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PERMUTATIONS AVOIDING CONSECUTIVE PATTERNS

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Abstract. Given some permutation $\sigma \in S_m \quad (m \geq 3)$ denote by $B(n, m; \sigma)$ the set of all permutations $\pi \in S_n$ with the property that

$$\pi(k + \sigma(1)) < \ldots < \pi(k + \sigma(m)) \text{ is false for } 0 \leq k \leq n - m.$$

We conjecture that to any $\sigma \in S_m$ there are constants $c(\sigma) > 0, \ \xi(\sigma) > 1$ such that

$$|B(n, m; \sigma)|/n! \sim c(\sigma)\xi(\sigma)^{-n} \quad \text{as} \quad n \to \infty.$$

We prove this for $m = 3$ and for the particular $\sigma \in S_m$ being the identity: $\sigma(j) = j \quad (1 \leq j \leq m)$.

1. Introduction

Let S_n denote the set of all permutations π of $1, \ldots, n$. Let $m \geq 3$ and some permutation $\sigma \in S_m$ be fixed, ϵ the identity ($\epsilon(j) = j$ for $1 \leq j \leq m$).

1. Let $A(n, m; \sigma)$ denote the set of all $\pi \in S_n$ with the property that

$$\pi\left(j_{\sigma(1)}\right) < \ldots < \pi\left(j_{\sigma(m)}\right) \quad \text{is false for all } 1 \leq j_1 < \ldots < j_m \leq n.$$

In particular $A(n, m; \epsilon)$ is the set of all those $\pi \in S_n$ with the property that

$$\pi(j_1) < \ldots < \pi(j_m) \text{ is false for all } 1 \leq j_1 < \ldots < j_m \leq n.$$

The Stanley-Wilf conjecture states that to any $m \geq 3$ there is some constant $K(m) > 1$ such that

$$|A(n,m;\sigma)| \leq K(m)^n \quad \text{for all} \quad \sigma \in S_m.$$

The following facts are known:

$$(*) \qquad \begin{aligned} |A(n,3;\sigma)| &= \frac{1}{n+1}\binom{2n}{n} \quad \text{for all} \quad \sigma \in S_3, \\ |A(n,m;\epsilon)| &\leq (m-1)^{2n}, \\ |A(n,m;\epsilon)| &\sim c(m)\frac{(m-1)^{2n}}{n^{m(m-2)/2}} \quad \text{as } n \to \infty. \end{aligned}$$

$|A(n,m;\sigma)|$ varies considerably with σ already for $m = 4$.

References for this can be found in Richard Arratia's paper "On the Stanley-Wilf conjecture for the number of permutations avoiding a given pattern", *Electronic J. Combin.* 6 (1999), #N1. The right guess for $K(m)$ should be $(m-1)^2$.

We mention that we could sharpen $(*)$:

$$|A(n,m;\epsilon)| \leq \frac{(m-1)^{2n}}{\prod_{k=1}^{m-2}\left(1+\frac{1}{k}\right)^k}.$$

2. Let $B(n,m;\sigma)$ denote the set of all $\pi \in S_n$ with the property that

$$\pi(k+\sigma(1)) < \ldots < \pi(k+\sigma(m)) \quad \text{is false for} \quad 0 \leq k \leq n-m.$$

In particular $B(n,m;\epsilon)$ is the set of all $\pi \in S_n$ with the property that

$$\pi(k) < \ldots < \pi(k+m-1) \quad \text{is false for} \quad 1 \leq k \leq n-m+1.$$

Since the conditions imposed on π there concern only m consecutive values of the argument of π they are less restrictive than those met before; hence $A(n,m;\sigma) \subset B(n,m;\sigma)$. Our results for B are at this stage as incomplete as those for A. Put

$$\begin{aligned} b(n,m;\sigma) &:= \frac{|B(n,m;\sigma)|}{n!}, \quad b(0,m;\sigma) := 1, \\ b(n,m) &:= b(n,m;\epsilon), \quad b(n) := b(n,3), \\ \tilde{b}(n) &:= b(n,3;\sigma_1), \quad \text{where } \sigma_1 := \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}. \end{aligned}$$

We obtained:

(1) $b(n,m;\sigma) \le \left(1-\frac{1}{m!}\right)^{[\frac{n}{m}]}$ for all $\sigma \in S_m$;

(2) $\lim\limits_{n\to\infty} b(n,m;\sigma)^{\frac{1}{n}}$ exists and is equal to $\inf\limits_{n} b(n,m;\sigma)^{\frac{1}{n}}$ for all $\sigma \in S_m$;

(3) $\sum\limits_{n=0}^{\infty} b(n,m)z^n = P_m(z)^{-1}$, where

$$P_m(z) := \sum_{\substack{k=0\\ k\equiv 0(m)}}^{\infty} \frac{z^k}{k!} - \sum_{\substack{k=0\\ k\equiv 1(m)}}^{\infty} \frac{z^k}{k!};$$

(4) $b(n,m) \sim c(m)\xi(m)^{-n}$ as $n \to \infty$, where

$$c(m) > 0 \quad\text{and}\quad \xi(m) = 1 + \frac{1}{m!}(1+o(1)) \quad\text{as}\quad m \to \infty;$$

in particular

(5) $b(n) \sim \exp\left(\frac{\pi}{\sqrt{3}}\right)\left(\frac{3\sqrt{3}}{2\pi}\right)^{n+1}$ as $n \to \infty$;

