THE MAXIMAL ORDER OF A CLASS OF MULTIPLICATIVE ARITHMETICAL FUNCTIONS

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Dedicated to Professor Karl-Heinz Indlekofer on his sixtieth birthday

Abstract. We prove simple theorems concerning the maximal order of a large class of multiplicative functions. As an application, we determine the maximal orders of certain functions of the type $\sigma_A(n) = \sum_{d\in A(n)} d$, where $A(n)$ is a subset of the set of all positive divisors of n, including the divisor-sum function $\sigma(n)$ and its unitary and exponential analogues. We also give the minimal order of a new class of Euler-type functions, including the Euler-function $\phi(n)$ and its unitary analogue.

1. Introduction

Let $\sigma(n)$ and $\phi(n)$ denote, as usual, the sum of all positive divisors of n and the Euler function, recpectively. It is well-known, that

$$\limsup_{n\to\infty} \frac{\sigma(n)}{n\log\log n} = e^{\gamma}, \tag{1}$$

$$\liminf_{n\to\infty} \frac{\phi(n)\log\log n}{n} = e^{-\gamma}, \tag{2}$$

The first author supported partially by the Hungarial National Foundation for Scientific Research under grant OTKA T031877.

where γ is Euler's constant. These results go back to the work of T.H. Gronwall [5] and E. Landau [7] and have been established for a number of modified σ- and ϕ-functions.

One such modification relates to unitary divisors d of n, notation $d\|n$, meaning that $d|n$ and $(d, n/d) = 1$. The corresponding σ- and ϕ-functions are defined by $\sigma^*(n) = \sum_{d\|n} d$ and $\phi^*(n) = \#\{1 \le k \le n;\ (k,n)_* = 1\}$, where $(k,n)_*$ denotes the largest divisor of k which is a unitary divisor of n. These functions are multiplicative and for prime powers p^ν given by $\sigma^*(p^\nu) = p^\nu + 1$, $\phi^*(p^\nu) = p^\nu - 1$, see [3, 8]. They are treated, along with other multiplicative functions, in [2] with the result that

$$\limsup_{n\to\infty} \frac{\sigma^*(n)}{n \log\log n} = \frac{6}{\pi^2} e^\gamma, \tag{3}$$

while ϕ^* gives again (2). (Actually (3) is written incorrectly in [2] with the factor $6/\pi^2$ missing.)

In [4] it is shown that (3) holds also for $\sigma^{(e)}(n)$, the sum of exponential divisors of n. (A number $d = \prod p^{\delta_p}$ is called an exponential divisor of $n = \prod p^{\nu_p}$ if $\delta_p|\nu_p$ for all p.)

These and a number of similar results from literature refer to rather special functions. Textbooks dealing with the extremal order of arithmetic functions also treat only particular cases, see [6, 1, 11]. It should be mentioned that a useful result concerning the maximal order of a class of prime-independent functions, including the number of all divisors, unitary divisors and exponential divisors, is proved in [10].

In the present paper we develop easily applicable theorems for determining

$$L = L(f) := \limsup_{n\to\infty} \frac{f(n)}{\log\log n},$$

where f are nonnegative real-valued multiplicative functions. Essential parameters are

$$\rho(p) = \rho(f,p) := \sup_{\nu \ge 0} f(p^\nu)$$

for the primes p, and the product

$$R = R(f) := \prod_p \left(1 - \frac{1}{p}\right) \rho(p).$$

These theorems can, in particular, be used to obtain the maximal or minimal order, respectively, of generalized σ- and ϕ-functions which arise in connection with Narkiewicz-convolutions of arithmetic functions.

2. General results

We formulate the conditions for lower and upper estimates for L separately. Note that $\rho(p) \geq f(p^0) = 1$ for all p.

