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CHARACTERIZATION OF ALMOST–PERIODIC q–MULTIPLICATIVE FUNCTIONS

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Dedicated to Prof.Dr. Dr.h.c.mult. K.-H. Indlekofer on his 60th birthday

Abstract. We give a complete characterization of q-multiplicative functions that are almost-periodic.

1. Introduction and results

The starting point of the definition of (classical) multiplicative functions is the unique representation of the natural numbers

$$n = \prod_{p \in \mathbb{P}} p^{\alpha_p(n)}, \quad \alpha_p(n) = \max\{\alpha : p^\alpha | n\}$$

as a product of prime numbers. Then $f : \mathbb{N} \to \mathbb{C}$ is called *multiplicative* in case

$$f(n) = \prod_{p \in \mathbb{P}} f(p^{\alpha_p(n)}).$$

Now, let $q \geq 2$ be an integer and $\mathbb{A} = \{0, 1, \dots, q-1\}$. The *q-ary expansion* of some $n \in \mathbb{N}_0$ is defined as the unique sequence $\varepsilon_0(n), \varepsilon_1(n), \dots$ for which

$$n = \sum_{j=0}^{\infty} \varepsilon_j(n) q^j, \quad \varepsilon_j(n) \in \mathbb{A} \tag{1}$$

holds. $\varepsilon_0(n), \varepsilon_1(n), \ldots$ are called the *digits* in the q-ary expansion of n. A function $f : \mathbb{N}_0 \to \mathbb{C}$ is called *q-multiplicative* if $f(0) = 1$, and for every $n \in \mathbb{N}_0$,

$$f(n) = \prod_{j=0}^{\infty} f(\varepsilon_j(n) q^j). \tag{2}$$

For $f : \mathbb{N}_0 \to \mathbb{C}$ define, for any real number $\alpha \geq 1$,

$$\|f\|_\alpha := \left(\limsup_{N\to\infty} \frac{1}{N} \sum_{0\leq n<N} |f(n)|^\alpha \right)^{\frac{1}{\alpha}}, \tag{3}$$

and let

$$\mathcal{L}^\alpha := \{f : \mathbb{N}_0 \to \mathbb{C}, \quad \|f\|_\alpha < \infty\}.$$

An arithmetical function[1] $f : \mathbb{N}_0 \to \mathbb{C}$ is called *uniformly summable* in case

$$\lim_{K\to\infty} \sup_{N\geq 1} \frac{1}{N} \sum_{\substack{n\leq N \\ |f(n)|\geq K}} |f(n)| = 0.$$

The set of all uniformly summable functions, denoted $\mathcal{L}^*$, is a proper subset of $\mathcal{L}^1$. Obviously $(\alpha > 1)$

$$\mathcal{L}^\alpha \underset{\neq}{\subset} \mathcal{L}^* \underset{\neq}{\subset} \mathcal{L}^1.$$

Let $e(\beta) := \exp(2\pi i \beta)$. f is called *α-almost-periodic*, if for every $\varepsilon > 0$ there is a linear combination h of exponential functions[2] e_β, $\beta \in \mathbb{R}$, such that $\|f - h\|_\alpha \leq$ $\leq \varepsilon$. The linear space of α-almost-periodic functions is denoted by $\mathcal{A}^\alpha$. If h can always be chosen to be periodic then f is called *α-limit-periodic*. The linear space of α-limit-periodic functions is denoted by $\mathcal{D}^\alpha$. We have the inclusions

$$\mathcal{D}^1 \underset{\neq}{\subset} \mathcal{A}^1 \underset{\neq}{\subset} \mathcal{L}^*.$$

For every function $f \in \mathcal{A}^1$, the *mean value*

$$M(f) := \lim_{N\to\infty} \frac{1}{N} \sum_{0\leq n<N} f(n)$$

[1] If f is defined on $\mathbb{N}$ we may extend f to $\mathbb{N}_0$ by putting $f(0) = 0$.

