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ON THE MEAN VALUES OF MULTIPLICATIVE FUNCTIONS OVER RATIONAL NUMBERS

V. Stakėnas (Vilnius, Lithuania)

Professor K.-H. Indlekofer zum 60. Geburtstag gewidmet

Abstract. Two mean value theorems for the multiplicative functions of rational arguments are proved. One of them may be viewed as an analogue of the Delange's theorem well-known in the theory of arithmetical functions.

Definitions and results

A complex valued function $f(m/n)$, defined on the set of positive rational numbers, is called multiplicative, if for all irreducible fractions m/n

$$f\left(\frac{m}{n}\right) = f(m) \cdot f\left(\frac{1}{n}\right)$$

holds with $g(n) = f(n)$, $h(n) = f(1/n)$ being arithmetical multiplicative functions. Using the unique expression of m/n as a product of powers of primes (positive and negative ones) we may write

$$f\left(\frac{m}{n}\right) = \prod_{p^\nu || \frac{m}{n}} f(p^\nu);$$

here $p^\nu || \frac{m}{n}$ for $\nu > 0$ means that $p^\nu || m$ and for $\nu < 0$ it means that $p^{-\nu} || n$.

Let for $x \geq 1$ $\alpha = \alpha_x$, $\beta = \beta_x$ and $0 < \alpha < \beta$. With the system of intervals

$$I_x = (\alpha, \beta), \quad 0 < \alpha < \beta, \quad x \geq 1,$$

we define the following subsets of rational numbers

$$\mathcal{F}_x = \left\{ \frac{m}{n} \in I_x : n \le x \right\};$$

here and in the following all fractions are supposed to be irreducible.

We are interested in the asymptotical behavior of the sums

$$S(f,x) = \sum_{\frac{m}{n} \in \mathcal{F}_x} f\left(\frac{m}{n}\right),$$

where f is a multiplicative function.

This problem was considered in particular cases in the papers [4], [7]. It is our aim to prove in this work the following analogue of the Delange's mean value theorem for the multiplicative functions defined for rational numbers.

Theorem 1. *Let with some $\xi \in (0,1)$ and $\zeta > 0$ for intervals $I_x = (\alpha, \beta)$ the following condition*

$$\max\{x^{-\xi}, \alpha/x\} \ll \beta - \alpha \ll x^{\zeta} \quad (0 < \alpha < \beta) \tag{1}$$

be satisfied. If for a complex valued multiplicative function $f(m/n)$, $|f(m/n)| \le$ ≤ 1, the series

$$\sum_p \frac{2 - \mathrm{Re} f(p) - \mathrm{Re} f(p^{-1})}{p} \tag{2}$$

converges, then

$$\frac{1}{\#\mathcal{F}_x} S(f,x) = \Pi_1(x) \cdot \Pi_2(x) \cdot \Pi_3(x) + \mathrm{o}(1) \quad (x \to \infty) \tag{3}$$

holds with $\Pi_i(x)$ defined as follows

$$\Pi_1(x) = \prod_{p \le x_*} \left(1 - \frac{2}{p+1}\right) \sum_{|\nu| \ge 0} \frac{f(p^\nu)}{p^{|\nu|}}, \quad x_* = \min\{(\beta - \alpha)x, x\},$$

$$\Pi_2(x) = \prod_{x_* < p \le x} \left(1 - \frac{1}{p}\right)\left(1 + \frac{f(p^{-1})}{p}\right),$$

$$\Pi_3(x) = \prod_{x_* < p,\ p \in P_*} \left(1 - \frac{1}{p}\right)\left(1 + \frac{f(p)}{p}\right),$$

where $P_* = \{p : \text{there exists } m/n \in \mathcal{F}_x, p || m\}$.

With some stronger constraints on intervals the asymptotics may be written in more simple form.

Corollary. *Let for every* $\epsilon > 0$

$$x^{-\epsilon} \ll \beta - \alpha < \beta \ll x^{\epsilon}$$

hold. If the condition (2) is satisfied, then

$$\frac{1}{\#\mathcal{F}_x} S(f,x) = \prod_{p \le x} \left(1 - \frac{2}{p+1}\right) \sum_{|\nu| \ge 0} \frac{f(p^\nu)}{p^{|\nu|}} + o(1) \quad (x \to \infty).$$

The proof of Theorem 1 is based on the following statement.

Theorem 2. *Let with some* $\xi \in (0,1)$ *and* $\zeta > 0$ *for intervals* I_x *the conditions (1) be satisfied. Then for the complex valued multiplicative functions* $f(m/n)$ *with the conditions*

$$\left| f\left(\frac{m}{n}\right)\right| \le 1, \quad f(p^\nu) = 1 \quad \text{as} \quad |\nu| \ge 1,\ p \ge y,\ y = \exp\left\{c_1 \frac{\log x}{\log_2 x}\right\}$$

the asymptotics

$$\frac{1}{\#\mathcal{F}_x} S(f,x) = \prod_{p \le y} \left(1 - \frac{2}{p+1}\right) \sum_{|\nu| \ge 0} \frac{f(p^\nu)}{p^{|\nu|}} + o(\log^{-B} x) \quad (x \to \infty) \tag{4}$$

holds uniformly with an arbitrary constant $B > 0$.

This theorem is formulated in a slightly different form in [4]. A sketch of the proof is given in that paper, too. We present here the proof in details.

Proofs

We start with the proof of the Theorem 2 and then, using it prove the Theorem 1.