Theorem 1. *Suppose that $\rho(p) < \infty$ for all primes p and that the product R converges unconditionally (i.e. irrespectively of order), improper limits being allowed, then*

$$L \leq e^\gamma R. \tag{4}$$

A different assumption uses

Theorem 2. *Suppose that $\rho(p) < \infty$ for all p and that the product R converges, improper limits being allowed, and that*

$$\rho(p) \leq 1 + o\left(\frac{\log p}{p}\right), \tag{5}$$

then (4) holds.

Remark. Neither does condition (5) plus convergence of R imply unconditional convergence of R nor vice versa.

To establish $e^\gamma R$ also as the lower limit more information is required: The suprema $\rho(p)$ must be sufficiently well approximated at not too large powers of p.

Theorem 3. *Suppose that $\rho(p) < \infty$ for all primes p, that for each prime p there is an exponent $e_p = p^{o(1)} \in \mathbb{N}$ such that*

$$\prod_p f\left(p^{e_p}\right) \rho(p)^{-1} > 0, \tag{6}$$

and that the product R converges, improper limits being allowed. Then

$$L \geq e^\gamma R.$$

Corollary 1. *If for all p we have $\rho(p) \leq (1 - 1/p)^{-1}$ and there are e_p such that $f\left(p^{e_p}\right) \geq 1 + 1/p$, then*

$$L = e^\gamma R.$$

In other words: The maximal order of $f(n)$ is $e^\gamma R \log\log n$.

Formally R becomes infinite if there is a nonempty set $\mathcal{S}$ of primes for which $\rho(p) = \infty$. So one might expect that the assumptions of Theorem 3 taken for all p with finite $\rho(p)$ would imply $L = \infty$. Surprisingly enough this is true only for rather thin sets $\mathcal{S}$. But note that for $p \in \mathcal{S}$ there is no substitute for the $f(p^{e_p})$ approximating $\rho(p)$.

We begin by stating what the above theorems imply if one ignores the numbers with prime factors from a given set $\mathcal{S}$ of primes. For any such set define

$$N(\mathcal{S}) := \{n : n \in \mathbb{N},\ p|n \Rightarrow\ p \in \mathcal{S}\}, \quad C(\mathcal{S}) := \{n : n \in \mathbb{N},\ p|n \Rightarrow p \notin \mathcal{S}\}.$$

Corollary 2. *Modify the assumptions of Theorems 1, 2 and 3 by replacing R with*

$$R_{\mathcal{S}} = R_{\mathcal{S}}(f) := \prod_{p \notin \mathcal{S}} \left(1 - \frac{1}{p}\right) \rho(p),$$

L with

$$L_{\mathcal{S}} = L_{\mathcal{S}}(f) := \limsup_{n\to\infty,\ n \in C(\mathcal{S})} \frac{f(n)}{\log\log n},$$

condition (5) with

$$\rho(p) \le 1 + o\left(\frac{\log p}{p}\right) \quad \text{for} \quad p \notin \mathcal{S}, \tag{7}$$

and (6) with

$$\prod_{p \notin \mathcal{S}} f(p^{e_p})\, \rho(p)^{-1} > 0. \tag{8}$$

Assume further that

$$\sum_{p \in \mathcal{S}} \frac{1}{p} < \infty.$$

Then

$$L_{\mathcal{S}} \le e^{\gamma} \prod_{p \in \mathcal{S}} \left(1 - \frac{1}{p}\right) \cdot R_{\mathcal{S}}, \quad L_{\mathcal{S}} \ge e^{\gamma} \prod_{p \in \mathcal{S}} \left(1 - \frac{1}{p}\right) \cdot R_{\mathcal{S}},$$

respectively. This applies even if $\rho(p) = \infty$ for some or all of the $p \in \mathcal{S}$.

Theorem 4. *Let $\mathcal{S}$ be a set of primes such that*

$$\sum_{p \in \mathcal{S}} \frac{1}{p} < \infty. \tag{9}$$

If $\rho(p) = \infty$ exactly for the $p \in \mathcal{S}$, if (8) holds and $R_{\mathcal{S}} > 0$, then $L = \infty$. Condition (9) must not be waived.

In fact there are counter-examples for any set $\mathcal{S}$ for which $\sum 1/p$ diverges.