[2] $e_\beta : \mathbb{N} \to \mathbb{C}$ with $e_\beta(n) = e(\beta n)$ is a q-multiplicative function.

and, for every $\beta \in \mathbb{R}$, the *Fourier coefficient*

$$\hat{f}(\beta) := \lim_{N\to\infty} \frac{1}{N} \sum_{0\le n<N} f(n)e_{-\beta}(n)$$

exist (see, for example, W. Schwarz and J. Spilker [7] Chap. IV and VI).

For $f \in \mathcal{L}^1$ the *Fourier-Bohr spectrum* $\sigma(f)$ is defined as

$$\sigma(f) = \left\{ \beta \in \mathbb{R}/\mathbb{Z} : \limsup_{N\to\infty} \left| \frac{1}{N} \sum_{n\le N} f(n)e_{-\beta}(n) \right| > 0 \right\}.$$

If $f \in \mathcal{A}^1$ then $\beta \in \sigma(f)$ if and only if $\hat{f}(\beta) \neq 0$.

In his paper [4] K.-H. Indlekofer gives s complete characterization of α-almost-periodic multiplicative functions. He proved the following results.

Proposition 1. ([4], Theorem 1) *Let $f \in \mathcal{A}^1$ be multiplicative. Then $M(|f|) = 0$ if and only if $\sigma(f) = \emptyset$.*

Proposition 2. ([4], Theorem 2) *Let $f \in \mathcal{A}^\alpha$ be multiplicative. Then f is α-limit-periodic.*

Proposition 3. ([4], Corollary 1) *Let $f : \mathbb{N} \to \mathbb{C}$ be multiplicative. Then the following assertions are equivalent.*

(i) *$f \in \mathcal{A}^\alpha$ and $\|f\|_1 > 0$.*
(ii) *$f \in \mathcal{A}^\alpha$ and the spectrum $\sigma(f)$ of f is non-empty.*
(iii) *$f \in \mathcal{L}^\alpha \cap \mathcal{L}^*$ and there exists a Dirichlet-character χ such that the mean-value $M(f\chi)$ of $f\chi$ exists and is different from zero.*
(iv) *There exists a Dirichlet-character χ such that the series*

$$\sum_{p} \frac{f(p)\chi(p) - 1}{p}, \quad \sum_{\substack{p \\ |f(p)|\le 3/2}} \frac{|f(p)\chi(p) - 1|^2}{p} \tag{4}$$

and

$$\sum_{\substack{p \\ ||f(p)|-1|>1/2}} \frac{|f(p)|^\lambda}{p}, \quad \sum_{p} \sum_{k\ge 2} \frac{|f(p^k)|^\lambda}{p^k} \tag{5}$$

converge for all λ with $1 \le \lambda \le \alpha$.

Remark 1. The equivalence of (ii) and (iv) was proved by H. Daboussi [1]. The equivalence of (ii), (iii) and (iv) was shown by K.-H. Indlekofer in [5], Corollary 7.

The aim of this paper is to find corresponding characterizations for q-multiplicative functions belonging to $\mathcal{D}^1$ and $\mathcal{A}^1$, respectively. A first step in this direction was done recently by J. Spilker [8] who proved the following

Proposition 4. ([8], Theorem 4) *Let f be q-multiplicative and the following two series*

$$\sum_{r=0}^{\infty}\sum_{a=0}^{q-1}(f(aq^r)-1) \tag{6}$$

and

$$\sum_{r=0}^{\infty}\sum_{a=0}^{q-1}|f(aq^r)-1|^2 \tag{7}$$

converge. Then

(*i*)
$$f\in\mathcal{D}^\alpha,\ \ \alpha\geq 1.$$

(*ii*)
$$M(f)=\prod_{r=0}^{\infty}\left(\frac{1}{q}\sum_{a=0}^{q-1}f(aq^r)\right).$$