Proof of Theorem 2. We denote

$$S_n = \sum_{\substack{\alpha n < m < \beta n \\ (m,n)=1}} f(m).$$

Let θ be a real number, $\xi < \theta < 1$. Then for $n \le x^\theta$

$$S_n \ll (\beta - \alpha) x^\theta.$$

Hence, from the obvious equality

$$S(f, x) = \sum_{n \le x} f(n^{-1}) S_n$$

we obtain

$$S(f, x) = \sum_{x^\theta < n \le x} f(n^{-1}) S_n + O((\beta - \alpha) x^{2\theta}). \tag{5}$$

For the natural number n we denote

$$n^* = \prod_{\substack{p^\nu \| n \\ p \le y}} p^\nu, \quad n^{**} = \frac{n}{n^*}.$$

Now

$$S_n = \sum_{\substack{\alpha n < m < \beta n \\ (m, n^*) = 1}} f(m) + O\left(\sum_{\substack{\alpha n < m < \beta n \\ (m, n^{**}) > 1}} 1 \right).$$

The sum under the O-sign can be estimated as

$$\ll \sum_{p | n^{**}} \sum_{\substack{\alpha n < m < \beta n \\ p | m}} 1 \ll \frac{(\beta - \alpha) x}{y} \cdot \frac{\log x}{\log y}.$$

Hence,

$$S_n = S_n^* + O\left(\frac{(\beta - \alpha) x}{y} \cdot \frac{\log x}{\log y} \right), \quad S_n^* = \sum_{\substack{\alpha n < m < \beta n \\ (m, n^*) = 1}} f(m). \tag{6}$$

We estimate now the sum

$$S_n^* = \sum_{\substack{\alpha n < m < \beta n \\ (m, n^*) = 1}} f(m) = \sum_{\substack{\alpha n < m < \beta n \\ (m, n^*) = 1}} f(m^*)$$

for $x^\theta < n \le x$. Let us choose a small number $\tau \in (0,1)$ and taking $w = x^\tau$ split the sum into two parts S^*_{n1}, S^*_{n2} according to whether which of the two conditions $m^* \le w$ or $w < m^*$ is satisfied. Hence,

$$(7) \quad S^*_n = S^*_{n1} + S^*_{n2}, \quad S^*_{n1} = \sum_{\substack{\alpha n < m < \beta n \\ (m,n^*)=1 \\ m^* \le w}} f(m^*), \quad S^*_{n2} = \sum_{\substack{\alpha n < m < \beta n \\ (m,n^*)=1 \\ w < m^*}} f(m^*).$$

We shall use the notations $p^-(m),\ p^+(m)$ for the smallest, respectively, largest prime divisor of $m,\ m > 1$, i. e.

$$\begin{aligned} p^-(m) &= \max\{q: \text{ if } p < q, \quad \text{then } p \nmid m\}, \\ p^+(m) &= \min\{q: \text{ if } p > q, \quad \text{then } p \nmid m\}, \end{aligned}$$

here p, q are primes. We set formally $p^-(1) = +\infty,\ p^+(1) = 0$. We estimate S^*_{n2} as follows

$$(8) \quad \begin{aligned} |S^*_{n2}| &\le \left| \sum_{\substack{w < m < \beta n \\ (m,n^*)=1 \\ p^+(m) \le y}} f(m) \# \left\{ l : \frac{\alpha n}{m} < l < \frac{\beta n}{m}, p^-(l) > y \right\} \right| \ll \\ &\ll \sum_{\substack{w < m < \beta n \\ (m,n^*)=1 \\ p^+(m) \le y}} \# \left\{ l : \frac{\alpha n}{m} < l < \frac{\beta n}{m}, p^-(l) > y \right\}. \end{aligned}$$

Represent now m in the last sum as $m = m' \cdot m''$, where $w < m' \le w^2$ and $p^+(m') \le p^-(m'')$. For the uniqueness choose m' the largest possible. Hence, the sum in (8) is bounded by

$$\sum_{\substack{w < m' \le w^2 \\ (m',n^*)=1 \\ p^+(m') \le y}} \sum_{\substack{w/m' < m'', (m'',n^*)=1 \\ p^+(m') \le p^-(m'') \\ p^+(m'') \le y}} \# \left\{ m''l : \frac{\alpha n}{m'} < lm'' < \frac{\beta n}{m'}, p^-(l) > y \right\}.$$

For fixed m' and different m'' the sets

$$\left\{ m''l : \frac{\alpha n}{m'} < lm'' < \frac{\beta n}{m'}, p^-(l) > y \right\}$$

do not intersect themselves. This observation leads to the bound

$$S_{n2}^* \ll \sum_{\substack{w<m'\le w^2\\ (m',n^*)=1\\ p^+(m')\le y}} \#\left\{m : \frac{\alpha n}{m'} < m < \frac{\beta n}{m'}, (m,n^*)=1, p^+(m') \le p^-(m)\right\}.$$

We estimate the summands on the right-hand side using the folowing sieve result (see [1], Theorem 2.5).