(6) $\sum\limits_{n=0}^{\infty} \tilde{b}(n)z^n = \tilde{P}(z)^{-1}$, where

$$\tilde{P}(z) := 1 - \int_0^z \exp\left(-\frac{1}{2}w^2\right) dw;$$

(7) $\tilde{b}(n) \sim \dfrac{\exp\left(\frac{1}{2}\xi^2\right)}{\xi^{n+1}}$ as $n \to \infty$, where ξ is given by

$$\int_0^{\xi} \exp\left(-\frac{1}{2}t^2\right) dt = 1 \quad (1 < \xi < \infty);$$

(8) $b(n,3;\sigma) = b(n)$ for $\sigma = \epsilon,\ \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$,

$b(n,3;\sigma) = \tilde{b}(n)$ for the 4 remaining $\sigma \in S_3$.

We conjecture that to any $\sigma \in S_m$ $(m \ge 3)$ there are constants $c(\sigma) > 0$, $\xi(\sigma) > 1$ such that

$$b(n,m;\sigma) \sim c(\sigma)\xi(\sigma)^{-n} \quad\text{as}\quad n \to \infty.$$

In the last section of our paper we report on $m = 4$, where we could achieve only partial results.

3. **Acknowledgement.** The author thanks Arnold Knopfmacher and Helmut Prodinger for bringing to his attention the PhD problem of their student Albert Tshifhurmulo on permutations avoiding consecutive patterns.

Tshifhumulo in his thesis will study these problems mainly using an approach based on geometrically distributed random variables. (3) of our list of results will appear in the thesis. Here I will study Tshifhumulo's problem from a different (elementary) perspective based on a direct treatment of permutations.

Finally I am obliged to Arnold Knopfmacher for localizing and bringing to my attention Arratia's paper.

2. The proofs of (1.1), (1.2)

Since $B(m, m; \sigma) = S_m - \{\sigma^{-1}\}$, we get

$$b(m, m; \sigma) = 1 - \frac{1}{m!}. \tag{1}$$

Put $B(k) := |B(k, m; \sigma)|$ and $b(k) := \dfrac{B(k)}{k!}$, $b(0) = 1$. Then we have

$$b(n) \leq b(p)b(n-p) \quad \text{for} \quad 0 \leq p \leq n. \tag{2}$$

Proof. This is true for $p = 0, n$. Assume $0 < p < n$. From

$$B(n, m; \sigma) = \bigcup_{\substack{P \subset [1,n] \\ |P| = p}} \{\pi \in B(n, m; \sigma) \mid \pi([1, p]) = P\}$$

and

$$\# \{\pi \in B(n, m; \sigma) \mid \pi([1, p]) = P\} \leq B(p)B(n-p)$$

we obtain

$$B(n) \leq \binom{n}{p} B(p)B(n-p)$$

which gives (2).

From (2), (1) we infer

$$b(n) \le b(m)^{[\frac{n}{m}]} b\left(n - \left[\frac{n}{m}\right] m\right) \le b(m)^{[\frac{n}{m}]} = \left(1 - \frac{1}{m!}\right)^{[\frac{n}{m}]}$$

which is (1.1).

From (2) we get

$$b(p+q) \le b(p)b(q) \quad \text{for all} \quad p, q \ge 0,$$

which yields (1.2).

3. A formula for $b(n, m)$ and the proof of (1.3)

We shall need two formulae whose proofs are left to the reader.

(1)
$$\sum_{\substack{\delta_s \ge 0 \ (1 \le s \le t) \\ \sum_{s=1}^{t} \delta_s \le N}} \left(N - \sum_{s=1}^{t} \delta_s\right) = \binom{t+N}{t+1}.$$

Put

$$S(h, q, b) := \sum_{\substack{0 \le \delta_i \le q \ (1 \le i \le h) \\ \sum_{i=1}^{h} \delta_i = b}} 1.$$

Put $m = q + 2$. Then

(2)
$$f_m(a) := \sum_{h=0}^{a} (-1)^{h+1} S(h, q, a-h) = \begin{cases} -1 & \text{if } a \equiv 0(m), \\ 1 & \text{if } a \equiv 1(m), \\ 0 & \text{else.} \end{cases}$$

Put $p = m - 1$. The sieve formula yields

$$b(n,m) = 1 + \sum_{j=1}^{n-p} (-1)^j \sum_{1 \le k_1 < \ldots < k_j \le n-p} \frac{D(k_1, \ldots, k_j)}{n!},$$

where

$$D(k_1, \ldots, k_j) = \# \{\pi \in S_n \mid \pi(k_i) < \ldots < \pi(k_i + p) \text{ for } 1 \le i \le j\}.$$

We find

$$D(k_1, \ldots, k_j) = \frac{n!}{M(k_1, \ldots, k_j)},$$

where $M(k_1, \ldots, k_j)$ is defined in the following way:

$j = 1:$ $M(k) = m!$

$j \ge 2$: Put

$$L(k_1, \ldots, k_j) := \{i \mid 1 \le i \le j, \; k_{i+1} - k_i \ge m\}.$$

If $L(k_1, \ldots, k_j) = \emptyset$ then

$$M(k_1, \ldots, k_j) = (k_j - k_1 + m)!$$

If $L(k_1, \ldots, k_j) \ne \emptyset$ write

$$L(k_1, \ldots, k_j) = \{\ell_1, \ldots, \ell_t\} \text{ with } 1 \le \ell_1 < \ldots < \ell_t < j.$$

Furthermore, put $\ell_0 = 0, \; \ell_{t+1} = j$. Then

$$M(k_1, \ldots, k_j) = \prod_{s=1}^{t+1} \left(k_{\ell_s} - k_{\ell_{s-1}+1} + m\right)!$$

This, too, should be verified by the reader.

