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MULTIPLICATIVE FUNCTIONS CLOSE TO THE DIVISOR FUNCTION ON SHIFTED PRIMES

K.-H. Indlekofer (Paderborn, Germany)
N.M. Timofeev †

Dedicated to Professor Imre Kátai on his 65th birthday

Abstract. We consider complex-valued multiplicative arithmetical functions which are close to 2 on the set of primes and analyse the asymtotic behaviour of the sum

$$\sum_{p\leq x} f(p+1).$$

In particular we prove

Corollary 1. *Let $f: \mathbb{N} \to \mathbb{C}$ be multiplicative. Let $c_1 \geq 4$ and $c_2 > 0$ be constants such that the inequality*

$$\sum_{|f(p)|\leq c_1} \frac{|f(p)^2|\log^2 p}{p} + \sum_p \sum_{r\leq 2} \frac{|f(p^r)|^2}{p^r} \leq c_2$$

holds, and assume that the series

$$\sum_p \frac{|2-f(p)|}{p}$$

converges. Then

$$\sum_{p\leq x} f(p+1) = x\prod_p \left(1-\frac{2}{p}+\left(1-\frac{1}{p}\right)\sum_{r=1}^{\infty}\frac{f(p^r)}{p^r}\right)+o(x).$$

1. There are many results concerning the mean behaviour of multiplicative arithmetical functions f on the set $\{p+1\}$ of shifted primes, especially in the case when the values $f(p)$ are close to 1 for "almost all" primes p. As a typical result we mention a theorem of I. Kátai [1].

Proposition 1 (see [1]). *If $|f| \leq 1$ and if the series*

$$\sum_p \frac{1-f(p)}{p}$$

converges then the mean-value

$$m_p(f) := \lim_{x\to\infty} \frac{1}{\pi(x)} \sum_{p\leq x} f(p+1)$$

exists.

The wish to abandon the restriction on the size of f led to the investigation of multiplicative functions which belong to the class $\mathcal{L}^q$, $q \geq 1$. Here, for $1 \leq q < \infty$,

$$\mathcal{L}^q := \{f : \mathbb{N} \to \mathbb{C}, ||f||_q < \infty\}$$

denotes the linear space of arithmetic functions with bounded seminorm

$$||f||_q := \left\{ \limsup_{x\to\infty} x^{-1} \sum_{n\leq x} |f(n)|^q \right\}^{\frac{1}{q}}.$$

Many results in this context are due to K.-H. Indlekofer - N.M. Timofeev ([2], [3], [4]) and A. Hildebrand [5].

The following proposition describes a typical situation.

Proposition 2 (see [2]). *Suppose that*

$$\limsup_{x\to\infty} \frac{1}{x} \sum_{n\leq x} |f(n)|^2 < \infty, \quad \sum_{p\leq x} |f(p)|^2 \ll x \log^{-\rho} x$$

where $0 < \rho \leq 1$. If there is a Dirichlet-character χ_d mod d such that

$$M(f\chi_d) := \lim_{x\to\infty} \frac{1}{x} \sum_{n\leq x} f(n)\chi_d(n)$$

exists and is different from zero, then $m_p(f)$ exists, too. If the mean-value $M(|f|)$ exists, then the same is true for the mean-value $m_p(|f|)$.

We shall extend this result to a further class of functions. For this let us consider the *divisor function* τ, i.e. $\tau(n)$ denotes the number of divisors of n. It is well known that

$$\sum_{n\le x}\tau(n)=x\log x+(2\gamma-1)x+R(x)$$

(γ: Euler's constant) where $R(x)$ can easily be estimated by $O(\sqrt{x})$, and the question of finding better bounds for $R(x)$ is referred to as *Dirichlet's divisor problem.*

Concerning the average order of τ on the set of shifted primes it is known (see, for example [6], Theorem 3.9) that

$$\sum_{p\le x}\tau(p+1)=\prod_{p}\left(1+\frac{1}{p(p-1)}\right)x+O\left(\frac{x\log\log x}{\log x}\right).$$

This result motivated the present investigations. Observe that τ is multiplicative and $\tau(p)=2$ for all primes p. In this paper we deal with multiplicative functions f the values $f(p)$ of which are close to 2 for "almost all" primes p, and analyse the asymptotic behaviour of the sum

$$\sum_{p\le x}f(p+1).$$

We prove

Theorem. *Let f be a complex-valued multiplicative arithmetical function. Let $c_1\ge 4$ and c_2 be positive constants such that the inequality*

$$\sum_{|f(p)|\ge c_1}\frac{|f(p)|^2\log^2 p}{p}+\sum_{p}\sum_{r\ge 2}\frac{|f(p^r)|^2}{p^r}\le c_2 \tag{1}$$

holds. Suppose further that

$$\lim_{x\to\infty}\frac{1}{\log x}\sum_{p\le x}\frac{|f(p)-2|}{p}\log p=0. \tag{2}$$

Then

$$\sum_{p\le x}f(p+1)=x\prod_{p\le x}\left(1-\frac{2}{p}+\left(1-\frac{1}{p}\right)\sum_{r=1}^{\infty}\frac{f(p^r)}{p^r}\right)+$$

$$+o\left(x\prod_{p\le x}\left(1-\frac{2}{p}+\left(1-\frac{1}{p}\right)\sum_{r=1}^{\infty}\frac{|f(p^r)|}{p^r}\right)\right)$$

as $x\to\infty$.

Immediate consequences of the Theorem are the following corollaries.