Lemma 1. *Let* $2 \le s < v < u$, $\epsilon_p \in \{0,1\}$ *for all primes* p,

$$P_s = \prod_{p\le s} p^{\epsilon_p},$$

and

$$S(u,v,P_s) = \#\{m : u-v < m \le u, (m,P_s)=1\}.$$

Then

$$S(u,v,P_s) = v\prod_{p|P_s}\left(1-\frac{1}{p}\right)(1+O(R)),$$

uniformly for all sequencies $\{\epsilon_p\}$, *where*

$$R = \exp\left\{-c_2\cdot\frac{\log v}{\log s}\log\left(\frac{\log v}{\log s}\right)\right\} + \exp\{-\log^{1/2} v\}.$$

We apply this Lemma with

$$u = \frac{\beta n}{m'}, \quad v = \frac{(\beta-\alpha)n}{m'}, \quad s = y = \exp\left\{c_1\frac{\log x}{\log_2 x}\right\},$$

$$P_s = \prod_{\substack{p<p^+(m')\\ or\ p|n^*}} p.$$

If τ is chosen such that $\sigma = \theta - \xi - 2\tau > 0$, then

$$v = \frac{(\beta-\alpha)n}{m'} > x^{\theta-\xi-2\tau} = x^\sigma.$$

It is easy to check, that the Lemma 1 is applicable. Because of $\log v > \sigma \log x$, we have for the remainder term R the following bound

$$R \ll \exp\{-c_3 \log_2 x\cdot\log_3 x\}.$$

We have then

$$|S_{n2}^*| \ll (\beta-\alpha)n \sum_{\substack{w<m'<w^2\\(m',n^*)=1\\p^+(m')\le y}} \frac{1}{m'} \prod_{\substack{p<p^+(m')\\ or\ \ p|n^*}} \left(1-\frac{1}{p}\right).$$

Let $q=p^+(m')$; then $m'=q\cdot l,\ p^+(l)\le q$ and

(9) $$|S_{n2}^*| \ll (\beta-\alpha)n \sum_{\substack{q\le y\\(q,n^*)=1}} \frac{1}{q} \prod_{\substack{p<q\\ or\ \ p|n^*}} \left(1-\frac{1}{p}\right) \sum_{\substack{w/q<l<w^2/q\\(l,n^*)=1\\p^+(l)\le q}} \frac{1}{l}.$$

Due to $q\le y$ we have with some small constant $\epsilon>0$

(10) $$\sum_{\substack{w/q<l<w^2/q\\(l,n^*)=1\\p^+(l)\le q}} \frac{1}{l} \le \sum_{\substack{w^{1-\epsilon}<l\\(l,n^*)=1\\p^+(l)\le q}} \frac{1}{l}.$$

For estimating this last sum we apply the technique used in [5], [3] as well as in [4] for similar sums. With some z,q,t and $\delta\in(0,1)$ consider the sum

$$\sum_{\substack{z<b,p^+(b)\le q\\(b,t)=1}} \frac{1}{b} \le \frac{1}{z^{1-\delta}} \sum_{\substack{z<b,p^+(b)\le q\\(b,t)=1}} \frac{1}{b^\delta} \le \frac{1}{z^{1-\delta}} \prod_{\substack{p\le q\\(p,t)=1}} \left(1+\frac{1}{p^\delta}+\frac{1}{p^{2\delta}}+\dots\right) \le$$

$$\le \exp\left\{-(1-\delta)\log z + \sum_{\substack{p\le q\\(p,t)=1}} \left\{\frac{1}{p}+c_p(\delta)\right\} + \sum_{p\le q}\left(\frac{1}{p^\delta}-\frac{1}{p}\right)\right\},$$

where

$$c_p(\delta)=\sum_{m\ge2}\frac{1}{p^{m\delta}}.$$

Using the obvious arguments we have

$$\sum_{p\le q}\left(\frac{1}{p^\delta}-\frac{1}{p}\right) \le \sum_{p\le q}\frac{p^{1-\delta}-1}{p} = \sum_{p\le q}\frac{1}{p}\sum_{n=1}^{\infty}\frac{((1-\delta)\log p)^n}{n!} \le$$

$$\le \sum_{n=1}^{\infty}\frac{(1-\delta)^n\log^{n-1}q}{n!}\sum_{p\le q}\frac{\log p}{p} \le 2\sum_{n=1}^{\infty}\frac{(1-\delta)^n\log^n q}{n!} < 2q^{1-\delta}.$$

Hence,

$$\sum_{\substack{z<b,\ p^+(b)\le q\\ (b,t)=1}} \frac{1}{b} \le \exp\left\{-(1-\delta)\log z + 2q^{1-\delta} + \sum_{\substack{p\le q\\ (p,t)=1}} \left\{\frac{1}{p} + c_p(\delta)\right\}\right\}.$$

We use this bound with $z = w^{1-\epsilon} = x^{c_4}, t = n^*$. Because of $q \le y = = \exp\{c_1 \log x / \log_2 x\}$ taking

$$1-\delta = \frac{1}{c_1}\frac{\log_2 x}{\log x}\log_3 x$$

we get

$$\sum_{\substack{w^{1-\epsilon}<l\\ (l,n^*)=1\\ p^+(l)\le q}} \frac{1}{l} \ll \exp\left\{\sum_{\substack{p\le q\\ (p,n^*)=1}} \frac{1}{p} - c_5 \log_2 x \log_3 x\right\}. \tag{11}$$

We obtain now from (9), (10) and (11)

$$|S^*_{n2}| \ll$$

$$\ll (\beta-\alpha)n\exp\big\{-c_5\log_2 x\log_3 x\big\} \sum_{\substack{q\le y\\ (q,n^*)=1}} \frac{1}{q} \prod_{\substack{p<q\ \ or\\ p|n^*}} \left(1-\frac{1}{p}\right) \exp\left\{\sum_{\substack{p\le q\\ (p,n^*)=1}} \frac{1}{p}\right\}.$$