(*iii*)
$$\hat{f}(\beta)=\begin{cases}\displaystyle\prod_{r=0}^{\infty}\left(\frac{1}{q}\sum_{a=0}^{q-1}f(aq^r)e_{-\frac{c}{b}}(aq^r)\right) & \text{if }\ \beta=\dfrac{c}{b},\\ 0 & \text{if } \beta \text{ irrational.}\end{cases}$$

Remark 2. Assertion (iii) of Proposition 4 is not correct as it stands. Choose, for example, $f=1$ and $\beta=\dfrac{1}{p}$, where p is a prime which does not divide q. Then $\hat{f}(\beta)=0$ and for all $r\in\mathbb{N}_0$, $\displaystyle\sum_{a=0}^{q-1}f(aq^r)e_{-\frac{1}{p}}(aq^r)=\frac{1-e(q^{r+1}/p)}{1-e(q^r/p)}\neq 0$, i.e. the infinite product $\displaystyle\prod_{r=0}^{\infty}\left(\frac{1}{q}\sum_{a=0}^{q-1}f(aq^r)e_{-\frac{1}{p}}(aq^r)\right)$ does not converge in this case.

We shall characterize the q-multiplicative functions $f\in\mathcal{D}^1$ and $f\in\mathcal{A}^1\backslash\mathcal{D}^1$ by their respective spectrum $\sigma(f)$. First we show that the spectrum is empty only in the trivial case. We prove

Theorem 1. *Let $f \in \mathcal{A}^1$ be q-multiplicative. Then $M(|f|) = 0$ if and only if $\sigma(f) = \emptyset$.*

Recently K.-H. Indlekofer, Y.-W. Lee and R. Wagner [6] could describe the mean behaviour of uniformly summable q-multiplicative functions. In the special case that the mean value exists and is different from zero their results can be summarized in the following

Proposition 5. (see [6], Corollary 1) *Let f be q-multiplicative. Then the following assertions hold.*

(i) Let $f \in \mathcal{L}^*$. If the mean value $M(f)$ exists and is different from zero then the series (6) and (7) converge and

$$\sum_{a=1}^{q-1} f(aq^r) \neq 0 \quad for\ each\ r \in \mathbb{N}_0.$$

(ii) If the series (6) and (7) converge then $f \in \mathcal{L}^*$, the mean value $M(f)$ exists,

$$M(f) = \prod_{r=0}^{\infty} \left(\frac{1}{q} \sum_{a=0}^{q-1} f(aq^r) \right)$$

and $\|f - f_R\|_1 \to 0$ as $R \to \infty$, where

$$f_R(n) = \prod_{r \leq R} f(\varepsilon_r(n) q^r).$$

(iii) Let $f \in \mathcal{L}^*$. If the mean value $M(f)$ exists and is different from zero then the mean value $M(|f|^\alpha)$ of $|f|^\alpha$ exists for each $\alpha \geq 1$ (and is different from zero).

Using Proposition 5 we shall obtain

Theorem 2. *For every q-mutiplicative function f, the following assertions are equivalent:*

(a) $f \in \mathcal{D}^1$ and the mean value $M(f)$ is nonzero.

(b) The series (6) and (7) are both convergent and $\sum_{a=1}^{q-1} f(aq^r) \neq 0$ for each $r \in \mathbb{N}_0$.

(c) $f \in \mathcal{L}^*$ and the mean value $M(f)$ exists and is nonzero.

(d) $f \in \mathcal{D}^\alpha$ for all $\alpha \geq 1$ and the mean value $M(f)$ is nonzero.

(e) $f \in \mathcal{A}^1$ and the mean-value $M(f)$ is nonzero.

(f) $f \in \mathcal{A}^\alpha$ for all $\alpha \geq 1$ and the mean value $M(f)$ is nonzero.