Corollary 1. *Let the multiplicative arithmetical function* f *satisfy condition* (1), *and assume that the series*

$$\sum_{p}\frac{|2-f(p)|}{p} \tag{3}$$

converges. Then

$$\lim_{x\to\infty}\frac{1}{x}\sum_{p\le x}f(p+1)=\prod_{p}\left(1-\frac{2}{p}+\left(1-\frac{1}{p}\right)\sum_{r=1}^{\infty}\frac{f(p^r)}{p^r}\right).$$

Corollary 2. *Let* f *be multiplicative. Suppose that* $|f(p)|\le 2$ *and the two series*

$$\sum_{p}\sum_{r\ge 2}\frac{|f(p^r)|^2}{p^r},\qquad \sum_{p}\frac{2-f(p)}{p}$$

converge. Then

$$\lim_{x\to\infty}\frac{1}{x}\sum_{p\le x}f(p+1)=\prod_{p}\left(1-\frac{2}{p}+\left(1-\frac{1}{p}\right)\sum_{r=1}^{\infty}\frac{f(p^r)}{p^r}\right).$$

Proof of Corollary 1. An application of Cauchy's inequality gives

$$\sum_{p\le x}\frac{|f(p)-2|}{p}\log p\le\sum_{p\le y}\frac{(2+c_1)\log p}{p}+\sum_{\substack{|f(p)|\le c_1,\\ p>y}}\frac{|f(p)-2|}{p}\log x+$$

$$+\left(\sum_{|f(p)|\ge c_1}\frac{4|f(p)|^2}{p}\sum_{p\le y}\frac{\ln^2 p}{p}\right)^{\frac{1}{2}}+\left(\sum_{\substack{|f(p)|\ge c_1,\\ p\ge y}}\frac{4|f(p)|^2}{p}\sum_{p\le x}\frac{\log^2 p}{p}\right)^{\frac{1}{2}}.$$

Letting $y(x) \to \infty$ such that $\dfrac{\log y(x)}{\log x} \to 0$ as $x \to \infty$, and making use of the hypothesis (3) and (1) we establish (2). If the series (3) converges then the series

$$\sum_p \frac{|2 - |f(p)||}{p}$$

converges, too. Hence both products occuring in the assertion of Theorem are convergent, and this ends the proof of Corollary 1.

Proof of Corollary 2. The hypothesis of Corollary 2 implies (1). Using Cauchy's inequality we conclude

$$\sum_{p\le x} \frac{|f(p)-2|}{p}\log p \le 4\sum_{p\le y}\frac{\log p}{p} + \left(2\sum_{p\ge y}\frac{2-\mathrm{Re}f(p)}{p}\sum_{p\le x}\frac{\log^2 p}{p}\right)^{\frac12}.$$

In the same way as above we complete the proof of Corollary 2.

2. First we establish some preliminary results.

Lemma 1. (see [3], Theorem 1) *Let* f_i, $(i = 1, 2, \ldots, k)$, *be complex-valued multiplicative functions satisfying the conditions*

$$A(n) := \sum_{i=1}^{k} \alpha_i f_i(n) \ge 0 \quad (n = 1, 2, \ldots) \tag{4}$$

where $\alpha_i \in C$ *for* $i = 1, \ldots, k$. *Further, assume that there are constants* $c_3 > 0$, $c_4 \ge 0$, $c_5 > 0$ *and* $0 < \rho \le 1$ *such that the inequalities*

$$\sum_{n\le x} |f_i(n)|^2 \le c_3 x(\log x)^{c_4} \quad (i = 1, \ldots, k) \tag{5}$$

and

$$\sum_{p\le x} |f_i(p)|^2 \le c_5 x(\log x)^{-\rho} \quad (i = 1, \ldots, k) \tag{6}$$

hold. Then for $u = (\log x)^A$ *with* $A \ge 8(9c_4 + 8\cdot 3^4 + 150)$ *we have*

$$\frac{1}{\pi(x)}\sum_{p\le x} A(p+1) \ll \frac{\log u}{x} \sum_{\substack{n\le x,\\ (n-1,P(u))=1}} A(n) + (\log\log x)^{1+\frac{c_4}{2}}(\log x)^{-\frac{\rho}{2}},$$

where $P(u) := \prod_{p\le u} p$. *The constant implied in* $\ll$ *depends at most on* c_3 *and* c_5.

Lemma 2. *Let* f *be multiplicative. Further, let* $c_1 \ge 4$ *and* $c_2 > 0$ *be constants so that (1) holds, i.e.*

$$\sum_{|f(p)|\ge c_1} \frac{|f(p)|^2 \log^2 p}{p} + \sum_p \sum_{r\ge 2} \frac{|f(p^r)|^2}{p^r} \le c_2 \quad .$$

Then, with Euler's φ*-function,*

$$\sum_{n\le x} \frac{|f(n)|^2}{\varphi(n)} \ll (\log x)^{c_1^2} \tag{7}$$

and

$$\sum_{n\le x} |f(n)| \ll \frac{x}{\log x} \sum_{n\le x} \frac{|f(n)|}{n}, \tag{8}$$

where the constants implied in $\ll$ *depend at most on* c_1 *and* c_2.

Proof. It is easy to see that

$$\sum_{n\le x} \frac{|f(n)|^2}{\varphi(n)} \le \prod_{p\le x} \left(1 + \sum_{r=1}^{\infty} \frac{|f(p^r)|^2}{\varphi(p^r)}\right) \ll \exp\left(\sum_{p\le x} \sum_{r=1}^{\infty} \frac{|f(p^r)|^2}{p^r}\right),$$

and using (1) we obtain (7). As an immediate consequence we deduce

$$\sum_{n\le x} |f(n)|^2 \le x \sum_{n\le x} \frac{|f(n)|^2}{n} \ll x(\log x)^{c_1^2}.$$

Application of the Cauchy inequality and (1) show that

$$\sum_{p^r\le x} |f(p^r)| \log p^r \le$$

$$\le c_1 \sum_{p\le x} \log p + x \left(\sum_p \sum_{r\ge 2} \frac{|f(p^r)|^2}{p^r}\right)^{\frac{1}{2}} \left(\sum_p \sum_{r\ge 2} \frac{(\log p^r)^2}{p^r}\right)^{\frac{1}{2}} +$$

$$+x\sum_{|f(p)|\geq c_1}\frac{|f(p)|^2}{p}\log p \ll x.$$

Combining the last two estimates gives

$$\sum_{n\leq x}|f(n)| \ll$$

$$\ll \sqrt{x}+\frac{1}{\log x}\sum_{n\leq x}|f(n)|\log n \ll \sqrt{x}+\frac{1}{\log x}\sum_{n\leq x}|f(n)|\sum_{p^r\leq \frac{x}{n}}|f(p^r)|\log p^r \ll$$

$$\ll \frac{x}{\log x}\sum_{n\leq x}\frac{|f(n)|}{n}.$$

This completes the proof of Lemma 2.