3. The proofs

Proof of Theorem 1. An arbitrary $n = \prod p^{\nu_p}$ we write as $n = n_1 n_2$ with $n_1 := \prod\limits_{p \le \log n} p^{\nu_p}$. Mertens's formula $\prod\limits_{p \le x} (1 - 1/p)^{-1} \sim e^{\gamma} \log x$ and the definition of $\rho(p)$ imply

$$\begin{aligned} f(n_1) &= \prod_{p \le \log n} f(p^{\nu_p}) \le \prod_{p \le \log n} \rho(p) = \\ &= \prod_{p \le \log n} \left(1 - \frac{1}{p}\right)^{-1} \cdot \prod_{p \le \log n} \left(1 - \frac{1}{p}\right) \rho(p), \\ f(n_1) &\le (1 + o(1)) e^{\gamma} R \log \log n \quad \text{as } n \to \infty. \end{aligned} \tag{10}$$

Let a denote the number of prime divisors in n_2. Then $a \le \log n / \log \log n$. There is nothing to prove if $R = \infty$, so let $R < \infty$. Using the unconditional convergence

$$\begin{aligned} f(n_2) &\le \prod_{p|n,\ p > \log n} \left(1 - \frac{1}{p}\right) \rho(p) \cdot \prod_{p|n,\ p > \log n} \left(1 - \frac{1}{p}\right)^{-1} \le \\ &\le (1 + o(1)) \cdot \left(1 - \frac{1}{\log n}\right)^{-a} = \\ &= (1 + o(1)) e^{O(1/\log \log n)} \to 1. \end{aligned} \tag{11}$$

Combining (10) and (11) finishes the proof.

Proof of Theorem 2. There is no change in the estimation of $f(n_1)$. For n_2 we have

$$f(n_2) \le \left(1 + o\left(\frac{\log \log n}{\log n}\right)\right)^{\frac{\log n}{\log \log n}} = 1 + o(1).$$

Proof of Theorem 3. We treat the case of proper convergence only. There is nothing to prove if $R = 0$ and the changes for $R = \infty$ are obvious. For given ε take P so large that

$$\prod_{p>P} f(p^{e_p})\rho(p)^{-1} \geq 1 - \varepsilon \tag{12}$$

and choose exponents k_p for the $p \leq P$ such that

$$\prod_{p\leq P} f(p^{k_p}) \geq (1-\varepsilon) \prod_{p\leq P} \rho(p). \tag{13}$$

Keeping P and the k_p fixed let x tend to infinity and consider

$$n(x) := \prod_{p\leq P} p^{k_p} \prod_{P<p\leq x} p^{e_p}.$$

Now on the one hand, using (12) and (13), we see

$$f(n(x)) \prod_{p\leq x} \left(1 - \frac{1}{p}\right) \geq (1-\varepsilon) \prod_{p\leq x} \left(1 - \frac{1}{p}\right) \rho(p) \cdot \prod_{P<p\leq x} f(p^{e_p})\rho(p)^{-1} \geq$$
$$\geq (1-\varepsilon)^2 (1 + o(1))R$$

and with Mertens's formula again

$$f(n(x)) \geq (1-\varepsilon)^2(1+o(1))Re^{\gamma} \log x. \tag{14}$$

On the other hand, since $e_p = p^{o(1)}$, we have

$$\log n(x) \leq \sum_{p\leq P} k_p \log p + \sum_{P<p\leq x} e_p \log p \leq x^{o(1)} \sum_{p\leq x} \log p = x^{1+o(1)},$$

and therefore

$$\log\log n(x) \leq (1+o(1)) \log x.$$

Together with (14) this yields the lower bound

$$\limsup_{x\to\infty} \frac{f(n(x))}{\log\log n(x)} \geq (1-\varepsilon)^2 Re^{\gamma}$$

with arbitrary $\varepsilon > 0$.

Proof of Corollary 1. Apply Theorems 1 (or 2) and 3.

