligned} \tag{6}$$

Using the ε-subadditivity of p_0 and the δ-superadditivity of q_0 and Lemma 1, we have that

$$p_0\left(k\ell u+\ell\sum_{i=1}^{n}x_i+k\sum_{j=1}^{m}y_j\right)\leq \ell\sum_{i=1}^{n}p_0(x_i)+kp_0\left(lu+\sum_{j=1}^{m}y_j\right)+(\ell n+k-1)\varepsilon$$

and

$$k\sum_{j=1}^{m}q_0(y_j)+\ell q_0\left(ku+\sum_{i=1}^{n}x_i\right)-(km+\ell-1)\delta\leq q_0\left(k\ell u+\ell\sum_{i=1}^{n}x_i+k\sum_{j=1}^{m}y_j\right).$$

Thus, it follows from (6) and from these inequalities that

$$\begin{aligned} p_0\left(k\ell u+\ell\sum_{i=1}^{n}x_i+k\sum_{j=1}^{m}y_j\right)+(k-1)(\ell-1)\varepsilon< \\ < q_0\left(k\ell u+\ell\sum_{i=1}^{n}x_i+k\sum_{j=1}^{m}y_j\right)-(k-1)(\ell-1)\delta, \end{aligned}$$

which contradicts $q_0\leq p_0$, the nonnegativity of ε,δ, and $k,\ell\geq 1$. The proof is complete.

2. Approximate decomposition theorem

First we introduce the notion of *recession semigroup* for an arbitrary subset A of the semigroup S by

$$\mathrm{rec}A=\{x\in S\cup\{0\}\mid A+x\subseteq A\}.$$

It is immediate to see that rec$A$ is always a (nonempty) subsemigroup of $S\cup\{0\}$. If S is a real linear space and A is a convex subset then one can also see that rec$A$ is a convex cone usually called the *recession (or asymptotic) cone of A* (cf. Holmes [6, p.34]).

Now we define the notion of an approximate subsemigroup. For, let α in $S\cup\{0\}$ be fixed. A subset A of S is called an *α-subsemigroup* of S if

$$A+A+\alpha\subseteq A$$

holds. In the case when the semigroup operation "+" is cancellative, it is easy to see that A is an α-subsemigroup if and only if $A+\alpha$ is a subsemigroup of S. Thus, for instance, the interval $[-\alpha,\infty[$ is an α-subsemigroup of $\mathbb{R}$ for each α in $\mathbb{R}$.

The next lemma obviously follows from the definition by induction.

Lemma 2. *Assume that A is an α-subsemigroup of S. Then, for all n in $\mathbb{N}$ and $x_1,\dots,x_n$ in A,*

$$x_1+\dots+x_n+(n-1)\alpha\in A.$$

The following result offers an approximate decomposition theorem.

Theorem 2. *Let A and B nonempty disjoint subsets of S. Let α and β be in* rec$A$ *and in* rec$B$*, respectively, and assume that A is an α-subsemigroup and B is a β-subsemigroup of S. Then there exists a pair of sets (A_0,B_0) such that*

$$\begin{gathered} A\subseteq A_0,\quad \mathrm{rec}A\subseteq\mathrm{rec}A_0,\quad B\subseteq B_0,\quad \mathrm{rec}B\subseteq\mathrm{rec}B_0,\\ A_0\cap B_0=\emptyset,\quad A_0\cup B_0=S, \end{gathered}\tag{7}$$

and A_0 is an α-subsemigroup, B is a β-subsemigroup of S.

We note that the particular case $\alpha=\beta=0$ of this result reduces to a slight generalization of the Stone-type decomposition theorem obtained in [15].

Proof. Using Zorn's lemma, we can find a maximal (with respect to the componentwise inclusion) pair (A_0,B_0) such that

$$A\subseteq A_0,\qquad \mathrm{rec}A\subseteq\mathrm{rec}A_0,\qquad B\subseteq B_0,\qquad \mathrm{rec}B\subseteq\mathrm{rec}B_0,\qquad A_0\cap B_0=\emptyset,$$

and A_0 is an α-subsemigroup, B is a β-subsemigroup of S.

The rest of the proof is to show that $A_0\cup B_0=S$.

Assume, on the contrary, that there exists $x\notin A_0\cup B_0$. Consider the following set

$$\begin{aligned}\overline{A}:=\big\{a_1+\dots+a_n+kx+(n+k-1)\alpha+u\,\big|\\ \big|\, n,k\ge 0,\, n+k\ge 1,\, a_1,\dots,a_n\in A_0,\, u\in\mathrm{rec}A_0\big\}.\end{aligned}$$

Then, with $n=1$, $k=0$, $u=0$, we get $A_0\subseteq\overline{A}$. The choice $n=0$, $k=1$, $u=0$ shows that x is in $\overline{A}$ and it is also immediate to see that $\overline{A}$ is an α-subsemigroup. Furthermore, $\mathrm{rec}A_0\subseteq\mathrm{rec}\overline{A}$ also holds.

The inclusion $A_0 \subset \overline{A}$ being proper, the maximality of the pair (A_0, B_0) yields that $\overline{A}$ cannot be disjoint from B_0. Thus, there exist nonnegative integers $n, k \geq 0$ with $n + k \geq 1$ and $a_1, \ldots, a_n$ in A_0, u in $\mathrm{rec}A_0$ such that

$$b_0 = a_1 + \cdots + a_n + kx + (n + k - 1)\alpha + u \in B_0.$$

Here k cannot be zero since then, by Lemma 2,

$$b_0 = a_1 + \cdots + a_n + (n - 1)\alpha + u \in A_0 + u \subseteq A_0,$$

contradicting the disjointness of A_0 and B_0. Therefore $k \geq 1$.