Lemma 3. *Let f_1, f_2 be complex-valued multiplicative functions such that the squares f_1^2, f_2^2 satisfy the condition of Lemma 2. Then*

$$\frac{1}{\pi(x)}\sum_{p\leq x}|f_1(p+1)-f_2(p+1)|^2 \ll \frac{1}{\log x}\sum_{n\leq x}\frac{|f_1(n)-f_2(n)|^2}{\varphi(n)}+$$

$$+\frac{1}{\log x}\sum_{p\leq x}\frac{|f_1(p)-f_2(p)|^2}{p}\log p\max_{t\leq x}\frac{1}{\log 2t}\sum_{n\leq t}\frac{|f_2(n)|^2}{\varphi(n)}+$$

$$+\frac{1}{(\log x)^{1+\delta}}\sum_{n\leq x}\frac{|f_1(n)|^2+|f_2(n)|^2}{n}+\frac{1}{(\log x)^{\delta}},$$

where $\delta>0$ and the implied constant depends at most on c_1, c_2 and δ.

Proof. We first apply Lemma 1 with

$$A(n)=|f_1(n)-f_2(n)|^2=|f_1(n)|^2-\overline{f_1}(n)f_2(n)-f_1(n)\overline{f_2}(n)+|f_2(n)|^2.$$

The condition (6) of Lemma 1 is satisfied with $\rho=1$, and hypothesis (5) follows directly from (7). So, we deduce from Lemma 1 that

(9)

$$\frac{1}{\pi(x)}\sum_{p\leq x}|f_1(p+1)-f_2(p+1)|^2 \ll \frac{\log u}{x}\sum_{\substack{n\leq x,\\ (n-1,P(u))=1}}|f_1(n)-f_2(n)|^2+(\log x)^{-\frac{1}{2}+\varepsilon}$$

where $u = \log^A x$. Since (5) holds for f_1 and f_2 we conclude in the same way as above

$$\sum_{\substack{n\le x,\\ (n-1,P(u))=1}} |f_1(n)-f_2(n)|^2 \ll$$

$$\ll \sqrt{x} + \frac{1}{\log x} \sum_{\substack{np^r\le x,\\ (np^r-1,P(u))=1}} |f_1(n)f_1(p^r)-f_2(n)f_2(p^r)|^2 \log p^r.$$

Now we split the last sum into three parts $\sum_1$, $\sum_2$, $\sum_3$, where $\sum_1$ is the sum over $np^r \le x$ with $p^r \le y$, $\sum_2$ is the sum over $np^r \le x$ such that $p^r > y$, $r \ge 2$ and $p \ge y$, $|f_1(p)| \ge c_1$ or $|f_2(p)| \ge c_1$ and $\sum_3$ contains the remaining summands, respectively.

By (8) and (1) the sum $\sum_1$ can be estimated by

$$\sum\nolimits_1 \ll x \sum_{p^r\le y} \frac{|f_1(p^r)|^2+|f_2(p^r)|^2}{p^r} \log p^r \frac{1}{\log \frac{x}{y}} \sum_{n\le x} \frac{|f_1(n)|^2+|f_2(n)|^2}{n} \ll$$

$$\ll x \frac{\log y}{\log \frac{x}{y}} \sum_{n\le x} \frac{|f_1(n)|^2+|f_2(n)|^2}{n}.$$

From the Cauchy inequality we deduce that

$$\sum\nolimits_2 \ll x \sum_{n\le x} \frac{|f_1(n)|^2+|f_2(n)|^2}{n} \left(\frac{1}{\log y} \sum_{i=1}^{2} \sum_{|f_i(p)|\ge c_1} \frac{|f_i(p)|^2}{p} \log^2 p + \right.$$

$$\left. + \left(\sum_{r\ge 2} \sum_{p} \frac{|f_1(p^r)|^4+|f_2(p^r)|^4}{p^r} \right)^{\frac{1}{2}} \left(\sum_{r\ge 2} \sum_{p^r\ge y} \frac{(\log p^r)^2}{p^r} \right)^{\frac{1}{2}} \right).$$

Putting $y = \exp(\log^\alpha x)$, $0 < \alpha < \frac{1}{2}$, we obtain

$$\sum\nolimits_1 + \sum\nolimits_2 \ll \frac{x}{(\log x)^\alpha} \sum_{n\le x} \frac{|f_1(n)|^2+|f_2(n)|^2}{n}.$$

The contribution of $\sum_3$ is bounded by

$$\sum\nolimits_3 \ll \sum\nolimits_{31} + \sum\nolimits_{32} + \sum\nolimits_{33},$$

where

$$\sum\nolimits_{31} = \sum_{n \le \frac{x}{y}} |f_1(n) - f_2(n)|^2 \sum_{\substack{p \le \frac{x}{n}, \\ (np-1, P(u))=1}} \log p,$$

$$\sum\nolimits_{32} = \sum_{n \le t} |f_2(n)|^2 \sum_{\substack{p \le \frac{x}{n}, \\ (np-1, P(u))=1}} \log p$$

and

$$\sum\nolimits_{33} = \sum_{\substack{y < p \le x/t, \\ f_i(p) \le c_1, \\ i=1,2}} |f_1(p) - f_2(p)| \log p \sum_{\substack{n \le x/p, \\ (np-1, P(u))=1}} |f_2(n)|^2,$$

respectively. Here t is given by $\log t = \log u \cdot \log y$. To estimate $\sum_{31}$ and $\sum_{32}$ we employ Selberg's sieve (see [6]). Choosing $z = u$, $\xi^2 = \sqrt[4]{\frac{x}{n}} \ge \sqrt[4]{y} > u$ in Theorem 6.2 of [6] we obtain

$$\left| \left\{ p \le \frac{x}{n}, (np-1, P(u)) = 1 \right\} \right| \ll \frac{x}{\varphi(n) \log u \log \frac{x}{n}}. \tag{10}$$

Hence

$$\sum\nolimits_{31} + \sum\nolimits_{32} \ll \frac{x}{\log u} \sum_{n \le x} \frac{|f_1(n) - f_2(n)|^2}{\varphi(n)} + \frac{x}{\log u} \sum_{n \le t} \frac{|f_2(n)|^2}{\varphi(n)}.$$