Proof of Corollary 2. To see this one applies the theorems to the multiplicative function f^* defined by $f^*(n) = f(n)$ for $n \in C(\mathcal{S})$ and $f(n) = 1$ for $n \in N(\mathcal{S})$. One finds

$$L(f^*) = L_{\mathcal{S}}(f), \quad R(f^*) = R_{\mathcal{S}}(f) \prod_{p \in \mathcal{S}} \left(1 - \frac{1}{p}\right),$$

and (8) implies (6) for f^* because $\prod_{p \in \mathcal{S}} (1 - 1/p)$ converges absolutely.

Note also that for any sequence of numbers $n = n_1 n_2$ tending to ∞, where $n_1 \in N(\mathcal{S})$, $n_2 \in C(\mathcal{S})$, we have $f^*(n)/\log\log n = f(n_2)/\log\log(n_1 n_2)$, hence $\limsup f^*(n)/\log\log n = 0$ if n_2 stays bounded, and

$$\le \limsup_{n_2} f(n_2)/\log\log n_2$$

otherwise, with equality if n_1 is bounded. Thus $L(f^*) = L_{\mathcal{S}}$.

Proof of Theorem 4. I. Assume (9). With any $n_1 \in N(\mathcal{S})$ we have

$$L \ge \limsup_{n_2 \in C(\mathcal{S})} \frac{f(n_1) f(n_2)}{\log\log(n_1 n_2)} = f(n_1) L_{\mathcal{S}}.$$

From Corollary 2, as it refers to Theorem 3, we have $L_{\mathcal{S}} > 0$ and $f(n_1)$ can be chosen arbitrarily large.

II. Assume that (9) does not hold. We shall construct a counter-example. The assumption implies that

$$g(x) := \prod_{p \in \mathcal{S},\ p \le x} \left(1 + \frac{1}{p}\right)$$

tends to ∞ as $x \to \infty$. Choose an increasing sequence of numbers $q_j = p_j^{\nu_j}$ with $p_j \in \mathcal{S}$ and ν_j so large that $g(\log q_j) \ge j^j$ for all j, and such that every prime $p \in \mathcal{S}$ occurs infinitely often in the sequence of the p_j. Put $f(q_j) = j$ for all $j \in \mathbb{N}$ and $f(p^\nu) = 1 + 1/p$ for all p^ν that are not among the q_j. Then, obviously, $\rho(p) = \infty$ for $p \in \mathcal{S}$ and $\rho(p) = 1 + 1/p$ for $p \notin \mathcal{S}$. The product

$$R_{\mathcal{S}} = \prod_{p \notin \mathcal{S}} \left(1 - \frac{1}{p}\right)\left(1 + \frac{1}{p}\right)$$

converges absolutely and so does (choosing $e_p = 1$) $\prod_{p \notin \mathcal{S}} f(p^1)/\rho(p) = 1$. Any $n \in \mathbb{N}$ can be written as $n = n_1 n_2$, where n_1 collects from the canonical representation of n those prime powers that occur among the q_j, while the rest compose n_2. For given n let $k := \max\{j; q_j \| n_1\}$. Then $f(n_1) \le k! = o(k^k) = o(g(\log q_k)) = o(g(\log n))$ by construction. Now for any $n \in \mathbb{N}$

$$f(n) = f(n_1)f(n_2) = \prod_{p|n,\ p\notin\mathcal{S}} \left(1 + \frac{1}{p}\right) \cdot o(g(\log n)) =$$

$$= o\left(\prod_{p \le \log n} \left(1 + \frac{1}{p}\right)\right) \cdot \prod_{p|n,\ p \ge \log n} \left(1 + \frac{1}{p}\right) \le$$

$$\le o(\log\log n) \cdot \left(1 + \frac{1}{\log n}\right)^{\frac{\log n}{\log\log n}} = o(\log\log n),$$

hence $L = 0$.