(g) $f \in \mathcal{L}^\alpha$ *for all* $\alpha \geq 1$ *and the mean value* $M(f)$ *exists and is nonzero.*

Concerning the description of the spectrum $\sigma(f)$ for q-multiplicative functions $f \in \mathcal{D}^1$ or $f \in \mathcal{A}^1 \setminus \mathcal{D}^1$ we establish

Theorem 3. *Let* $f \in \mathcal{D}^1$ *be* q*-multiplicative with non-empty spectrum* $\sigma(f)$*.*

(a) If $M(f) \neq 0$ *then*

$$\sigma(f) = \Big\{ \beta \mid \beta = \frac{c}{b} \, mod\, 1, \quad \frac{c}{b} \in \mathbb{Q}; \; p\, prime, \; p|b \Rightarrow p|q;$$

$$\sum_{a=0}^{q-1} f(aq^r) e_{-\beta}(aq^r) \neq 0 \quad for\ all \quad r \in \mathbb{N}_0 \Big\} .$$

(b) If $M(f) = 0$ *then there exists some* $\beta_0 \in \mathbb{Q}/\mathbb{Z}$ *such that*

$$\sigma(f) = \Big\{ \beta \mid \beta = \beta_0 + \frac{c}{b} \, mod\, 1, \quad \frac{c}{b} \in \mathbb{Q}; \quad p\, prime, \quad p|b \Rightarrow p|q;$$

$$\sum_{a=0}^{q-1} f(aq^r) e_{-\beta}(aq^r) \neq 0 \quad for\ all \quad r \in \mathbb{N}_0 \Big\} .$$

Corollary 1. *Let* $f \in \mathcal{A}^1 \setminus \mathcal{D}^1$ *be* q*-multiplicative with non-empty spectrum* $\sigma(f)$*. Then there exists some* $\beta_0 \in (R \setminus \mathbb{Q})/\mathbb{Z}$ *such that*

$$\sigma(f) = \{ \beta \mid \beta = \beta_0 + \frac{c}{b} \, mod\, 1, \quad \frac{c}{b} \in \mathbb{Q}; \quad p\, prime, \quad p|b \Rightarrow p|q;$$

$$\sum_{a=0}^{q-1} f(aq^r) e_{-\beta}(aq^r) \neq 0 \quad for\ all \quad r \in \mathbb{N}_0 \} .$$

Example. Let $f = e_\beta$ where $\beta \in (R \setminus \mathbb{Q})/\mathbb{Z}$. Then, obviously, the mean value $M(f)$ equals zero and $\sigma(f) = \{\beta\}$.

2. Proof of Theorem 1 and Theorem 2

We use the following well-known result.

Lemma 1. (see [7] Chap. VI.8. Proposition 8.2) *For* $\alpha \geq 1$ *and every arithmetical function* f, $f \in \mathcal{A}^\alpha$ *if and only if* $f \in \mathcal{A}^1$ *and* $|f| \in \mathcal{A}^\alpha$.

Proof of Theorem 2. The implications "(a)⇒(e)⇒(c)" are obvious and "(c)⇒(b)⇒(a)" hold by Proposition 5, (i) and (ii). Using Lemma 1 together with Proposition 5 for $|f|^\alpha$, $\alpha \geq 1$, gives "(c)⇒(d)", whereas the implications "(d)⇒(f)⇒(g)⇒(c)" are again obvious. This proves Theorem 2.

Proof of Theorem 1. If $M(|f|) = 0$ then obviously $\sigma(f) = \emptyset$. Assume that $M(|f|) \neq 0$. Then, by Theorem 2, $|f| \in \mathcal{A}^2$ and $M(|f|^2) \neq 0$, and Lemma 1 implies $f \in \mathcal{A}^2$. By Parseval's equation $M(|f|^2) = \sum\limits_{\beta \in \sigma(f)} |M(f \cdot e_{-\beta})|^2$, and $\sigma(f) = \emptyset$ implies $M(|f|) = M(|f|^2) = 0$. This contradiction proves Theorem 1.

