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HIGH ORDER MEAN–VALUE THEOREMS FOR MULTIPLICATIVE FUNCTIONS VIA HALÁSZ'S METHOD

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Dedicated to Professor Karl-Heinz Indlekofer on his 60-th birthday

1. Introduction

Mean-value theorems for multiplicative functions f on $\mathbb{N}$ via Halász's method are classical in probabilistic number theory [8, 7]. The method was first presented in [8]. These theorems give information about mean-values

$$m_f := \lim_{x\to\infty} x^{-1} \sum_{n\le x} f(n) \tag{1.1}$$

of functions f of which the generating function

$$\hat{F}(s) := \sum_{n=1}^{\infty} f(n) n^{-s}$$

satisfies

$$\hat{F}(s) = \frac{A}{s-1} + o\left(\frac{|s|}{\sigma-1}\right), \quad s = \sigma + it \tag{1.2}$$

uniformly for $\sigma > 1$ and $-\infty < t < \infty$. A shortcoming of these theorems is that they do not convey information about the "higher order" mean-values of functions of which

$$\hat{F}(s) = \frac{A}{(s-1)^{\tau}} + o\left(\frac{|s|}{(\sigma-1)^{\tau}}\right) \tag{1.3}$$

with $\tau > 1$. A variety of multiplicative functions f satisfy condition (1.3), but not (1.2). A trivial example is the divisor function $f(n) = d(n)$, which satisfies the three conditions of Halász's general mean-value theorem and (1.3) with $\tau = 2$. In particular, the functions $\omega(d)$ in the theory of sieves of dimension $\kappa > 1$ are of this variety [9]. For these functions f, instead of the mean-value m_f defined by (1.1), one should consider the "higher order" mean-values

$$m_f := \lim_{x\to\infty} \frac{1}{x(\log x)^{\tau-1}} \sum_{n\le x} f(n), \tag{1.4}$$

which we call, when it exists, an order τ mean-value of the function f. Notice that, in classical theory, only order one ($\tau = 1$) mean-value has been discussed.

2. A general mean-value theorem

This lacuna is removed principally in [12]. Actually, a general mean-value theorem for multiplicative functions f defined on a set $\mathcal{N}$ of generalized integers n_j, associated with a set $\mathcal{P}$ of generalized primes p_j (henceforth, g-integers, g-primes, etc.) in Beurling's sense [1, 2], is proved as follows.

Let $f(n_j)$ be a complex-valued function on $\mathcal{N}$ and

$$F(x) := \sum_{n_j\le x} f(n_j).$$

Also, let $\Lambda(n_j)$ be the analog of the classical von Mongoldt function and

$$\psi(x) := \sum_{n_j\le x} \Lambda(n_j)$$

be the Chebyshev function associated with $\mathcal{P}$.

Theorem 1. (i) *Suppose there exist a constant c, real constants α and $\tau > 0$, and a measurable slowly oscillating function $L(u)$ with $|L(u)| = 1$ such that*

$$F(x) = \frac{cx^{1+i\alpha}(\log x)^{\tau-1}}{\Gamma(\tau)(1+i\alpha)} L(\log x) + o\left(x \log^{\tau-1} x\right), \tag{2.1}$$

where $\Gamma(x)$ *is Euler's gamma function. Then the asymptotic formula*

$$\hat{F}(s) = \frac{c}{(s-1-i\alpha)^{\tau}} L\left(\frac{1}{\sigma-1}\right) + o\left(\frac{|s|}{(\sigma-1)^{\tau}}\right) \tag{2.2}$$

holds as $\sigma \to 1+$ *uniformly for* $-\infty < t < \infty$.

(ii) *Conversely, suppose*

(1) There exist positive constants δ *and* K_δ *such that*

$$\limsup_{\sigma \to 1+} (\sigma-1) \sum_{p_j} \max\left\{1, |f(p_j)|^{1+\delta}\right\} p_j^{-\sigma} \log p_j = K_\delta;$$

(2)

$$\sum_{p_j} \sum_{k \geq 2} |f(p_j^k)| p_j^{-k} < \infty;$$

and

(3)

$$1 + \sum_{k=1}^{\infty} f(p_j^k) p_j^{-k(1+it)} \neq 0$$

for every $p_j \in \mathcal{P}$ *and* $-\infty < t < \infty$.