Lemma 2 shows that the second summand is $O\left(\frac{x}{\log u} (\log t)^{c_1^2} \right)$. Turning to (9) we arrive at

$$\frac{1}{\pi(x)} \sum_{p \le x} |f_1(p+1) - f_2(p+1)|^2 \ll \frac{1}{\log x} \sum_{n \le x} \frac{|f_1(n) - f_2(n)|^2}{\varphi(n)} + \frac{\log u}{x \log x} \sum\nolimits_{33} +$$

$$+x (\log x)^{\alpha c_1^2 - 1} (\log u)^{c_1^2 - 1} + \frac{\log u}{(\log x)^{1+\alpha}} \sum_{n \le x} \frac{|f_1(n)|^2 + |f_2(n)|^2}{n}. \tag{11}$$

Let $0 < \alpha < \frac{1}{2c_1^2}$. We are now ready to give an upper bound for $\sum_{33}$. First

$$\sum_{\substack{n \le x/p, \\ (np-1, P(u))=1}} |f_2(n)|^2 \ll \sqrt{\frac{x}{p}} + \frac{1}{\log \frac{x}{p}} \sum_{\substack{q^r n \le x/p, \\ (q^r np-1, P(u))=1}} |f_2(q^r)|^2 |f_2(n)|^2 \log q^r.$$

The three parts of the last sum where $q^r \le y$ or $q^r \ge y$, $r \ge 2$ or $q > y$, $|f(q)| \ge c_1$, respectively, can be estimated in the same way as $\sum_1$ and $\sum_2$. The contribution of these parts are

$$\ll \frac{x \log y}{p \log \frac{x}{py}} \sum_{n \le x/p} \frac{|f_2(n)|^2}{n} + \frac{x}{p \log y} \sum_{n \le x/p} \frac{|f_2(n)|^2}{n} \ll \frac{x}{p \log u} \sum_{n \le x/p} \frac{|f_2(n)|^2}{n}.$$

Recall that $p \le x/t$, $\log t = \log u \log y$, $u = \log^A x$, $\log y > \log u$.

For the remaining summands we use Selberg's sieve as above (see (10)). Thus

$$\log \frac{x}{p} \cdot \sum_{\substack{n \le x/p \\ (np-1, P(u))=1}} |f_2(n)|^2 \ll$$

$$\ll \frac{x}{p \log u} \sum_{n \le x/p} \frac{|f_2(n)|^2}{n} + \sum_{n \le \frac{x}{pu}} |f_2(n)|^2 \sum_{\substack{q \le \frac{x}{np}, \\ (qnp-1, P(\sqrt[8]{u}))=1}} \log q \ll$$

$$\ll \frac{x}{p \log u} \sum_{n \le x/p} \frac{|f_2(n)|^2}{\varphi(n)}$$

and

$$\sum\nolimits_{33} \ll \frac{x}{\log u} \sum_{p \le x} \frac{|f_1(p) - f_2(p)|^2}{p} \log p \frac{1}{\log \frac{x}{p}} \sum_{n \le x/p} \frac{|f_2(n)|^2}{\varphi(n)}.$$

Combining this estimate with (11) gives the assertion of Lemma 3.

Lemma 4. *Let f be a complex-valued multiplicative function satisfying the condition of Lemma 2, and put*

$$\prod(f, x) := \prod_{p \le x} \left(1 + \sum_{r=1}^{\infty} \frac{f(p^r)}{p^r}\right).$$

Then

$$\prod(|f|, x) \asymp \exp\left(\sum_{p \le x} \frac{|f(p)|}{p}\right). \tag{12}$$

If, in addition, f satisfies condition (2) then

$$\max_{2 \le t \le x} \frac{1}{\log t} \prod(|f|, t) \ll \frac{1}{\log x} \prod(|f|, x). \tag{13}$$

Proof. Hypothesis (1) implies the convergence of the product

$$\prod_p \left(1+\sum_{r=1}^{\infty}\frac{|f(p^r)|}{p^r}\right)\exp\left(-\frac{|f(p)|}{p}\right).$$

Thus (12) holds. If $\sqrt{x}\le y\le x$ then (1) and (2) yield

$$\frac{\prod(|f|,x)}{\prod(|f|,y)}=\exp\left(\sum_{\substack{y<p\le x,\\ |f(p)|\le c_1}}\frac{|f(p)|}{p}\right)(1+o(1))=\left(\frac{\log x}{\log y}\right)^2(1+o(1)).$$

Hence

$$\frac{\prod(|f|,x)}{\log x}\frac{\log y}{\prod(|f|,y)}\ge\frac{1}{2}\quad\text{and}\quad\frac{\prod(|f|,x)}{\log x}\frac{\log\sqrt{x}}{\prod(|f|,\sqrt{x})}\ge\frac{3}{2}$$

for $\sqrt{x}\le y\le x$ if $x\ge x_0$. This implies (13), which ends the proof of Lemma 4.

Lemma 5. (see [7], Theorem 5) *Let $z\le\sqrt{x}$. For any positive constant A there is a constant $B=B(A)$ such that with $Q=\sqrt{x}(\log x)^{-B}$*

$$\sum_{d\le Q}\max_{y\le x}\max_{(a,d)=1}\Big|\left|\{n:n\le y,(n,P(z))=1,n\equiv a\pmod d\}\right|-$$

$$-\frac{1}{\varphi(d)}\left|\{n:n\le y,(n,P(z)d)=1\}\right|\Big|\ll x(\log x)^{-A}.$$

Lemma 6. *Let $d\le x^2$, $z\le\sqrt[4]{x}$, $y\le\sqrt[4]{x}$. Then*

$$|\{n:n\le x,(n,P(z))=1,(dn-1,P(y))=1\}|\ll\frac{d}{\varphi(d)}\frac{x}{\log z\log y}+\frac{x\log^3 x}{z}.$$

Proof. By Selberg's sieve (see [6], Theorem 6.2) we get

$$|\{n:n\le x,(n,P(z))=1,(dn-1,P(y))=1\}|\ll\prod_{\substack{p\le y,\\ p\nmid d}}\left(1-\frac{1}{p}\right)\sum_{\substack{n\le x,\\ (n,P(z))=1}}1+$$

$$+\sum_{\substack{\delta\le\xi^2,\\(\delta,d)=1}} 3^{\omega(\delta)}\Big|\,|\{n: n\le x, (n,P(z))=1, nd\equiv 1 \pmod{\delta}\}|-$$

$$-\frac{1}{\varphi(\delta)}\,|\{n: n\le x,(n,P(z))=1\}|\Big|, \tag{14}$$

where $\xi^2=\sqrt[3]{x}$. Now,

$$\sum_{\delta\le\xi^2}\frac{3^{\omega(\delta)}}{\varphi(\delta)}\,||\{n:n\le x,(n,P(z))=1\}|-|\{n:n\le x,(n,P(z)\delta)=1\}||\ll$$

$$\ll x\sum_{p>z}\frac{1}{p^2}\sum_{\delta\le\xi^2}\frac{3^{\omega(\delta)}}{\varphi(\delta)}\ll\frac{x(\log x)^3}{z}.$$