Using

$$\prod_{\substack{p\le q\ \ or\\ p|n^*}} \left(1-\frac{1}{p}\right) \exp\left\{\sum_{\substack{p\le q\\ (p,n^*)=1}} \frac{1}{p}\right\} \ll \prod_{p|n^*}\left(1-\frac{1}{p}\right),$$

we get

$$|S^*_{n2}| \ll (\beta-\alpha)n\exp\big\{-c_5\log_2 x\log_3 x\big\} \prod_{p|n^*}\left(1-\frac{1}{p}\right) \sum_{\substack{q\le y\\ (q,n^*)=1}} \frac{1}{q},$$

and, finally,

$$|S^*_{n2}| \ll (\beta-\alpha)n\exp\big\{-c_6\log_2 x\log_3 x\big\}. \tag{12}$$

Let us now for $x^\theta < n \le x$ investigate the sum

$$S^*_{n1} = \sum_{\substack{\alpha n < m < \beta n \\ (m,n^*)=1 \\ m^* \le w}} f(m^*).$$

We have

$$S^*_{n1} = \sum_{\substack{l<w \\ (l,n^*)=1 \\ p^+(l)\le y}} f(l) \# \left\{ k : \frac{\alpha n}{l} < k < \frac{\beta n}{l}, p^-(k) > y \right\}.$$

Now we apply the sieve result formulated in Lemma 1 above with $u = \beta n/l,\ v = (\beta-\alpha)n/l, s = y$ and $P_s = \prod_{p\le s} p$. The inequality $v > x^{\theta-\xi}/w > x^\epsilon$ for some $\epsilon > 0$ ensures that the remainder term is bounded uniformly by the same term as above in (12). Then

$$S^*_{n1} = (1+O(R))(\beta-\alpha)n \prod_{p\le y}\left(1-\frac{1}{p}\right) \sum_{\substack{l<w \\ (l,n^*)=1 \\ p^+(l)\le y}} \frac{f(l)}{l},$$

$$R = \exp\{-c_6 \log_2 x \cdot \log_3 x\}.$$

Using for the sum an obvious relation

$$\sum_{\substack{l<w \\ (l,n^*)=1 \\ p^+(l)\le y}} \frac{f(l)}{l} = \sum_{\substack{(l,n^*)=1 \\ p^+(l)\le y}} \frac{f(l)}{l} + O\left(\sum_{\substack{l>w \\ p^+(l)\le y}} \frac{1}{l}\right),$$

one gets expanding the first sum into product of primes and estimating the remainder term (using, for example, (11) with $n^* = 1, q = y$) the following expression

$$\sum_{\substack{l<w \\ (l,n^*)=1 \\ p^+(l)\le y}} \frac{f(l)}{l} = \prod_{\substack{p\le y \\ (p,n^*)=1}} \sum_{\nu\ge 0} \frac{f(p^\nu)}{p^\nu} + O(R),$$

with $R = \exp\{-c_6 \log_2 x \cdot \log_3 x\}$. Now we can deal with the sum S^*_{n1} :

(13)

$$\begin{aligned} S^*_{n1} =& (1+O(R))(\beta-\alpha)n \prod_{p\le y}\left(1-\frac{1}{p}\right)\left(\prod_{\substack{p\le y \\ (p,n^*)=1}} \sum_{\nu\ge 0} \frac{f(p^\nu)}{p^\nu} + O(R)\right) = \\ =& (\beta-\alpha)n \prod_{p\le y}\left(1-\frac{1}{p}\right) \prod_{\substack{p\le y \\ (p,n^*)=1}} \sum_{\nu\ge 0} \frac{f(p^\nu)}{p^\nu} + (\beta-\alpha)nO(R). \end{aligned}$$

We may due to (6), (7), (12) and (13) write now uniformly for $x^\theta < n \le x$

$$S_n = (\beta-\alpha)n \prod_{p\le y}\left(1-\frac{1}{p}\right) \prod_{\substack{p\le y\\(p,n^*)=1}} \sum_{\nu\ge 0} \frac{f(p^\nu)}{p^\nu} + O((\beta-\alpha)xR).$$

With this equality and (5) one gets

(14)
$$S(f,x) = (\beta-\alpha) \prod_{p\le y}\left(1-\frac{1}{p}\right) \sum_{n\le x} nf(n^{-1}) \prod_{\substack{p\le y\\(p,n^*)=1}} \sum_{\nu\ge 0} \frac{f(p^\nu)}{p^\nu} + O((\beta-\alpha)x^2R);$$

we extended the range of summation, but this affected only the remainder term.

We proceed further by considering the cases

$$\sum_{\nu\ge 0}\frac{f(2^\nu)}{2^\nu} \ne 0 \quad \text{and} \quad \sum_{\nu\ge 0}\frac{f(2^\nu)}{2^\nu} = 0.$$

Consider the non-zero case first. We can rewrite the equality (14) then as

$$S(f,x) = (\beta-\alpha) \prod_{p\le y}\left(1-\frac{1}{p}\right) \sum_{\nu\ge 0}\frac{f(p^\nu)}{p^\nu} \sum_{n\le x} ng(n) + O((\beta-\alpha)x^2R), \tag{15}$$

where $R = \exp\{-c_6 \log_2 x \cdot \log_3 x\}$ and $g(n)$ is a multiplicative function defined on the powers of primes as follows:

$$g(p^\nu) = \begin{cases} f(p^{-\nu}) = 1, & \text{if } p > y,\\ f(p^{-\nu}) \left(\sum\limits_{\nu\ge 0}\frac{f(p^\nu)}{p^\nu}\right)^{-1}, & \text{if } p \le y. \end{cases} \tag{16}$$

Note, that $g(p^\nu)$ is bounded on the powers of primes. We proceed using the following Lemma (see [2], Theorem 02).