3. Proof of Theorem 3 and Corollary 1

Let $f \in \mathcal{D}^1$ be q-multiplicative and let the mean value $M(f)$ be nonzero. Then the series (6) and (7) both converge for f. Let $\beta \in \sigma(f)$. Then $\beta \in \mathbb{R}/\mathbb{Z}$ and the mean value $M(f \cdot e_{-\beta})$ is nonzero. Putting $g = f \cdot e_{-\beta}$ implies that

$$\sum_{r=0}^{\infty} \sum_{a=0}^{q-1} |g(aq^r) - 1|^2 \tag{8}$$

is convergent. We show that this happens if and only if $\beta = c/b$ is a rational number and each prime divisor of b divides q. We consider three cases.

- **Case 1:** Let β be irrational. The function $e_{-\beta}$ is q-multiplicative and its absolute value is equal to 1. By Delange's result [2] for q-multiplicative functions f of absolute value less or equal to 1, whose mean value $M(f)$ exists, the series

$$\sum_{r=0}^{\infty} \frac{1}{q} \sum_{a=1}^{q-1} |e_{-\beta}(aq^r) - 1|^2 \tag{9}$$

converges if and only if the representation

$$M(e_{-\beta}) = \prod_{r=0}^{\infty} \left(\frac{1}{q} \sum_{a=1}^{q-1} (e_{-\beta}(aq^r)) \right)$$

holds. Since $M(e_\beta) = 0$ and $\frac{1}{q}\sum_{a=1}^{q-1}(e_{-\beta}(aq^r)) \neq 0$ for all $r \in \mathbb{N}_0$ the series (9) diverges.

- **Case 2:** Let $\beta = c/b$ be rational and assume there is a prime p which divides b, but does not divide q. Then for all r the numbers $\frac{c}{b}q^r$ are not integers. This implies

$$\left|e\left(-\frac{c}{b}q^r\right) - 1\right| \geq \left|1 - e\left(-\frac{1}{b}\right)\right|,$$

and the series

$$\sum_{r=0}^{\infty}\frac{1}{q}\sum_{a=1}^{q-1}|e_{-\frac{c}{b}}(aq^r) - 1|^2 \tag{10}$$

diverges.

- **Case 3:** Let $\beta = \dfrac{c}{b}$ be rational, and assume that for each prime divisor of b divides q, too. Then for all $a = 1, 2, \cdots, q-1$ and all $r \geq r_0$, we have $e_{-\beta}(aq^r) = 1$. Now

$$|1 - e_{-\beta}(aq^r)|^2 \ll |1 - g(aq^r)|^2 + |1 - f(aq^r)|^2.$$

Since the series (7) and (8) converge, cases 1 and 2 can not occur. Therefore, the mean value $M(f \cdot e_{-\beta})$ is zero for the cases 1 and 2. In case 3 the series

$$\sum_{r=0}^{\infty}\frac{1}{q}\sum_{a=0}^{q-1}(g(aq^r) - 1)^2$$

and

$$\sum_{r=0}^{\infty}\frac{1}{q}\sum_{a=0}^{q-1}(g(aq^r) - 1)$$

converge. Then

$$M(g) = \prod_{r=0}^{\infty}\frac{1}{q}\sum_{a=0}^{q-1}g(aq^r) \tag{11}$$

and the mean value $M(g)$ is nonzero if and only if each factor of (11) is nonzero. This proves (a).

For the proof of (b) and Corollary 1 let the mean value of f be zero, and let $\beta_0 \in \mathbb{R}/\mathbb{Z}$ such that the mean value of $f \cdot e_{-\beta_0}$ is nonzero . Then $f \cdot e_{-\beta_0} \in \mathcal{D}^1$. Since $f \in \mathcal{A}^1 \setminus \mathcal{D}^1$ if and only if β_0 is irrational, (a) yields (b) and Corollary 1.

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