Hence, using Cauchy's inequality and Lemma 5, we obtain that the second sum on the right-hand side of (14)

$$\ll\left(x\sum_{\delta\le\xi^2}\frac{9^{\omega(\delta)}}{\varphi(\delta)}\right)^{\frac12}\times$$

$$\times\Bigg(\sum_{\delta\le\xi^2}\max_{(a,d)=1}\Big|\,|\{n:n\le x,(n,P(z))=1, n\equiv a \pmod{\delta}\}|-$$

$$-\frac{1}{\varphi(d)}\,|\{n:n\le x,(n,\delta P(z))=1\}|\Big|\Bigg)^{\frac12}+\frac{x(\log x)^3}{z}\ll\frac{x}{\log^A x}+\frac{x(\log x)^3}{z},$$

where A is an arbitrary positive constant. Now, again by Selberg's sieve,

$$|\{n:n\le x,(n,P(z))=1\}|\ll\frac{x}{\log z}.$$

Substituting these results into (14) yields Lemma 6.

3. Proof of Theorem. For $y\ge 1$ define the multiplicative function f_y by

$$f_y(p^r)=\begin{cases} f(p^r) & \text{if } p^r=y,\\ r+1 & \text{if } p^r>y.\end{cases}$$

Assume that $y = y(x) \to \infty$ and $\dfrac{\log y}{\log x} \to 0$ as $x \to \infty$. The function f_y satisfies hypothesis (1). Let f_1 and f_2 be multiplicative functions such that $f_1(p^r) = \sqrt{f(p^r)}$ and $f_2(p^r) = \sqrt{f_y(p^r)}$, $r = 1, 2, \ldots$, respectively, where $\sqrt{z} = \sqrt{|z|}\exp\left(\frac{1}{2} i \arg z\right)$. The functions f_1 and f_2 satisfy the hypothesis of Lemma 3. Applying Lemma 3 gives

$$\frac{1}{\pi(x)} \sum_{p \le x} |f_1(p+1) - f_2(p+1)|^2 \ll \frac{1}{\log x} \sum_{n \le x} \frac{|f_1(n) - f_2(n)|^2}{\varphi(n)} +$$

$$+ \frac{1}{\log x} \sum_{p \le x} \frac{|f_1(p) - f_2(p)|^2}{p} \log p \max_{t \le x} \frac{1}{\log 2t} \sum_{n \le x} \frac{|f(n)|}{\varphi(n)} +$$

$$+ \frac{1}{(\log x)^{\delta}} + \frac{1}{(\log x)^{1+\delta}} \sum_{n \le x} \frac{|f(n)| + |f_y(n)|}{\varphi(n)}.$$

From (2) we conclude that there is a function ε such that $\varepsilon(x) \downarrow 0$ but $\varepsilon(x)\sqrt{\log x}$ tends to infinity as $x \to \infty$ and

$$\sum_{p \le x} \frac{|2 - f(p)|}{p} \log p \le \varepsilon(x) \log x. \tag{15}$$

Let $\log y \ge \sqrt{\varepsilon(x)} \log x$. Then

$$\sum_{y < p \le 2} \frac{||f(p)| - 2|}{p} \le \frac{1}{\log y} \sum_{p \le 2} \frac{|f(p) - 2|}{p} \log p \le \sqrt{\varepsilon(x)} \tag{16}$$

and therefore $\prod(|f)|, x) \asymp \prod(|f_y|, x)$. Note that $|\sqrt{f(p)} + \sqrt{2}| \ge \sqrt{2}$. Then, by (1) and (2) we obtain

$$\sum_{p \le x} \frac{|f_1(p) - f_2(p)|^2}{p} \log p \le \frac{1}{2} \sum_{p \le x} \frac{|f(p) - 2|^2}{p} \log p \le \frac{c_1 + 2}{2} \varepsilon(x) \log x +$$

$$+ \sum_{|f(p)| \ge c_1} \frac{|f(p)|^2}{p} \log p \ll \varepsilon(x) \log x.$$

Applying Lemma 4 shows that

$$\frac{1}{\pi(x)} \sum_{p \le x} |f_1(p+1) - f_2(p+1)|^2 \ll \tag{17}$$

$$\ll \frac{1}{\log x} \sum_{n\le x} \frac{|f_1(n)-f_2(n)|^2}{\varphi(n)} + o\left(\frac{1}{\log x}\prod(|f|,x)\right).$$

Next we prove

$$\sum_{n\le x} \frac{|f_1(n)-f_2(n)|^2}{\varphi(n)} = o\left(\prod(|f|,x)\right). \tag{18}$$

For this we divide the sum on the left into two parts $\sum_1'$ and $\sum_2'$, where $\sum_1'$ denotes the sum over $n \le x$ such that $p^r|n$, $r \ge 2$, $p^r \ge y$ or $p||n$, $p > y$, $|f(p)| \ge c_1$ and $\sum_2$ contains the remaining summands, respectively. By (1) we have

$$\sum\nolimits_1' \ll \left(\sum_{\substack{p^r\ge y,\\ r\ge 2}} \frac{|f(p^r)|+(r+1)}{p^r} + \sum_{\substack{p^r\ge y,\\ |f(p)\ge c_1}} \frac{|f(p)|}{p}\right)\prod(|f|,x) = o\left(\prod(|f|,x)\right).$$

Now write each number n occuring in $\sum_2'$ in the form $n = n_1 p_1 \dots p_t$, where n_1 has only prime divisors $\le y$ and $p_i > y$ with $|f(p_i)| \le c_1$. Then

$$|f_1(n)-f_2(n)| = |f_1(n_1)|\left|f_1(p_1)\dots f_1(p_t) - \left(\sqrt{2}\right)^t\right| \le$$

$$\le |f_1(n_1)|c_1^{t-1}\left(|f_1(p_1)-\sqrt{2}| + \ldots + |f_1(p_t)-\sqrt{2}|\right).$$