Lemma 2. *Let $A > 0$ and $0 < \alpha < 1$. Then there exists $y_0 = y_0(A,\alpha)$ so that for every complex valued multiplicative function $h(m)$ such that*

$$|h(p)| \le B, \quad \sum_{p,\nu\ge 2}\frac{|h(p^\nu)|}{p^{\alpha\nu}} \le C, \quad h(p) = 1 \quad \text{as } p > y \ge y_0,$$

we have uniformly for $z \ge \exp\left\{A^{-1}(B+1)\log y \log_2 y\right\}$, $y \ge y_0$,

$$\sum_{n\le z} h(n) = z \prod_{p\le y}\left(1-\frac{1}{p}\right) \sum_{\nu\ge 0}\frac{h(p^\nu)}{p^\nu} + O\left(z \exp\left\{-A\frac{\log z}{\log y}\right\}\right)$$

and the implied constant depends only on A, B and C.

We have $g(p) = 1$ as $p \geq y, y = \exp\{c_1 \log x / \log_2 x\}$. Then with chosen $A > 0$ we may apply Lemma 2 for $h(n) = g(n)$ as $z \geq \exp\left\{\frac{c_7}{A} \log x\right\}$. We have in this range then

$$(17) \quad G(z) = \sum_{n \leq z} g(n) = z \prod_{p \leq y} \left(1 - \frac{1}{p}\right) \sum_{\nu \geq 0} \frac{g(p^\nu)}{p^\nu} + O\left(z \exp\left\{-A \frac{\log z}{\log y}\right\}\right).$$

It is enough for us that we can use (17) for $x/(\log x)^{2A} \leq z \leq x$ with A as large as it is needed the remaining term in (17) still being $O(x \exp\{-A_1 \log_2 x\})$, $\quad A_1 = A/c_8$.

Using the Cauchy-Buniakowski inequality and the bound for the sum of values of multiplicative function we estimate

$$\left| \sum_{n \leq x/(\log x)^{2A}} n g(n) \right| \leq \left\{ \sum_{n \leq x/(\log x)^{2A}} n^2 \sum_{n \leq x/(\log x)^{2A}} |g(n)|^2 \right\}^{1/2} \ll$$
$$\ll \frac{x^2}{\log^{2A} x} \log^{c_9} x \ll \frac{x^2}{\log^A x}.$$

Hence,

$$(18) \qquad \sum_{n \leq x} n g(n) = \sum_{x/(\log x)^{2A} \leq n \leq x} n g(n) + O\left(\frac{x^2}{\log^A x}\right).$$

Integrating by parts we have now

$$\sum_{x/(\log x)^{2A} \leq n \leq x} n g(n) =$$

$$= \int_{x/(\log x)^{2A}}^{x} z dG(z) = \frac{1}{2} x^2 \prod_{p \leq y} \left(1 - \frac{1}{p}\right) \sum_{\nu \geq 0} \frac{g(p^\nu)}{p^\nu} + O\left(\frac{x^2}{\log^{A_*} x}\right)$$

with $A_* = \min(A, A_1)$.

Using this in (18) we derive from (15) taking into account that the product over primes is bounded by $\log^{c_{10}} x$, we obtain that with an arbitrary constant B

$$(19) \quad S(f, x) = \frac{1}{2} x^2 (\beta - \alpha) \prod_{p \leq y} \left(1 - \frac{1}{p}\right)^2 \sum_{|\nu| \geq 0} \frac{f(p^\nu)}{p^{|\nu|}} + O\left((\beta - \alpha) \frac{x^2}{\log^B x}\right).$$

If we take $h\left(\frac{m}{n}\right) = 1$, then $S(h,x) = \#\mathcal{F}_x$ and (19) is applicable for the function $f = h$. Hence after some routine calculation we have

$$(20)\quad \#\mathcal{F}_x = \frac{1}{2}x^2(\beta-\alpha)\prod_{p\le y}\left(1-\frac{1}{p}\right)^2\left(1-\frac{2}{p+1}\right)^{-1} + O\left((\beta-\alpha)\frac{x^2}{\log^B x}\right).$$

It is now almost straightforward to derive (4) from (19) and (20).

Let now

$$(21)\qquad \sum_{\nu\ge 0}\frac{f(2^\nu)}{2^\nu} = 0.$$

We write the sum over n in (14), then

$$\sum_{n\le x} nf(n^{-1}) \prod_{\substack{p\le y\\(p,n^*)=1}}\sum_{\nu\ge 0}\frac{f(p^\nu)}{p^\nu} = \prod_{2<p\le y}\sum_{\nu\ge 0}\frac{f(p^\nu)}{p^\nu}\sum_{n\le x} ng^*(n),$$

where $g^*(n) = 0$, as $2\nmid n$ and $g^*(n) = g(n)$ as $2|n$ with $g(n)$ defined in (16).