We have

$$mt \le \sum_{i=1}^{j-1} (k_{i+1} - k_i) = k_j - k_1 \le n - m.$$

Therefore $0 \le t \le T = \left[\frac{n}{m}\right] - 1$. We get

$$b(n,m) = 1 + \sum_{t=0}^{T} X(t),$$

where

$$X(0) = -\frac{n-p}{m!} + \sum_{j=2}^{n-p}(-1)^j \sum_{\substack{1 \le k_1 < \ldots < k_j \le n-p \\ L(k_1,\ldots,k_j)=\emptyset}} M(k_1,\ldots,k_j)^{-1}$$

and $(t \geq 1)$

$$X(t) = \sum_{j=t+1}^{n-p} (-1)^j \sum_{\substack{1 \le k_1 < \ldots < k_j \le n-p \\ |L(k_1,\ldots k_j)|=t}} M(k_1,\ldots,k_j)^{-1}.$$

We have to evaluate $(j \geq 2,\ t \geq 0)$ the inner sums

$$X_j(t) := \sum_{\substack{1 \le k_1 < \ldots < k_j \le n-p \\ |L(k_1,\ldots,k_j)|=t}} M(k_1,\ldots,k_j)^{-1}.$$

We begin with $t = 0$.

$$X_j(0) = \sum_{\substack{1 \le k_1 < \ldots < k_j \le n-p \\ k_{i+1}-k_i \le p \ \ (1 \le i < j)}} \frac{1}{(k_j - k_1 + m)!} =$$

$$= \sum_{j-1 \le d \le n-m} \frac{1}{(d+m)!} \sum_{\substack{1 \le k_1 < \ldots, k_j \le n-p \\ k_{i+1}-k_i \le p \ \ (1 \le i < j) \\ k_j - k_1 = d}} 1,$$

$$X_j(0) = \sum_{j-1 \le d \le n-m} \frac{n-p-d}{(d+m)!} S(j-1, q, d-(j-1)).$$

Now let $t \geq 1$.

$$X_j(t) = \sum_{1 \le \ell_1 < \ldots < \ell_t < j} \ \sum_{\substack{1 \le k_1 < \ldots < k_j \le n-p \\ L(k_1,\ldots,k_j)=\{\ell_1,\ldots,\ell_t\}}} M(k_1,\ldots,k_j)^{-1}.$$

Denote the inner sum by $S_j(\ell_1, \ldots \ell_t)$ and write $L(t) := \{\ell_1, \ldots \ell_t\}$. Then

$$S_j(\ell_1,\ldots,\ell_t) = \sum_{\substack{1 \le k_1 < \ldots < k_j \le n-p \\ L(k_1,\ldots,k_j)=L(t)}} \left(\sum_{s=1}^{t+1} (k_{\ell_s} - k_{\ell_{s-1}+1} + m)! \right)^{-1}.$$

We introduce the differences $d_i = k_{i+1} - k_i$ $(1 \le i \le j)$. They satisfy

$$d_i \le p \quad (i \notin L(t)) \quad \text{and} \quad d_{\ell_s} \ge m \quad (1 \le s \le t).$$

We have

$$k_{\ell_s} - k_{\ell_{s-1}+1} = \sum_{i=\ell_{s-1}+1}^{\ell_s - 1} d_i$$

and

$$\sum_{\substack{i=1 \\ i\notin L(t)}}^{j-1} d_i = k_j - k_1 - \sum_{\ell \in L(t)} d_\ell \le n - m(t+1).$$

Therefore

$$S_j(\ell_1, \ldots, \ell_t) = \sum_{\substack{1 \le d_i \le p \ (i \notin L(t)) \\ D := \sum_{i \notin L(t)} d_i \le n - m(t+1)}} \left(\prod_{s=1}^{t+1} \left(\sum_{i=\ell_{s-1}+1}^{\ell_s - 1} d_i + m \right)! \right)^{-1} \cdot X,$$

where

$$X = \sum_{\substack{d_{\ell_s} \ge m \ (1 \le s \le t) \\ \sum_{s=1}^{t} d_{\ell_s} \le n - m - D}} \sum_{\substack{1 \le k_1 < \ldots < k_j \le n-p \\ k_{i+1} - k_i = d_i \ (1 \le i < j)}} 1 =$$

$$= \sum_{\substack{d_{\ell_s} \ge m \ (1 \le s \le t) \\ \sum_{s=1}^{t} d_{l_s} \le n - m - D}} \left(n - p - D - \sum_{s=1}^{t} d_{\ell_s} \right) =$$

$$= \binom{n - p(t+1) - D}{t+1} \quad \text{by} \quad (1).$$

Therefore

$$S_j(\ell_1, \ldots, \ell_t) = \sum_{\substack{1 \le d_i \le p (i \notin L(t)) \\ \sum_{i \notin L(t)} d_i \le n - m(t+1)}} \binom{n - p(t+1) - \sum_{i \notin L(t)} d_i}{t+1} \times$$