This gives

$$|f_1(n)-f_2(n)|^2 \le |f(n_1)|c_1^{2t} t\left(|f(p_1)-2| + \ldots + |f(p_t)-2|\right),$$

and therefore

$$\sum\nolimits_2' \ll c_1^{2\frac{\log x}{\log y}} \frac{\log x}{\log y} \sum_{y<p\le x} \frac{|f(p)-2|}{p} \exp\left(\sum_{p\le y}\frac{|f(p)|}{p} + \sum_{y<p\le x}\frac{1}{p}\right).$$

Choosing $y \ge x^{\sqrt{\varepsilon(x)}}$, $\dfrac{\log x}{\log y} \le \dfrac{1}{4\log c_1}\log\dfrac{1}{\varepsilon(x)}$, we obtain by (15)

$$\sum\nolimits_2' \ll c_1^{2\frac{\log x}{\log y}} \left(\frac{\log x}{\log y}\right)^3 \varepsilon(x)\prod(|f|,x) \ll \sqrt{[3]}\varepsilon(x)\prod(|f|,x).$$

Thus, (18) holds. Using (17) and Lemmas 3 and 4 we see

$$\frac{1}{\pi(x)}\sum_{p\le x}|f_1^2(p+1)-f_2^2(p+1)|\le\left(\frac{1}{\pi(x)}\sum_{p\le x}|f_1(p+1)-f_2(p+1)|^2\right)^{\frac12}\times$$

$$\times\left(\frac{1}{\pi(x)}\sum_{p\le x}|f_1(p+1)+f_2(p+1)|^2\right)^{\frac12}=o\left(\frac{1}{\log x}\prod(|f|,x)\right).$$

Therefore

$$\frac{1}{\pi(x)}\sum_{p\le x}f(p+1)=\frac{1}{\pi(x)}\sum_{p\le x}f_y(p+1)+o\left(\frac{1}{\log x}\prod(|f|,x)\right), \tag{19}$$

where $y(x)\to\infty$ and $\dfrac{\log y(x)}{\log x}\to 0$ as $x\to\infty$. The last sum in (19) can be written in the form

$$\sum_{p\le x}f_y(p+1)=\sum_{\substack{p+1=n_1n_2\le x,\\ n_1\le y_1}}f(n_1)\tau(n_2)+O\left(\sum_{\substack{p+1=n_1n_2\le x,\\ n_1>y_1}}|f(n_1)|\tau(n_2)\right), \tag{20}$$

where $(n_2,P(y))=1$ and where n_1 denotes an even number whose prime divisors are not larger than y. Let us apply Lemma 3 for f_1 and f_2 where $f_1^2(n)=f(n)$ and $f_2^2(n)=0$ if $n>1$. Then

$$\frac{1}{\pi(x)}\sum_{p+1=n_1\le x}|f(n_1)|\ll\frac{1}{\log x}\sum_{n_1\le x}\frac{|f(n_1)|}{\varphi(n_1)}+$$

$$+\frac{1}{\log x}\sum_{p\le y}\frac{|f(p)|}{p}\log p\frac{1}{\log x}\prod(|f|,x)+\frac{1}{(\log x)^{1+\delta}}\prod(|f|,x).$$

Choose $\log y\ge\sqrt{\varepsilon(x)}\log x$. Using Lemma 4 together with (1) and (16) we see that

$$\sum_{n_1\le x}\frac{|f(n_1)|}{\varphi(n_1)}\ll\left(\frac{\log y}{\log x}\right)^2\prod(|f|,x).$$

Hence

$$\frac{1}{\pi(x)}\sum_{p+1=n_1\le x}|f(n_1)|\ll\frac{\log y}{\log x}\cdot\frac{1}{\log x}\prod(|f|,x)=o\left(\frac{1}{\log x}\prod(|f|,x)\right) \tag{21}$$

and

$$\frac{1}{\pi(x)}\sum_{\substack{p+1=n_1n_2\le x,\\ n_1>y_1,n_2>y}}|f(n_1)|\tau(n_2)\le\frac{2}{\pi(x)}\sum_{y_1<n_1\le x/y}|f(n_1)|\times$$

$$\times\sum_{n_2\le\sqrt{\frac{x}{n_1}}}\left(\left|\left\{n:n\le\frac{x}{n_2n_1},(n_2n(n_1n_2n-1),P(\sqrt[8]{y}))=1\right\}\right|+\sqrt[8]{y}\right).$$

Applying Selberg's sieve (see [6], Theorem 6.2) gives

$$\left|\left\{n:n\le\frac{x}{n_2n_1},(n_2n(n_1n_2n-1),P(\sqrt[8]{y}))=1\right\}\right|\ll\frac{x}{\varphi(n_2)\varphi(n_1)\log^2 y}.$$

Therefore

$$\frac{1}{\pi(x)}\sum_{\substack{p+1=n_1n_2\le x,\\ n_1>y_1,n_2>y}}|f(n_1)|\tau(n_2)\ll\frac{\log x}{\log^2 y}\sum_{y_1<n_1\le x/y}\frac{|f(n_1)|}{\varphi(n_1)}\sum_{n_2\le x}\frac{1}{\varphi(n_2)}. \tag{22}$$

By Cauchy's inequality and the hypothesis (1) we get

$$\sum_{y_1<n_1\le x}\frac{|f(n_1)|}{\varphi(n_1)}\le\frac{1}{\log y_1}\sum_{y_1<n_1\le x}\frac{|f(n_1)|}{\varphi(n_1)}\log n_1\le$$

$$\le\frac{1}{\log y_1}\sum_{p^r\le 2y}\frac{|f(p^r)|}{\varphi(p^r)}\log p^r\sum_{\frac{y_1}{2y}<n_1\le x}\frac{|f(n_1)|}{\varphi(n_1)}+$$

$$+\frac{1}{\log y_1}\left(\sum_{r\ge 2,p}\frac{|f(p^r)|^2}{\varphi(p^r)}\right)^{\frac{1}{2}}\left(\sum_{\substack{p^r\ge y,\\ r\ge 2}}\frac{(\log p^r)^2}{p^r}\right)^{\frac{1}{2}}\sum_{n_1\le x}\frac{|f(n_1)|}{\varphi(n_1)}.$$