Let

$$S = \sum_{n\le x} ng^*(n).$$

Then

$$(22)\quad S(f,x) = (\beta-\alpha)\left(1-\frac{1}{2}\right)\prod_{2<p\le y}\left(1-\frac{1}{p}\right)\sum_{\nu\ge 0}\frac{f(p^\nu)}{p^\nu}S + O((\beta-\alpha)x^2R).$$

We proceed with the expression

$$(23)\qquad S = \sum_{n\le x} ng^*(n) = \sum_{2\le 2^m\le x} 2^m f\left(2^{-m}\right)\sum_{n\le x/2^m} nh(n)$$

with $h(n)$ being a multiplicative function, $h(2m) = 0$ and $h(2m+1) = g(2m+1)$. The Lemma 2 implies that for $z \ge \exp\left\{\frac{c_7}{A}\log x\right\}$

$$(24)\qquad H(z) =$$

$$= \sum_{n\le z} h(n) = z\left(1-\frac{1}{2}\right)\prod_{3\le p\le y}\left(1-\frac{1}{p}\right)\sum_{\nu\ge 0}\frac{g(p^\nu)}{p^\nu} + O\left(\exp\left\{-A\frac{\log z}{\log y}\right\}\right).$$

As above this asymptotical equality is needed for $z \geq x/(\log x)^{2A}$, the remaining term still being $O(x\exp\{-A_1\log_2 x\})$, $A_1 = \dfrac{A}{c_8}$. We split the sum (23) into two parts:

$$S = S_1 + S_2,$$

$$S_1 = \sum_{2^m \leq (\log x)^{2A}} 2^m f\left(2^{-m}\right) \sum_{n \leq x/2^m} nh(n),$$

$$S_2 = \sum_{(\log x)^{2A} \leq 2^m} 2^m f\left(2^{-m}\right) \sum_{n \leq x/2^m} nh(n).$$

Using the Cauchy-Buniakowski inequality

$$\left|\sum_{n\leq u} nh(n)\right| \leq \left(\sum_{n\leq u} n^2 \cdot \sum_{n\leq u} |h(n)|^2\right)^{1/2} \ll \left(u^3 \cdot u\log^{c_{11}} u\right)^{1/2} \ll u^2(\log u)^{c_{12}},$$

we have

$$S_2 \ll \sum_{(\log x)^{2A} \leq 2^m} 2^m \frac{x^2}{2^{2m}}(\log x)^{c_{12}} \ll \frac{x^2}{\log^A x}.$$

For S_1 we split the range $n \leq x/2^m$ of the inner sums into two intervals: $n \leq x/(\log x)^{2A}$ and $x/(\log x)^{2A} \leq n \leq x/2^m$. Let according to this partition $S_1 = S_{11} + S_{12}$. Using the Cauchy-Buniakowski inequality as above

$$S_{11} \ll \frac{x^2}{(\log x)^A}.$$

Hence

$$S_1 = \sum_{2^m \leq (\log x)^{2A}} 2^m f\left(2^{-m}\right) \sum_{x/(\log x)^{2A} \leq n \leq x/2^m} nh(n) + O\left(\frac{x^2}{\log^A x}\right) =$$

$$= \sum_{2^m \leq (\log x)^{2A}} 2^m f\left(2^{-m}\right) \int\limits_{x/(\log x)^{2A}}^{x/2^m} z\,dH(z) + O\left(\frac{x^2}{\log^A x}\right).$$

Using (24) and integrating by parts we derive

$$S_1 =$$

$$= \frac{1}{2}x^2\left(1-\frac{1}{2}\right)\prod_{3\leq p\leq y}\left(1-\frac{1}{p}\right)\sum_{\nu\geq 0}\frac{g(p^\nu)}{p^\nu}\sum_{2^m\leq(\log x)^{2A}}\frac{f\left(2^{-m}\right)}{2^m} + O\left(\frac{x^2}{\log^A x}\right).$$

If we extend the summation over all m the error introduced will be swallowed by the remainder term. Hence,

$$S_1 = \frac{1}{2}x^2\left(1-\frac{1}{2}\right)\sum_{m\geq 1}\frac{f\left(2^{-m}\right)}{2^m}\prod_{3\leq p\leq y}\left(1-\frac{1}{p}\right)\sum_{\nu\geq 0}\frac{g(p^\nu)}{p^\nu}+O\left(\frac{x^2}{\log^A x}\right),$$

and this asymptotics holds for the whole sum S in (23) as well. Then we derive with this asymptotics after some routine calculations from (25) the equality (4) required for the case (21).

The proof is finally complete.

Proof of Theorem 1. With $y \geq 2$ we define the multiplicative function $f_y(m/n)$ for the powers of primes taking the values

$$f_y(p^\nu) = \begin{cases} f(p^\nu), & \text{if } p < y, \\ 1, & \text{if } p \geq y. \end{cases}$$

With the notation

$$f(p^\nu) = r(p^\nu)\exp\{i\theta(p^\nu)\}, \quad -\pi < \theta(p^\nu) \leq \pi \tag{25}$$

we have

$$f\left(\frac{m}{n}\right) = r\left(\frac{m}{n}\right)\exp\left\{i\theta\left(\frac{m}{n}\right)\right\},$$

where $0 \leq r(m/n) \leq 1$ is a multiplicative, $\theta(m/n)$ an additive function, respectively.