$$\times \left(\prod_{s=1}^{t+1} \left(\sum_{i=\ell_{s-1}+1}^{\ell_s - 1} d_i + m \right)! \right)^{-1} =$$

$$= \sum_{\substack{a_1 + \ldots + a_{t+1} \le n - m(t+1) \\ a_s \ge \ell_s - \ell_{s-1} - 1 \ (1 \le s \le t+1)}} \binom{n - p(t+1) - \sum_{s=1}^{t+1} a_s}{t+1} \left(\prod_{s=1}^{t+1} (a_s + m)! \right)! \cdot Y,$$

where

$$Y = \sum_{\substack{1 \le d_i \le p \ \ (i \notin L(t)) \\ \sum_{i=\ell_{s-1}+1}^{\ell_s - 1} d_i = a_s \ \ (1 \le s \le t+1)}} 1 = \prod_{s=1}^{t+1} \sum_{\substack{1 \le d_i \le p \ \ (\ell_{s-1} < i < \ell_s) \\ \sum_{i=\ell_{s-1}+1}^{\ell_s - 1} d_i = a_s}} 1 =$$

$$= \prod_{s=1}^{t+1} S(\ell_s - \ell_{s-1} - 1,\ q,\ a_s - (\ell_s - \ell_{s-1} - 1)).$$

Therefore we obtain

$$S_j(\ell_1, \ldots, \ell_t) =$$

$$= \sum_{\substack{a_1 + \ldots + a_{t+1} \le n - m(t+1) \\ a_s \ge \ell_s \ell_{s-1} - 1 \ \ (1 \le s \le t+1)}} \binom{n - p(t+1) - \sum_{s=1}^{t+1} a_s}{t+1} \prod_{s=1}^{t+1} \frac{\sigma\left(a_s, \ell_s - \ell_{s-1}\right)}{(a_s + m)!},$$

where

$$\sigma(a_s, \ell_s - \ell_{s-1}) := S(\ell_s - \ell_{s-1} - 1,\ q,\ a_s - (\ell_s - \ell_{s-1} - 1)).$$

Now we have

$$X(t) = \sum_{j=t+1}^{n-p} (-1)^j X_j(t) = \sum_{j=t+1}^{n-p} (-1)^j \sum_{1 \le \ell_1 < \ldots < \ell_t < j} S_j(\ell_1, \ldots \ell_t) =$$

$$= \sum_{\substack{a_1 + \ldots + a_{t+1} \le n - m(t+1) \\ a_s \ge 0 \ \ (1 \le s \le t+1)}} \binom{n - p(t+1) - \sum_{s=1}^{t} a_s}{t+1} \cdot \prod_{s=1}^{t+1} \frac{1}{(a_s + m)!} \cdot F(a_1, \ldots, a_{t+1}),$$

where

$$F(a_1, \ldots, a_{t+1}) = \sum_{j=t+1}^{n-p} (-1)^j \sum_{\substack{1 \le \ell_1 < \ldots < \ell_t < j \\ \ell_s - \ell_{s-1} \le a_s + 1 \ \ (1 \le s \le t+1)}} \prod_{s=1}^{t+1} \sigma(a_s, \ell_s - \ell_{s-1}).$$

Since

$$j = \ell_{t+1} - \ell_0 = \sum_{s=1}^{t+1} (\ell_s - \ell_{s-1}),$$

we get

$$F(a_1,\dots,a_{t+1}) = \sum_{j=t+1}^{n-p} \sum_{\substack{1\le \ell_1<\dots<\ell_t<j \\ \ell_s-\ell_{s-1}\le a_s+1\ (1\le s\le t+1)}} \prod_{s=1}^{t+1} (-1)^{\ell_s-\ell_{s-1}} \sigma(a_s,\ell_s-\ell_{s-1}) =$$

$$= \prod_{s=1}^{t+1} \sum_{h=0}^{a_s} (-1)^{h+1} S(h,q,a_s-h) = \prod_{s=1}^{t+1} f_m(a_s) \quad \text{by (2).}$$

We summarize. Writing $t+1=k$ we arrive at the formula

$$b(n,m) = 1 + \sum_{k=1}^{\left[\frac{n}{m}\right]} S_k(n,m),$$

where

$$S_k(n,m) = \sum_{\substack{a_j\ge 0\ (1\le j\le k) \\ \sum_{j=1}^{k} a_j \le n-mk}} \binom{n-(m-1)k-\sum_{j=1}^{k} a_j}{k} \prod_{j=1}^{k} \frac{f_m(a_j)}{(a_j+m)!}$$

with

$$f_m(a) = \begin{cases} -1 & \text{if } a\equiv 0\ (m), \\ 1 & \text{if } a\equiv 1\ (m), \\ 0 & \text{else.} \end{cases}$$