A similar argument shows that there exist a nonnegative integer m, an integer $\ell \geq 1$ and $b_1, \ldots, b_m$ in B_0, v in $\mathrm{rec}B_0$ such that

$$a_0 = b_1 + \cdots + b_m + \ell x + (m + \ell - 1)\beta + v \in A_0.$$

Finally, to reach the final contradiction, we now construct an element that belongs to the intersection of A_0 and B_0. Let

$$w = k\ell x + \ell(a_1 + \cdots + a_n) + k(b_1 + \cdots + b_m) + \ell(n + k - 1)\alpha + k(m + \ell - 1)\beta + \ell u + kv.$$

Observe that we can write w in the form

$$w = \big[\ell b_0 + k(b_1 + \cdots + b_m) + (\ell + km - 1)\beta\big] + \big[(k - 1)(\ell - 1)\beta + kv\big].$$

Using Lemma 2, it follows that

$$\ell b_0 + k(b_1 + \cdots + b_m) + (\ell + km - 1)\beta \in B_0$$

and we also have (as $k, \ell \geq 1$ and β, v belong to $\mathrm{rec}B_0$) that

$$(k - 1)(\ell - 1)\beta + kv \in \mathrm{rec}B_0.$$

Hence w belongs to B_0. An analogous argument shows that it is also an element of A_0.

The contradiction obtained shows that $A_0 \cup B_0 = S$ completing the proof of the theorem.

3. Connection between sandwich and decomposition theorems

Analogously to what is known for the non-approximate case, we show that the Approximate Sandwich Theorem can be deduced from the Approximate Decomposition Theorem as well.

2nd Proof of Theorem 1. Assume that the conditions of Theorem 1 hold. If $q = -\infty$ (resp. $p = \infty$) then the function $f = -\infty$ (resp. $f = \infty$) is the desired ε-sub- and δ-superadditive separation. Thus we may assume that $q \not\equiv -\infty$ and $p \not\equiv \infty$. Now we define two subsets – the epigraph of p and the hypograph of q – as follows

$$A = \{(x,t) \in S \times \mathbb{R} \mid p(x) \le t\} \qquad \text{and} \qquad B = \{(x,t) \in S \times \mathbb{R} \mid t < q(x)\}.$$

Then A and B are nonempty disjoint subsets of $S \times \mathbb{R}$. Due to the ε-subadditivity of p, one can prove that A is an $\alpha = (0,\varepsilon)$-subsemigroup of $S \times \mathbb{R}$. One can also see that any element of the form $(0,c)$ where c is a nonnegative real number belongs to $\mathrm{rec}A$. Similarly, B is a $\beta = (0,\delta)$-subsemigroup and $(0,-c)$ belongs to $\mathrm{rec}B$ for all $c \ge 0$.

Now we are in the position to apply Theorem 2. Then there exist disjoint α- and β-subsemigroups A_0 and B_0 of $S \times \mathbb{R}$ such that (7) holds.

Since $(0,c)$ is in $\mathrm{rec}A_0$ for all $c \ge 0$, hence, for fixed x in S, the vertical line $\{x\} \times \mathbb{R}$ intersects A_0 in a possibly degenerate interval of $\overline{\mathbb{R}}$ which is unbounded from above. Similarly, the intersection of this line with the set B_0 is an interval of $\overline{\mathbb{R}}$ which is unbounded from below. Due to the disjointness of these intervals, there exists a unique value $f(x)$ of $\overline{\mathbb{R}}$ such that

$$(\{x\} \times \mathbb{R}) \cap A_0 \subset [f(x), \infty] \qquad \text{and} \qquad (\{x\} \times \mathbb{R}) \cap B_0 \subset [-\infty, f(x)].$$

Thus, we have that

$$A_0 \subset \{(x,t) \in S \times \mathbb{R} \mid f(x) \le t\} \quad \text{and} \quad B_0 = \{(x,t) \in S \times \mathbb{R} \mid t \le q(x)\}. \tag{8}$$

The last step of the proof of Theorem 1 is to show that the function f so obtained is both ε-sub- and δ-superadditive. For, let x, y in S be arbitrary. If either $f(x) = \infty$ or $f(y) = \infty$ then

$$f(x+y) \le f(x) \dot{+} f(y) + \varepsilon \tag{9}$$

is obvious. If $f(x), f(y) < \infty$ then choose t, s in $\mathbb{R}$ arbitrarily such that $f(x) < t$ and $f(y) < s$. The points (x,t), (y,s) cannot belong to B_0, therefore they are in A_0. By the $\alpha = (0,\varepsilon)$-subsemigroup property of A_0, we obtain that

$$(x+y, t+s+\varepsilon) = (x,t) + (y,s) + (0,\varepsilon) \in A_0.$$

Thus, by (8), $f(x+y) \le t+s+\varepsilon$, whence by taking the limits $t \downarrow f(x)$, $s \downarrow f(y)$, we arrive at (9), i.e. f is ε-subadditive. The δ-superadditivity of f follows by a completely similar argument.

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