Furthermore, suppose that the counting function $N(x)$ *of g-integers satisfies*

$$N(x) = x \sum_{r=1}^{m} A_r (\log x)^{\rho_r - 1} + O\left(x(\log x)^{-\gamma}\right) \tag{2.3}$$

with real constants $\rho_1 < \rho_2 < \ldots < \rho_m$ *and* $A_1, A_2, \ldots, A_m$ *such that* $\rho_m = \rho \geq$ $\geq 1,\ \rho_r \neq 0,\ A_m = A > 0$ *and real constants* $\gamma > \gamma_0$. *Also, suppose that*

$$\psi(x) = \left(\rho - 2\sum_{j=1}^{l} \alpha_j \cos(t_j \log x)\right) x + O\left(x(\log x)^{-M}\right) \tag{2.4}$$

holds with positive integers α_j, *real constants* t_j, $j = 1, \ldots, l$, *and constants* $M > M_0$. *Here constants* γ_0 *and* M_0 *depend on* ρ, δ, K_δ *and* τ *only. Then (2.2) with* $\tau \geq 1$ *entails (2.1).*

This theorem has the following corollary, which is a direct extension of Halász's general mean-value theorem [8].

Corollary 1. *Suppose that*

$$N(x) = Ax + O_k\left(x(\log x)^{-k}\right) \tag{2.5}$$

for every $k \in \mathbb{N}$. If f satisfies conditions (1), (2) and (3) in Theorem 1 then (2.2) with $\tau \geq 1$ entails (2.1).

3. High order mean-value theorems

On the basis of Theorem 1, one can characterize further the asymptotic behavior of the order τ mean-values of multiplicative functions with $\tau \geq 1$ [13]. These theorems assume conditions less restrictive than those of their counterparts in classical theory [10, 7] in some sense. In the following context, conditions (1), (2) and (3) of Theorem 1 are quoted as conditions (1), (2) and (3) without indicating Theorem 1 repeatedly. Also, p is used to denote the general g-prime p_j for convenience. This will not cause any confusion.

We first consider mean-value $m_f = 0$. Let

$$\log^+ |x| := \max\{0, \log |x|\}.$$

Theorem 2. *Suppose that (2.3) and (2.4) are satisfied and that f satisfies conditions (1) and (2).*

If

$$\sum_p p^{-1}\left(\frac{\tau}{\rho} - \Re\left(f(p)p^{-it}\right)\right) + \log^+ |t| \tag{3.1}$$

diverges to $+\infty$ uniformly for $-\infty < t < \infty$ then the order τ mean-value $m_f = 0$.

Conversely, suppose further that there exist a subset $\mathcal{P}_0$ of the set $\mathcal{P}$ of all g-primes and a constant $K > 0$ such that

$$\sum_{\substack{p \in \mathcal{P}_0 \\ p \leq x}} p^{-1}\left(\frac{\tau}{\rho} - \Re\left(f(p)p^{-it}\right)\right) \geq -K$$

uniformly for $0 \leq x < \infty$ and $-\infty < t < \infty$ and such that

$$\sum_{\substack{p \notin \mathcal{P}_0 \\ p \leq x}} |f(p)| \ll \frac{x}{\log x}.$$

If the order τ mean-value $m_f = 0$ then either there exist a real number t_0 and a g-prime p_0 such that

$$1 + \sum_{k=1}^{\infty} f(p_0^k) p_0^{-k(1+it_0)} = 0$$

or (3.1) diverges to $+\infty$ uniformly for $-\infty < t < \infty$.

Theorem 2 is a direct extension of the classical Halász-Wirsing theorem [6, 11].

The theorems on nonzero mean-values are more complicated. For simplicity, only the results under the condition (2.5) are presented here as follows.