Proof of (1.3). Put $S_0(n,m)=1$ and introduce

$$F_m(z) := \sum_{a=0}^{\infty} \frac{f_m(a)}{(a+m)!} z^a.$$

Then we have

$$\sum_{n=0}^{\infty} b(n,m) z^n = \sum_{n=0}^{\infty} \left(\sum_{0\le k\le \frac{n}{m}} S_k(n,m) \right) z^n =$$

$$= \sum_{k=0}^{\infty} \sum_{n\ge mk} S_k(n,m) z^n =$$

$$= \sum_{k=0}^{\infty} z^{mk} \sum_{\ell=0}^{\infty} S_k(\ell+mk,m) z^{\ell}.$$

The inner sum is equal to

$$\sum_{\ell=0}^{\infty}\left(\sum_{\substack{a_j\geq 0\ (1\leq j\leq k)\\ \sum_{j=1}^{k} a_j\leq \ell}}\binom{\ell+k-\sum_{j=1}^{k}a_j}{k}\prod_{j=1}^{k}\frac{f_m(a_j)}{(a_j+m!)}\right)z^{\ell}=(1-z)^{-(k+1)}F_m(z)^k.$$

Therefore

$$\sum_{n=0}^{\infty} b(n,m)z^n=\frac{1}{1-z}\sum_{k=0}^{\infty}\left(\frac{z^m F_m(z)}{1-z}\right)^k=\frac{1}{1-z-z^m F_m(z)}=P_m(z)^{-1}.$$

4. The proof of (1.5)

We have

$$\sum_{n=0}^{\infty} b(n)z^n=P_3(z)^{-1}\quad\text{with}\quad P_3(z)=\sum_{\substack{k=0\\ k\equiv 0\ (3)}}^{\infty}\frac{z^k}{k!}-\sum_{\substack{k=0\\ k\equiv 1\ (3)}}^{\infty}\frac{z^k}{k!}.$$

Putting $\xi:=\exp\left(\frac{2\pi i}{3}\right)$ we get

$$P_3(z)=\frac{1}{3}(\xi-1)\exp(\xi z)\left(\xi^{-1}-\exp(\xi(\xi-1)z)\right).$$

Therefore the zeros of $P_3(z)$ are

$$w_\ell=\frac{2\pi}{\sqrt{3}}\left(\frac{1}{3}-\ell\right)\quad(\ell=0,\pm1,\ldots).$$

We find

$$P_3'(w_\ell)=(-1)^{\ell+1}\exp\left(\frac{\pi}{\sqrt{3}}(3\ell-1)\right).$$

The residue theorem yields

$$b(n) \sim \sum_{\ell=-\infty}^{\infty} \frac{1}{-P_3'(w_\ell)} \frac{1}{w_\ell^{n+1}}.$$

In particular we get (1.5).

5. The proof of (1.4)

Since $m = 3$ has been settled we may assume $m \geq 4$. We show first that $P_m(z)$ has a zero on the open interval

$$I(m) = \left] 1 + \frac{1}{m!}(1 - \alpha(m)),\ 1 + \frac{1}{m!}(1 + \beta(m)) \right[,$$

where

$$\alpha(m) = \frac{m! + 1}{(m+1)! + 1}, \quad \beta(m) = \frac{2(m+1)}{m! - 2(m+1)}.$$

We have

$$P_m(x) = 1 - x + \frac{x^m}{m!} \left(1 - \frac{x}{m+1}\right) + S_m(x),$$

$$S_m(x) = \sum_{\ell=2}^{\infty} \frac{x^{m\ell}}{(m\ell)!} \left(1 - \frac{x}{m\ell + 1}\right).$$

Let $1 \leq x \leq 2$. Then $S_m(x) > 0$ and

$$S_m(x) < \sum_{\ell=2}^{\infty} \left(\frac{x^m}{m!}\right)^\ell = \left(\frac{x^m}{m!}\right)^2 \frac{1}{1 - \frac{x^m}{m!}}.$$

From this we see that for $1 \leq x \leq 2$ we have

(1) $$P_m(x) > 1 - x + \frac{1}{m!} \left(1 - \frac{x}{m+1}\right),$$

(2) $$P_m < \frac{1}{1 - \frac{x^m}{m!}} - x.$$

In (1) take $x = 1 + \frac{1}{m!}(1+\delta),\ 0 < \delta < 1$. Then

$$P_m(x) > \frac{1}{m!}\left(\delta\left(1 + \frac{1}{(m+1)!}\right) - \frac{1}{m+1}\left(1 + \frac{1}{m!}\right)\right) = 0 \ \text{ for } \ \delta = \alpha(m).$$

In (2) take $x = 1 + \dfrac{1}{m!}(1+\delta),\ 0 < \delta < 1$, and put $\epsilon = \dfrac{1+\delta}{m!}$. We get

$$P_m(x) < \frac{y}{m! - x^m},$$

where

$$\begin{aligned} y &= (1+\epsilon)^{m+1} - (1+\delta) \le \\ &\le \exp(\epsilon(m+1)) - (1+\delta) \le \\ &\le (1 + 2\epsilon(m+1)) - (1+\delta) = \\ &= \frac{2}{m!}(m+1) - \delta\left(1 - \frac{2}{m!}(m+1)\right) = \\ &= 0 \ \text{ for } \ \delta = \beta(m). \end{aligned}$$

Next we show that $P_m(z)$ has on the disk $|z| \le 1 + \dfrac{1}{m}$ just one zero and that this zero is simple.