4. Applications

A general frame for generalizations of the σ- and ϕ-functions mentioned in the introduction can be found in Narkiewicz [9]. Assume that for each n a set $A(n)$ of divisors of n is given and consider the A-convolution $*_A$ defined by

$$(f *_A g)(n) := \sum_{d \in A(n)} f(d) g\left(\frac{n}{d}\right). \tag{15}$$

Properties of convolution (15) and of arithmetical functions related to it have been studied extensively in the literature, see [9, 8]. The system A is called multiplicative if $A(n_1 n_2) = A(n_1)A(n_2)$ for coprime n_1, n_2, with elementwise multiplication of the sets, and not all $A(n)$ empty. Such a divisor system can be described by the sets $AE_p(\nu)$ of admissible exponents,

$$AE_p(\nu) := \{\delta;\ p^\delta \in A(p^\nu)\}.$$

The A-convolution of any two multiplicative functions f and g is multiplicative if and only if A is multiplicative. In particular multiplicativity of A implies multiplicativity of the modified divisor function

$$\sigma_A(n) := \sum_{d \in A(n)} d.$$

As natural means to define an Euler-function attached to A we consider the relation

$$\sum_{d\in A(n)} \phi_A(d) = n, \quad n \geq 1. \tag{16}$$

This need not be solvable; there is, however, the following

Theorem 5. *If the divisor system A is multiplicative then (16) has a solution if and only if $n \in A(n)$ for all $n \in \mathbb{N}$. In this case the solution ϕ_A is unique and is a multiplicative function with $1 \leq \phi_A(n) \leq n$ for all $n \in \mathbb{N}$.*

Proof. Suppose a solution exists. Then by induction on ν the recursion

$$\sum_{\delta\in AE_p(\nu)} \phi_A(p^\delta) = p^\nu \tag{17}$$

implies that $1 \leq \phi_A(p^\nu) \leq p^\nu$ and (therefore) $\nu \in AE_p(\nu)$ for all $\nu : p^\nu \in A(p^\nu)$. It follows from the multiplicativity of A that $n \in A(n)$ for all n. If, on the other hand, $n \in A(n)$ for all n, then (17) can be solved recursively and the multiplicative function defined from the $\phi_A(p^\nu)$ solves (16). This is in fact the only solution since $\phi_A(n) = n - \sum_{d\in A(n)\setminus\{n\}} \phi_A(d)$.

With suitable additional conditions on A we give the maximal and minimal orders of σ_A and ϕ_A, respectively. Extremal orders of such functions have not been investigated in the literature.

Obviously $\sigma_A(n) \leq \sigma(n)$ and if for any ν we have $p^\nu, p^{\nu-1} \in A(p^\nu)$ then $\sigma_A(p^\nu) \geq p^\nu + p^{\nu-1}$. So Corollary 1 applies to $f(n) = \sigma_A(n)/n$ and gives

Theorem 6. *Let the system A of divisors be multiplicative and suppose that for each prime p there is an exponent e_p such that*

$$p^{e_p}, \ p^{e_p-1} \in A(p^{e_p})$$

and $e_p = p^{o(1)}$. Then

$$\limsup_{n\to\infty} \frac{\sigma_A(n)}{n\log\log n} = e^\gamma \prod_p \left(1-\frac{1}{p}\right) \sup_{\nu\geq 0} \frac{\sigma_A(p^\nu)}{p^\nu},$$

where the product converges.

Remarks. The quotients $\sigma(p^\nu)/p^\nu$ are of the form $\sum \varepsilon_i p^{-i}$, $\varepsilon_i \in \{0,1\}$, and the set of such numbers is compact. Therefore each $\sup_\nu(\sigma_A(p^\nu)p^{-\nu})$ is itself of this form and we have for each prime p a finite or infinite sequence of exponents a_i such that $2 \le a_1 < a_2 < \dots$ and

$$\left(1-\frac{1}{p}\right)\sup_{\nu}\frac{\sigma_A(p^\nu)}{p^\nu} = 1-\frac{1}{p^{a_1}}+\frac{1}{p^{a_2}}-\frac{1}{p^{a_3}}+\dots .$$

The formulae (1) and (3) are obvious consequences of Theorem 6. In the standard case e_p is arbitrary, we have $(1-1/p)\rho(p)=1$ for all p, hence $R=1$. With unitary and exponential divisors the only admissible choices are $e_p=1$ and $e_p=2$, respectively, and $(1-1/p)\rho(p)=1-1/p^2$, hence $R=\zeta(2)^{-1}=6/\pi^2$ in both cases.