It is our aim to show, that the difference

$$\Delta(y|x) = \frac{1}{\#\mathcal{F}_x}\left|S(f,x)\exp\{-iA(x)\} - S(f_y,x)\exp\{-iA(y,x)\}\right| \tag{26}$$

vanishes as $y \leq x$, and $y \to \infty$. Here

$$A(y,x) = \sum_{\substack{p<y \\ p\in\Pi^*}}\frac{\theta(p^\nu)}{p}, \quad A(x) = \sum_{p\in\Pi^*}\frac{\theta(p^\nu)}{p},$$

$$\Pi^* = \left\{p^\nu : |\nu| = 1 \text{ there exists } \frac{m}{n}\in\mathcal{F}_x, p^\nu \| \frac{m}{n}\right\}.$$

Note, that for the set P^* from the formulation of Theorem 1 $P^* \subset \Pi^*$ holds. We start with the following obvious bound

$$\Delta(y|x) \leq \frac{1}{\#\mathcal{F}_x} \sum_{m/n \in \mathcal{F}_x} \left| \exp\{i(A(x) - A(y,x))\} \prod_{\substack{y \leq p \\ p^\nu \| \frac{m}{n}}} f(p^\nu) - 1 \right|.$$

Using the inequality $|u_1 \ldots u_l - 1| \leq |u_1 - 1| + \ldots + |u_l - 1|$, valid for the complex numbers $|u_j| \leq 1$, the notation

$$\theta_y\left(\frac{m}{n}\right) = \sum_{\substack{y \leq p \\ p^\nu \| \frac{m}{n}}} \theta(p^\nu)$$

and the expression (25) for $f(p^\nu)$ we write

(27)

$$\begin{aligned}\Delta(y|x) \ll & \frac{1}{\#\mathcal{F}_x} \sum_{m/n \in \mathcal{F}_x} \sum_{\substack{y \leq p \\ p^\nu \| \frac{m}{n}}} (1 - r(p^\nu)) + \\ & + \frac{1}{\#\mathcal{F}_x} \sum_{m/n \in \mathcal{F}_x} \left| \exp\left\{ i\theta_y\left(\frac{m}{n}\right) - i(A(x) - A(y,x)) \right\} - 1 \right| = \Delta_1 + \Delta_2.\end{aligned}$$

Changing the order of summation in the first term we obtain

$$\Delta_1 = \frac{1}{\#\mathcal{F}_x} \sum_{y \leq p} \sum_{|\nu| \geq 1} (1 - r(p^\nu)) \# \left\{ \frac{m}{n} : p^\nu \| \frac{m}{n}, \frac{m}{n} \in \mathcal{F}_x \right\}. \tag{28}$$

We proceed with the following bound valid for intervals satisfying conditions (1):

$$\# \left\{ \frac{m}{n} : p^\nu \| \frac{m}{n}, \frac{m}{n} \in \mathcal{F}_x \right\} \ll \# \left\{ \frac{m}{n} : p^{|\nu|} | mn, \frac{m}{n} \in \mathcal{F}_x \right\} \ll \frac{1}{p^{|\nu|}} \#\mathcal{F}_x. \tag{29}$$

This bound is proved, for example, in [6], p.125; for the interested reader we include here the proof.

Let u be a natural number. We consider the set

$$E(x, p^u) = \left\{ \frac{m}{n} \in \mathcal{F}_x : p^u | mn \right\} = \mathcal{D}_1(p^u, x) \cup \mathcal{D}_2(p^u, x),$$

$$\mathcal{D}_1(p^u, x) = \left\{ \frac{m}{n} \in \mathcal{F}_x : p^u | m \right\}, \quad \mathcal{D}_2(p^u, x) = \left\{ \frac{m}{n} \in \mathcal{F}_x : p^u | n \right\}.$$

We have to prove that

$$\#E(x,p^u) \ll \frac{1}{p^{|u|}}\#\mathcal{F}_x. \tag{30}$$

Evidently,

$$\#\mathcal{D}_1(p^u,x) = \left\{\frac{m}{n} : \frac{m}{n} \in \frac{1}{p^u}(\alpha,\beta),\ n \le x,\ (m,n)=(n,p)=1\right\}. \tag{31}$$

For the following we shall use the Lemma ([6], Theorem 1).

Lemma 3. *Let with the real numbers $0 \le \lambda_1 < \lambda_2$, and the natural numbers Q_0, Q_1 and Q_2 having no common prime factors (all numbers may depend on x)*

$$S(x,\lambda_1,\lambda_2,Q_0,Q_1,Q_2) =$$

$$= \#\left\{\frac{m}{n} \in (\lambda_1,\lambda_2) : (m,n)=(m,Q_0Q_1)=(n,Q_0Q_2)=1\right\}.$$

Then uniformly in Q_0, Q_1, Q_2 and $0 \le \lambda_1 < \lambda_2$, we have

$$S(x,\lambda_1,\lambda_2,Q_0,Q_1,Q_2) = \frac{3}{\pi^2}(\lambda_2-\lambda_1)x^2\prod_{p|Q_0}\left(1-\frac{2}{p+1}\right)\times$$

$$\times\prod_{p|Q_1Q_2}\left(1-\frac{1}{p+1}\right)\{1+B_\epsilon R(x,Q)\},$$

$$R(x,Q) = 2^{(2+\epsilon)\omega(Q)}\left(\frac{\log x}{x}+\frac{1}{x(\lambda_2-\lambda_1)}\right),$$

where $Q = Q_0Q_1Q_2$, $\omega(Q)$ denotes the number of distinct prime factors of Q, $\epsilon > 0$ is an arbitrary number, and the quantity B_ϵ is a bounded function with the bound depending only on ϵ.

Using this Lemma for (21) we obtain

$$\#\mathcal{D}_1(p^u,x) = \frac{3}{\pi^2}(\beta-\alpha)\frac{x^2}{p^u}\left(1-\frac{1}{p+1}\right)\left\{1+\frac{B\log x}{x}+\frac{Bp^u}{x(\beta-\alpha)}\right\}.$$

Since $p^u \ll \beta x$ we have $\#\mathcal{D}_1(p^u,x) \ll (\beta-\alpha)x^2p^{-u}$, provided that $\beta/(\beta-\alpha) \ll$ $\ll 1$, or, equivalently, $\alpha/(\beta-\alpha) \ll 1$. Let us now consider the case, where the last condition is not satisfied.