Theorem 3. *Suppose that (2.5) is satisfied. Let f be a multiplicative function satisfying condition (2) and*

$$1 + \sum_{k=1}^{\infty} f(p^k) p^{-k} \neq 0 \tag{3.2}$$

for all $p \in \mathcal{P}$. Suppose further that there exist a positive constant η such that both series

$$\sum_{|f(p)| \le \frac{\tau}{\rho} + \eta} p^{-1} \left| f(p) - \frac{\tau}{\rho} \right|^2 \tag{3.3}$$

and series

$$\sum_{|f(p)| > \frac{\tau}{\rho} + \eta} |f(p)| p^{-1} \tag{3.4}$$

converge. Then the order τ mean-value m_f exists for $\tau \ge 1$ and

$$m_f = \frac{(A)^\tau}{\Gamma(\tau)} \prod_p \left(1 - p^{-1}\right)^\tau \left(1 + \sum_{k=1}^{\infty} f(p^k) p^{-k}\right) \neq 0 \tag{3.5}$$

if and only if

$$\sum_p p^{-1} \left(\frac{\tau}{\rho} - f(p)\right) \tag{3.6}$$

converges.

Theorem 3 is a direct extension of the classical Delange-Halász theorem [4, 6, 8].

As an example, let $f(n) = (d(n))^2$, where $d(n)$ is the divisor function of $n \in \mathbb{N}$. The condition (2) and (3.2) are satisfied. Also, (3.6), (3.3) and (3.4) are satisfied with $\rho = 1,\ \tau = 4$ and $\eta = 1$ (the Riemann zeta function has order $\rho = 1$ and $A = 1$). Therefore the order 4 mean-value

$$m_f = \frac{(\Gamma(1))^2}{\Gamma(4)} \prod_p (1 - p^{-1})^4 \left(1 + \sum_{k=1}^{\infty} (k+1)^2 p^{-k}\right) = \\ = \frac{1}{6} \prod_p (1 - p^{-2}) = \frac{1}{\pi^2},$$

i.e.

$$\lim_{x\to\infty} \frac{1}{x(\log x)^3} \sum_{n\le x} (d(n))^2 = \frac{1}{\pi^2}.$$

This is well-known is elementary number theory.

4. A generalization of theorems of Elliott and Daboussi

The well-known theorems of Elliott [5] and Daboussi [3] can be extended to functions having high order mean-values [14].

Theorem 4. *Suppose that (2.5) is satisfied. Let f be a multiplicative function. Then the order τ mean-value m_f exists and is nonzero for $\tau \ge 1$ and the limit*

$$\lim_{x\to\infty} \frac{1}{x(\log x)^{\tau^\alpha - 1}} \sum_{n_j \le x} |f(n_j)|^\alpha$$

exists with some constant $\alpha > 1$ if and only if the series

$$\sum_p p^{-1}(\tau - f(p)), \qquad \sum_{|f(p)|\le \tau+\eta} p^{-1}|\tau - f(p)|^2,$$

$$\sum_{|f(p)|>\tau+\eta} p^{-1}|f(p)|^\alpha, \quad \sum_p \sum_{k\ge 2} p^{-k}|f(p^k)|^\alpha$$

converge with some constant $\eta > 0$ *and*

$$\sum_{k=0}^{\infty} p^{-k} f(p^k) \neq 0$$

for all $p \in \mathcal{P}$.

In case $\tau = 1$, Theorem 4 is the well-known theorem of Elliott and Daboussi.

The dual of the Turán-Kubilius inequality plays a key role in proofs of the necessity part of the Elliott-Daboussi theorem [7]. However, in case of higher order ($\tau > 1$) mean-values, the dual inequality fails. This is shown by the trivial example $f(n) = d^2(n)$. Hence the proof of the necessity part of Theorem 4 is based on an intrinsic connection between the order τ mean-value m_f and the order $\tau^\alpha - 1$ mean-value of $|f(n_j)|^\alpha$.

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