We turn to ϕ_A, assuming again that A is multiplicative and, in view of Theorem 5, that always $\nu \in AE_p(\nu)$. In order to determine the minimal order of ϕ_A consider the function $f(n):=n/\phi_A(n)$.

For all p and $\nu \ge 1$ we have $\phi_A(p^\nu) \ge p^\nu - \phi_A(p^{\nu-1}) - \dots - \phi_A(1) \ge p^\nu - -p^{\nu-1} - \dots - 1$, which gives

$$f(p^\nu) < \frac{p-1}{p-2}, \quad \rho(p) \le \frac{p-1}{p-2}.$$

Note that $\rho(2)$ may equal ∞. If moreover $e-1 \in AE_p(e)$ for some $e=e_p \ge 1$ then, on the other hand, $\phi_A(p^e) \le p^e - \phi_A(p^{e-1}) \le p^e - p^{e-1} + p^{e-2} + \dots + 1$ if $e \ge 2$, and $\phi_A(p) \le p-1$ if $e=1$. Therefore

$$f(p^e) \ge \frac{p(p-1)}{p^2-2p+2},$$
$$f(p^e)\rho(p)^{-1} \ge \frac{p(p-2)}{p^2-2p+2} = 1-\frac{2}{p^2-2p+2},$$

which is positive and yields a convergent product for $p \ge 3$.

Note that for powers of 2 there is no non-trivial lower estimate for $\phi_A(n)/n$ without further conditions on A. This is shown by the following example. Let $\mathcal{N}=\{n_1,n_2,\dots\} \subset \mathbb{N}$, $n_1<n_2<\dots$, and put $AE_2(n):=\{0,1,\dots,n\}$ for $n \in \mathcal{N}$ and $AE_2(n):=\{n\}$ for $n \notin \mathcal{N}$. Then the recursion gives $\phi_A(2^n)=2^n$ for $n \notin \mathcal{N}$ but $\phi_A(2^{n_j})=2^{n_{j-1}}$ for the $n \in \mathcal{N}$, where $n_0=0$. Hence it is possible to have $\rho(2)=\sup_\nu 2^\nu/\phi_A(2^\nu)=2^{\sup_j(n_j-n_{j-1})}=\infty$.

Thus applying Corollary 1 or Theorem 4 with $\mathcal{S}=\{2\}$ we obtain

Theorem 7. *Let A be multiplicative and $n \in A(n)$ for all n. Assume that for each prime $p > 2$ there is an exponent e_p such that $p^{e_p - 1} \in A(p^{e_p})$ and $e_p = p^{o(1)}$. Then*

$$\liminf_{n\to\infty} \frac{\phi_A(n) \log\log n}{n} = e^{-\gamma} \prod_p \left(1 - \frac{1}{p}\right)^{-1} \inf_\nu \frac{\phi_A(p^\nu)}{p^\nu}.$$

The product converges for $p > 2$; the first factor may vanish.

For the standard Euler function $\phi(n)$ and for its unitary analogue $\phi^*(n)$ we regain (2).

For the system of exponential divisors one has $\phi_A(1) = 1$ because of multiplicativity. The recursion $\sum_{\kappa|\nu} \phi_A(p^\kappa) = p^\nu$ is solved by $\phi_A(p^\nu) = \sum_{\kappa|\nu} \mu(\nu/\kappa)p^\kappa$. Again the minimum of $\phi_A(p^\nu)/p^\nu$ is $1 - 1/p$, it is taken for $\nu = e_p = 2$ and once more (2) follows.

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