This will follow from Rouché's theorem if

$$|P_m(z) - (1-z)| < |1-z| \quad \text{for} \quad |z| = 1 + \frac{1}{m}.$$

Since

$$\begin{aligned} |P_m(z) - (1-z)| &= \left|\sum_{\ell=1}^{\infty} \frac{z^{m\ell}}{(m\ell)!}\left(1 - \frac{z}{m\ell+1}\right)\right| \le \\ &\le \left(1 + \frac{|z|}{m+1}\right)\sum_{\ell=1}^{\infty}\left(\frac{|z|}{m!}\right)^{\ell} \end{aligned}$$

and since $|1-z| \ge |z| - 1$, it is enough to show that

$$\left(1 + \frac{|z|}{m+1}\right)\sum_{\ell=1}^{\infty}\left(\frac{|z|^m}{m!}\right)^{\ell} < |z| - 1 \quad \text{for} \quad |z| = 1 + \frac{1}{m}.$$

Since $\left(1+\frac{1}{m}\right)^m < 3$ the left side is

$$< \left(1+\frac{1}{m}\right)\frac{3}{m!}\sum_{\ell=0}^{\infty}\left(\frac{3}{4!}\right)^{\ell} \le \frac{5}{7}\frac{1}{m} < \frac{1}{m}.$$

Denote the unique zero of $P_m(z)$ on the disk $|z| \le 1+\frac{1}{m}$ by $\xi(m)$. Since it is simple we have $P'_m(\xi(m)) \ne 0$. Since

$$1+\frac{1}{m!}(1+\beta(m)) < 1+\frac{1}{m} \quad (m \ge 4)$$

we have $\xi(m) \in I(m)$.

The residue theorem yields

$$b(n,m) = \frac{c(m)}{\xi(m)^n} + O\left(\left(\frac{m}{m+1}\right)^n\right) \quad \text{as} \quad n \to \infty,$$

where

$$c(m) = -\left(\xi(m)P'_m(\xi(m))\right)^{-1}.$$

6. An invariance property of $b(n,m;\sigma)$

Let $\tau \in S_n$ be given by $\tau(k) = n+1-k$ $(1 \le k \le n)$. Let $\rho \in S_m$ be given by $\rho(\ell) = m+1-\ell$ $(1 \le \ell \le m)$. One finds

$$\pi \in \mathcal{B}(n,m;\sigma) \Longleftrightarrow \pi\circ\tau \in \mathcal{B}(n,m;\rho\circ\sigma),$$
$$\pi \in \mathcal{B}(n,m;\sigma) \Longleftrightarrow \tau\circ\pi \in \mathcal{B}(n,m;\sigma\circ\rho).$$

From this we deduce

$$b(n,m;\sigma) = b(n,m;\rho\circ\sigma) = b(n,m;\sigma\circ\rho) = b(n,m;\rho\circ\sigma\circ\rho).$$

We apply this on

$$S_3 = \left\{\begin{pmatrix}1&2&3\\1&2&3\end{pmatrix}, \begin{pmatrix}1&2&3\\3&2&1\end{pmatrix}, \begin{pmatrix}1&2&3\\2&1&3\end{pmatrix}, \begin{pmatrix}1&2&3\\2&3&1\end{pmatrix}, \begin{pmatrix}1&2&3\\1&3&2\end{pmatrix}, \begin{pmatrix}1&2&3\\3&1&2\end{pmatrix}\right\} =$$
$$= \{\epsilon, \rho, \sigma_1, \sigma_2, \sigma_3, \sigma_4\}$$

and see that $b(n,3;\sigma)$ is the same for $\sigma = \epsilon, \rho\ (= b(n))$ and for $\sigma = \sigma_1, \sigma_2, \sigma_3, \sigma_4$. We select σ_1 as a sample for our analysis.

$B(n,3;\sigma_1)$ is the set of all $\pi \in S_n$ with the property that

$$\pi(k+1) < \pi(k) < \pi(k+2) \ \text{ is false for } \ 1 \le k \le n-2.$$

The fraction

$$\tilde{b}(n) := \frac{|B(n,3;\sigma_1)|}{n!}$$

will be studied in the next section.

7. A formula for $\tilde{b}(n)$ and proof of (1.6), (1.7)

The sieve formula yields

$$\tilde{b}(n) = 1 + \sum_{j=1}^{n-2} (-1)^j \sum_{1 \le k_1 < \ldots < k_j \le n-2} \frac{\tilde{D}(k_1,\ldots,k_j)}{n!},$$

where

$$\tilde{D}(k_1,\ldots,k_j) = \#\left\{\pi \in S_n \mid \pi(k_i+1) < \pi(k_i) < \pi(k_i+2) \ \text{ for } \ 1 \le i \le j\right\}.$$

We find

$$\tilde{D}(k_1,\ldots,k_j) = \frac{n!}{\tilde{M}(k_1,\ldots,k_j)},$$

where $\tilde{M}(k_1,\ldots,k_j)$ is defined in the following way.

$j = 1 : \tilde{M}(k) = 3!$

$j \ge 2 : \tilde{D}(k_1,\ldots,k_j) = 0$ if there is some i_0 $(1 \le i_0 < j)$ such that $k_{i_0+1} - k_{i_0} = 1$.