Now, let $\dfrac{\log y_1}{\log y}\to\infty$ and $\dfrac{\log y_1}{\log x}\to 0$ as $x\to\infty$. Then

$$\sum_{y_1<n_1\le x}\frac{|f(n_1)|}{\varphi(n_1)}\ll\frac{\log y}{\log y_1}\frac{1}{\log y_1}\sum_{n_1\le x}\frac{|f(n_1)|}{\varphi(n_1)}\log n_1+\frac{1}{\sqrt[8]{y}\log y_1}\sum_{n_1\le x}\frac{|f(n_1)|}{\varphi(n_1)}\ll$$

$$\ll\left(\frac{\log y}{\log y_1}\right)^2\sum_{n_1\le x}\frac{|f(n_1)|}{\varphi(n_1)}$$

if $\sqrt[8]{y} \geq \frac{\log y_1}{(\log y)^2}$. Thus, by (22),

$$\frac{1}{\pi(x)} \sum_{\substack{p+1=n_1 n_2 \leq x, \\ n_1 > y_1, n_2 > y}} |f(n_1)|\tau(n_2) \ll \frac{\log x}{(\log y_1)^2} \sum_{n_1 \leq x} \frac{|f(n_1)|}{\varphi(n_1)} \sum_{n_2 \leq x} \frac{1}{\varphi(n_2)} \ll$$

$$\ll \left(\frac{\log x}{\log y_1}\right)^2 \frac{\log y}{\log x} \frac{1}{\log x} \prod(|f|, x) = o\left(\frac{1}{\log x} \prod(|f|, x)\right)$$

if $\frac{\log x}{\log y_1} \leq \sqrt[4]{\frac{\log y}{\log x}}$. Substituting this result and (21) into (20) shows

$$\frac{1}{\pi(x)} \sum_{p \leq x} f_y(p+1) =$$

$$= \frac{1}{\pi(x)} \sum_{n_1 \leq y_1} f(n_1) \sum_{n_2 \leq \sqrt{\frac{x}{n_1}}} \left(2I(x, n_1 n_2, y) - I(n_2 n_1 \sqrt{\frac{x}{n_1}}, n_1 n_2, y)\right) +$$

$$+ o\left(\frac{1}{\log x} \prod(|f|, x)\right)\Bigg),$$

where

$$I(x, n_1 n_2, y) = \left|\left\{p \leq x, p+1 \equiv 0 \pmod{n_1 n_2}, \left(\frac{p+1}{n_1 n_2}, P(y)\right) = 1\right\}\right|.$$

Now, let $Q = Q(x)$ such that $\frac{\log Q}{\log x} \to 0$ as $x \to \infty$. By Lemma 6 we have

$$\frac{1}{\pi(x)} \sum_{n_1 \leq y_1} |f(n_1)| \sum_{n_2 \leq \sqrt{\frac{x}{n_1}}} I\left(n_2 n_1 \sqrt{\frac{x}{n_1}}, n_1 n_2, y\right) \leq$$

$$\leq \frac{1}{\pi(x)} \sum_{n_1 \leq y_1} |f(n_1)| \sum_{n_2 \leq \sqrt{\frac{x}{n_1}}} \Bigg(\Bigg|\Bigg\{n : n \leq \sqrt{\frac{x}{n_1}}, (n, P(y)) = 1,$$

$$\left(n n_1 n_2 - 1, P\left(\sqrt[8]{x}\right)\right) = 1 \Bigg\}\Bigg| + \sqrt[8]{x}\Bigg) \ll$$

$$\ll \frac{1}{\sqrt{x} \log y} \sum_{n_1 \leq y_1} \frac{|f(n_1)\sqrt{n_1}}{\varphi(n_1)} \sum_{n_2 \leq \sqrt{\frac{x}{n_1}}} \frac{n_2}{\varphi(n_2)} \ll$$

$$\ll \frac{\log y}{\log x}\frac{1}{\log x}\prod(|f|,x).$$

Therefore

$$\frac{1}{\pi(x)}\sum_{p\le x} f_y(p+1) = \frac{2}{\pi(x)}\sum_{n_1\le y_1} f(n_1)\sum_{n_2\le Q^{-1}\sqrt{\frac{x}{n_1}}} I(x,n_1n_2,y)+$$

$$(23)\qquad +O\left(\frac{1}{\pi(x)}\sum_{n_1\le y_1}|f(n_1)|\sum_{\sqrt{\frac{x}{n_1}}Q^{-1}\le n_2\le\sqrt{\frac{x}{n_1}}} I(x,n_1n_2,y)\right)+$$

$$+o\left(\frac{1}{\log x}\prod(|f|,x)\right).$$

Using Lemma 6 shows that the second term may be estimated by

$$\le \frac{1}{\pi(x)}\sum_{n_1\le y_1}|f(n_1)|\left(\sum_{\sqrt{\frac{x}{n_1}}Q^{-1}\le n_2\le\sqrt{\frac{x}{n_1}}}\left|\left\{n: n\le\frac{x}{n_1n_2}, (n,P(y))=1,\right.\right.\right.$$

$$\left.\left.\left.(nn_1n_2-1, P\left(\sqrt[8]{x}\right))=1\right\}\right| + \sqrt[8]{x}\right)\ll$$

$$\ll \frac{1}{\log y}\sum_{n_1\le y_1}\frac{|f(n_1)|}{\varphi(n_1)}\sum_{\sqrt{\frac{x}{n_1}}Q^{-1}\le n_2\le\sqrt{\frac{x}{n_1}}}\frac{1}{\varphi(n_2)}.$$

We have

$$\sum_{n_2\le x}\frac{1}{\varphi(n_2)} = \sum_{d,(d,P(y))=1}\frac{\mu^2(d)}{d\varphi(d)}\sum_{n_2\le x/d}\frac{1}{n_2} = \sum_{n_2\le x}\frac{1}{n_2} + O\left(\frac{\log x}{y}\right).$$

Applying the Fundamental Lemma of sieve theory (see [6], Theorem 2.6) we conclude

$$\sum_{n_2\le u} 1 = u\prod_{p\le y}\left(1-\frac{1}{p}\right)\left(1+O\left(\left(\frac{\log y}{\log u}\right)^2\right)\right).$$

Then, by partial summation,

$$(24)\qquad \sum_{n_2\le x}\frac{1}{\varphi(n_2)} = \prod_{p\le y}\left(1-\frac{1}{p}\right)\log x\left(1+O\left(\left(\frac{\log y}{\log x}\right)^2\right)\right).$$

This shows that the first remainder term on the right side in (23) can be estimated by

$$\ll \frac{1}{\log y}\sum_{n_1\le y_1}\frac{|f(n_1)|}{\varphi(n_1)}\left(\frac{\log Q}{\log y}+\frac{\log y}{\log x}\right)\ll$$

$$\ll \frac{\log y}{\log x}\left(\frac{\log Q}{\log y}+\frac{\log y}{\log x}\right)\frac{1}{\log x}\prod(|f|,x)=o\left(\frac{1}{\log x}\prod(|f|,x)\right).$$

Hence

$$\frac{1}{\pi(x)}\sum_{p\le x}f_y(p+1)= \tag{25}$$

$$=\frac{2}{\pi(x)}\sum_{n_1\le y_1}f(n_1)\sum_{n_2\le Q^{-1}\sqrt{\frac{x}{n_1}}}I(x,n_1n_2,y)+o\left(\frac{1}{\log x}\prod(|f|,x)\right).$$

Now, Selberg' sieve (see [6], Theorem 7.1) gives

$$I(x,n_1n_2,y)=\frac{\pi(x)}{\varphi(n_1n_2)}\prod_{p\le y}\left(1-\frac{\varphi(n_1n_2)}{\varphi(n_1n_2p)}\right)\left(1+O\left(\exp\left(-\frac{\log\xi}{\log y}\right)\right)\right)+$$

$$+O\left(\sum_{\substack{d\le\xi^2,\\ d|P(y)}}3^{\omega(d)}\left|\pi(x,-1,n_1n_2d)-\frac{\mathrm{Li}x}{\varphi(n_1n_2d)}\right|\right).$$

Set $\xi^2=\dfrac{Q\sqrt{n_1}}{\log^A x}$. Then $n_1dn_2\le\dfrac{\sqrt{x}}{\log^A x}$. Applying Cauchy's inequality leads to

$$\sum_{n_1\le y_1}|f(n_1)|\sum_{n_2\le Q^{-1}\sqrt{\frac{x}{n_1}}}\sum_{\substack{d\le\xi^2,\\ d|P(y)}}3^{\omega(d)}\left|\pi(x,-1,n_1n_2d)-\frac{\mathrm{Li}x}{\varphi(n_1n_2d)}\right|\ll$$

$$\ll\left(\sum_{n\le x}\frac{x}{n}\left(\sum_{n_1dn_2=n}|f(n_1)|3^{\omega(d)}\right)^2\right)^{\frac12}\left(\sum_{d\le\frac{\sqrt{x}}{\log^A x}}\left|\pi(x,-1,d)-\frac{\mathrm{Li}x}{\varphi(d)}\right|\right)^{\frac12}.$$

The first sum is less than

$$\sum_{n\le x}\frac{\tau_3(n)}{n}\sum_{n\le x}\frac{1}{n}\sum_{n\le x}\frac{|f(n)|^2}{n}\sum_{n\le x}\frac{9^{\omega(n)}}{n}\ll(\log x)^{c_1^2+13}.$$

The second sum can be estimated by the Vinogradov-Bombieri theorem and is dominated by $O(x\log^{-(c_1^2+16)} x)$ if $A \geq A_0$. Recall that n_1 is an even number. Therefore

$$\prod_{p\le y}\left(1-\frac{\varphi(n_1n_2)}{\varphi(n_1n_2p)}\right)=\frac{1}{2}\prod_{2<p\le y}\left(1-\frac{1}{p-1}\right)\prod_{\substack{p|n_1,\\ p\ne 2}}\left(1-\frac{1}{p}\right)\left(1-\frac{1}{p-1}\right)^{-1}=$$

$$=2\frac{\varphi(n_1)}{n_1}\prod_{p\le y}\left(1-\frac{1}{p}\right)\prod_{2<p\le y}\left(1-\frac{1}{p}\right)\left(1-\frac{1}{p-1}\right)^{-1}\prod_{\substack{p|n_1,\\ p\ne 2}}\left(1+\frac{1}{p-2}\right).$$

Substituting these results into (25) we arrive at

$$\frac{1}{\pi(x)}\sum_{p\le x} f_y(p+1)=4\prod_{p\ne 2}\left(1-\frac{1}{p-1}\right)\left(1-\frac{1}{p}\right)^{-1}\prod_{p\le y}\left(1-\frac{1}{p}\right)\sum_{n_1\le y_1}\frac{f(n_1)}{n_1}\times$$

$$\times\prod_{\substack{p|n_1,\\ p\ne 2}}\left(1+\frac{1}{p-2}\right)\sum_{n_2\le\sqrt{\frac{x}{n_1}}Q^{-1}}\frac{1}{\varphi(n_2)}\times$$

$$\times\left(1+O\left(\exp\left(-\frac{\log\xi}{\log y}\right)\right)\right)\left(1+O\left(\frac{1}{y}\right)\right)+O\left(\frac{1}{\log x}\prod(|f|,x)\right).$$

Recall that $\dfrac{\log y_1}{\log x}\to 0$ and $\dfrac{\log Q}{\log x}\to 0$ as $x\to\infty$ and

$$\frac{\log\xi}{\log y}\ge\frac{\log Q}{\log y}-\frac{A\log\log x}{\log y}.$$

Let $y\ge\exp\left(\sqrt{\log x}\right)$ and $\dfrac{\log Q}{\log y}\to\infty$ as $x\to\infty$. By (24) we see that

$$\frac{1}{\pi(x)}\sum_{p\le x} f_y(p+1)=\prod_{p\le y}\left(1-\frac{2}{p}+(1-\frac{1}{p})\sum_{r=1}^{\infty}\frac{f(p^r)}{p^r}\right)\log x+$$

$$+o\left(\frac{1}{\log x}\prod(|f|,x)\right).$$

Choosing $y\ge x^{\sqrt{\varepsilon(x)}}$ gives

$$\sum_{y<p\le x}\frac{|f(p)-2|}{p}\to 0$$

as $x \to \infty$, which, together with (19), proves the assertion of the Theorem.

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K.-H. Indlekofer
Faculty of Computer Science, Electrical Engineering and Mathematics
University of Paderborn
Warburger Straße 100
D-33098 Paderborn, Germany