Now suppose that $k_{i+1} - k_i \ge 2$ $(1 \le i < j)$. Put

$$L(k_1,\ldots,k_j) := \{i \mid 1 \le i < j,\ k_{i+1} - k_i \ge 3\}.$$

If $L(k_1,\ldots,k_j) = \emptyset$ then

$$\tilde{M}(k_1,\ldots,k_j) = (2j+1)2^j j!.$$

If $L(k_1,\dots,k_j)\neq\emptyset$ write

$$L(k_1,\dots k_j)=\{\ell_1,\dots,\ell_t\}\ \text{ with }\ 1\le\ell_1<\dots<\ell_t<j.$$

Further put $\ell_0=0,\ \ell_{t+1}=j$ and define $d_s:=\ell_{s+1}-\ell_s$ $(0\le s\le t)$. Then

$$\tilde{M}(k_1,\dots,k_j)=\prod_{s=0}^{t}(2d_s+1)2^{d_s}d_s!.$$

We must leave the proof of this to the reader. We now have

$$\tilde{b}(n)=1-\frac{n-2}{3!}+\sum_{2\le j\le\frac{n-1}{2}}(-1)^jS(j),$$

where $(j\ge 2)$

$$S(j)=\sum_{\substack{1\le k_1<\dots<k_j\le n-2\\ k_{i+1}-k_i\ge 2\ (1\le i<j)}}\tilde{M}(k_1,\dots,k_j)^{-1}.$$

For $L\subset\{1,\dots,j-1\}$ put

$$S(j;L):=\sum_{\substack{1\le k_1<\dots<k_j\le n-2\\ k_{i+1}-k_i\ge 2\ (1\le i<j)\\ L(k_1,\dots,k_j)=L}}\tilde{M}(k_1,\dots,k_j)^{-1}.$$

Then we have $(j\ge 2)$

$$\begin{aligned}S(j)&=\sum_{L\subset\{1,\dots,j-1\}}S(j;L)=\\&=S(j;\emptyset)+\sum_{t=1}^{j-1}\sum_{1\le\ell_1<\dots<\ell_t<j}S(j;\{\ell_1,\dots,\ell_t\}).\end{aligned}$$

We have

$$S(j;\emptyset)=\sum_{\substack{1\le k_1<\dots<k_j\le n-2\\ k_{i+1}-k_i=2\ (1\le i<j)}}\frac{1}{(2j+1)2^jj!}=\frac{n-2j}{(2j+1)2^jj!}.$$

We have $(t\ge 1)$

$$S(j;\{\ell_1,\dots,\ell_t\})=\left(\prod_{s=0}^{t}(2d_s+1)2^{d_s}d_s!\right)^{-1}\cdot X,$$

where

$$X = \sum_{\substack{1\le k_1<\ldots<k_j\le n-2\\ k_{i+1}-k_i\ge 2\ (1\le i<j)\\ L(k_1,\ldots,k_j)=\{\ell_1,\ldots,\ell_t\}=:L}} 1 =$$

$$= \sum_{\vartheta_\ell\ge 3\ (\ell\in L)} \sum_{\substack{1\le k_1<\ldots<k_j\le n-2\\ k_{i+1}-k_i=2\ (i\notin L)\\ k_{\ell+1}-k_\ell=\vartheta_\ell\ (\ell\in L)}} 1 =$$

$$= \sum_{\substack{\delta_\ell\ge 0\ (\ell\in L)\\ \sum\limits_{\ell\in L}\delta_\ell\le n-2j-t}} \left(n-2j-t-\sum_{\ell\in L}\delta_\ell\right) =$$

$$= \binom{n-2j}{t+1} \quad \text{by} \quad (3.1).$$

Altogether we get $\left(2\le j\le \frac{n-1}{2}\right)$

$$S(j) = \sum_{0\le t\le \min\{j-1,n-2j-1\}} \binom{n-2j}{t+1}\sigma(j,t)$$

with

$$\sigma(j,0) = \frac{1}{(2j+1)2^j j!}$$

and $(t\ge 1)$

$$\sigma(j,t) = \sum_{1\le \ell_1<\ldots<\ell_t<j} \left(\prod_{s=0}^{t}(2d_s+1)2^{d_s}d_s!\right)^{-1} =$$

$$= \sum_{\substack{d_s\ge 1\ (0\le s\le t)\\ \sum\limits_{s=0}^{t} d_s=j}} \left(\prod_{s=0}^{t}(2d_s+1)2^{d_s}d_s!\right)^{-1}.$$

We arrive at the formula

$$\tilde{b}(n) = 1 + \sum_{0\le k\le \frac{n-3}{2}} (-1)^{k+1} \sum_{t=0}^{k} \binom{n-2k-2}{t+1} S(k,t),$$

where

$$S(k,t)=\sum_{\substack{a_s\ge 0\ (0\le s\le t)\\ \sum_{s=0}^{t} a_s=k-t}}\left(\prod_{s=0}^{t}(2a_s+3)2^{a_s+1}(a_s+1)!\right)^{-1}.$$

Proof of (1.6). The formula above gives

$$\sum_{n=0}^{\infty}\tilde{b}(n)z^n=\frac{1}{1-z}+S(z),$$

where

$$\begin{aligned}S(z)&=\sum_{n=3}^{\infty}\left(\sum_{0\le k\le\frac{n-3}{2}}(-1)^{k+1}\sum_{t=0}^{k}\binom{n-2k-2}{t+1}S(k,t)\right)z^n=\\&=\sum_{k=0}^{\infty}(-1)^{k+1}\sum_{t=0}^{k}S(k,t)\sum_{n=2k+3}^{\infty}\binom{n-2k-2}{t+1}z^n=\\&=z^3\sum_{k=0}^{\infty}(-1)^{k+1}z^{2k}\sum_{t=0}^{k}S(k,t)\frac{z^t}{(1-z)^{t+2}}=\\&=-\frac{z^3}{(1-z)^2}\sum_{t=0}^{\infty}\left(\frac{-z}{1-z}\right)^t\sum_{\ell=0}^{\infty}(-1)^{\ell}S(\ell+t,t)z^{2\ell}.\end{aligned}$$

Since

$$S(\ell+t,t)=\sum_{\substack{a_s\ge 0\ (0\le s\le t)\\ \sum_{s=0}^{t} a_s=\ell}}\left(\prod_{s=0}^{t}(2a_s+3)2^{a_s+1}(a_s+1)!\right)^{-1},$$

the inner sum above is equal to $F(z)^{t+1}$, where

$$F(z):=\sum_{a=0}^{\infty}\frac{(-1)^a}{(2a+3)2^{a+1}(a+1)!}z^{2a}.$$

Therefore

$$S(z) = -\frac{z^3}{(1-z)^2} \sum_{t=0}^{\infty} \left(-\frac{z^3}{1-z} F(z)\right)^t =$$
$$= -\frac{z^3}{1-z} \frac{F(z)}{1-z+z^3 F(z)}.$$

We conclude

$$\sum_{n=0}^{\infty} \tilde{b}(n) z^n = \frac{1}{1-(z-z^3 F(z))}.$$

But

$$z - z^3 F(z) = \int_0^z \exp\left(-\frac{1}{2}w^2\right) dw.$$

Proof of (1.7). Put

$$E(z) := \int_0^z \exp\left(-\frac{1}{2}w^2\right) dw.$$

Since $E(1) < 1$, $E(\infty) = \sqrt{\frac{\pi}{2}} > 1$ and $E(x)$ is strictly increasing for $x \geq 0$, there is a unique $\xi > 1$ with $E(\xi) = 1$. (1.7) follows with the theorem of residues if

(1) $$E(z) \neq 1 \text{ for all } z,\ |z| \leq \xi \text{ and } x \neq \xi.$$

We first show that

(2) $$z = x + iy,\ y \neq 0,\ |z| \leq \sqrt{\pi} \Rightarrow \mathrm{Im}E(z) \neq 0.$$

Namely from

$$\mathrm{Im}E(z) = \exp\left(-\frac{1}{2}x^2\right) \int_0^y \exp\left(\frac{1}{2}t^2\right) \cos(xt) dt$$

we see that

$$\mathrm{Im}E(z) \neq 0 \ if\ y \neq 0 \ and\ |xy| \leq \frac{\pi}{2}.$$

But $|xy| \leq \frac{\pi}{2}$ if $|z| \leq \sqrt{\pi}$.

From (2) we infer

(3) Let $0 < r \le \sqrt{\pi}$. Then

$$E(z) \neq E(r) \quad \text{for all} \quad z,\ |z| \le r \quad \text{and} \quad z \neq r.$$

Proof of (1). According to (3) it is enough to establish $\xi \le \sqrt{\pi}$. But this holds true since $E\left(\sqrt{\pi}\right) > 1$.

Concluding remark. Comparing (1.5), (1.7) one should know if $\frac{3\sqrt{3}}{2\pi} > \xi^{-1}$, $\frac{2\pi}{3\sqrt{3}} < \xi$. This is true if $E\left(\frac{2\pi}{3\sqrt{3}}\right) < 1$.

One has (courtesy Dr. Harald Schmidt, Universität Regensburg)

$$E\left(\frac{2\pi}{3\sqrt{3}}\right) = 0.9693304740\ldots\ .$$

8. $m = 4$, outlook

We identified 7 classes of S_4 on each of which $b(n, 4; \sigma)$ is the same:

$C_1 = \{e, \rho\}$
$C_2 = \{(1423), (4132), (2314), (3241)\} \ \cup\ \{(4123), (1432), (2341), (3214)\}$,
$C_3 = \{(1243), (4312), (2134), (3421)\}$,
$C_4 = \{(1342), (3124), (4213), (2431)\}$,
$C_5 = \{(1324), (4231)\}$,
$C_6 = \{(3142), (2413)\}$,
$C_7 = \{(3412), (2143)\}$.

We obtained

$$\sum_{n=0}^{\infty} b(n, 4; C_2) z^n = \left(1 - \int_0^z \exp\left(-\frac{1}{6} w^3\right) dw\right)^{-1}$$

from which we derive

$$b(n, 4; C_1) \sim \frac{\exp\left(\frac{1}{6}\xi_2^3\right)}{\xi_2^{n+1}} \quad \text{as } n \to \infty,$$

where $\xi_2 > 1$ is given by $\int\limits_0^{\xi_2} \exp\left(-\frac{1}{6}t^3\right) dt = 1$.

Further we got

$$\sum_{n=0}^{\infty} b(n,4;C_3)z^n = F(z)^{-1} = (1 - z - R(z))^{-1},$$

where

$$R(z) = \sum_{m=1}^{\infty} (-1)^m \frac{\prod\limits_{k=0}^{m-1}(3k+1)}{(3m+1)!} z^{3m+1}.$$

We have $F(1) > 0$, $F\left(\frac{3}{2}\right) > 0$ and $F(z)$ has on the disk $|z| \le \frac{3}{2}$ precisely one zero ξ_3 and it is simple. Therefore

$$b(n,4;C_3) \sim \frac{1}{(-F'(\xi_3))\xi_3^{n+1}} \quad \text{as } n \to \infty.$$

For the remaining classes we have no results.

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