Let $c_{13}(\beta-\alpha) < \alpha < c_{14}(\beta-\alpha)x$ with some positive constants c_{13}, c_{14}. We then have

$$\begin{aligned}\#\mathcal{D}_1(p^u, x) \leq & \sum_{m\leq \beta x/p^u} \sum_{\frac{mp^u}{\beta}\leq n\leq \frac{mp^u}{\alpha}} 1 \leq \\ \leq & \sum_{m\leq \beta x/p^u} \left\{ mp^u \frac{\beta-\alpha}{\alpha\beta} + 1 \right\} \leq \frac{\beta-\alpha}{\alpha\beta} p^u \left(\frac{\beta x}{p^u}\right)^2 + \frac{\beta x}{p^u} = \\ =& (\beta-\alpha)\frac{x^2}{p^u}\frac{\beta}{\alpha} + \frac{\beta x}{p^u} \ll (\beta-\alpha)\frac{x^2}{p^u};\end{aligned}$$

here we have used the inequalities $\beta/\alpha = 1 + (\beta-\alpha)/\alpha < 1 + 1/c_{13}$, and $\beta x = \alpha x + (\beta-\alpha)x \ll (\beta-\alpha)x^2$. Hence,

$$\#\mathcal{D}_1(p^u, x) \ll (\beta-\alpha)\frac{x^2}{p^u}$$

holds provided that $\alpha \ll (\beta-\alpha)x$.

For $\mathcal{D}_2(p^u, x)$ we have

$$\mathcal{D}_2(p^u, x) = \left\{ \frac{m}{n} : \frac{m}{n} \in p^u(\alpha, \beta),\ n \leq \frac{x}{p^u},\ (m, p) = 1 \right\}.$$

Lemma 3 now yields

$$\begin{aligned}\#\mathcal{D}_2(p^u, x) =& \frac{3}{\pi^2}(\beta-\alpha)\frac{x^2}{p^u}\left(1 - \frac{1}{p+1}\right) \times \\ & \times \left\{ 1 + B\frac{p^u \log(x/p^u)}{x} + \frac{1}{x(\beta-\alpha)} \right\} \ll (\beta-\alpha)\frac{x^2}{p^u}.\end{aligned}$$

Using the bounds for $\#\mathcal{D}_1(p^u, x)$, $\#\mathcal{D}_2(p^u, x)$, we obtain (30).

Hence, from (28) and (29) we get

$$\Delta_1 \ll \sum_{y\leq p} \frac{2 - \mathrm{Re}\ f(p) - \mathrm{Re}\ f(p^{-1})}{p} + \sum_{y\leq p} \frac{1}{p^2} = \delta(y), \tag{32}$$

where, due to condition (2), $\delta(y) \to 0$, as $y \to \infty$.

The function θ_y is additive with the values on powers of primes uniformly bounded. Using the inequality $|e^{iu}-1| \le |u|$, and then the Cauchy-Buniakowski inequality, we obtain the bound for Δ_2 in (27)

$$\begin{aligned}\Delta_2 \ll & \frac{1}{\#\mathcal{F}_x} \sum_{m/n\in\mathcal{F}_x} \left|\theta_y\left(\frac{m}{n}\right) - (A(x) - A(y,x))\right| \le \\ & \le \left\{\frac{1}{\#\mathcal{F}_x} \sum_{m/n\in\mathcal{F}_x} \left|\theta_y\left(\frac{m}{n}\right) - (A(x) - A(y,x))\right|^2\right\}^{1/2}.\end{aligned} \tag{33}$$

We proceed with the Kubilius inequality for additive functions, defined for rational numbers (see [8]).

Lemma 4. *Let an additive complex valued function $g(m/n)$ is bounded on powers of primes,*

$$A_g(x) = \sum_{p^\nu\in\Pi^*} \frac{g(p^\nu)}{p}, \quad B_g^2(x) = \sum_{p^\nu\in\Pi^*} \frac{|g(p^\nu)|^2}{p},$$

$$\Pi^* = \left\{p^\nu : |\nu| = 1,\ p^\nu \| \frac{m}{n} \text{ for some } \frac{m}{n} \in \mathcal{F}_x\right\}.$$

Then with the constraints (1) for the intervals (α,β) the following inequality holds

$$\frac{1}{\#\mathcal{F}_x} \sum_{m/n\in\mathcal{F}_x} \left|g\left(\frac{m}{n}\right) - A_g(x)\right|^2 \ll B_g^2(x).$$

Using this inequality in (33) with $g = \theta_y$ we get the bound

$$\Delta_2 \ll \sum_{\substack{y\le p\\ p^\nu\in\Pi^*}} \frac{\theta(p^\nu)^2}{p}.$$

Due to the bound $\theta^2(p^\nu) \ll 1 - \mathrm{Re}\, f(p^\nu)$ (see, for example, this inequality proved in [9], p. 368) and the convergence of (2), we obtain, that Δ_2 vanishes, as $y\to\infty$. From this fact and (27), (32) we have that $\Delta(y|x) \to 0$ as $y\to\infty$.

Then taking into account the form of $\Delta(y|x)$ (see, (26)) we get

$$\frac{1}{\#\mathcal{F}_x} S(f,x) = \exp\{i(A(x) - A(y,x))\} \sum_{m/n\in\mathcal{F}_x} f_y\left(\frac{m}{n}\right) + \mathrm{o}(1) \quad (x\to\infty). \tag{34}$$

As $y = \exp\{c_1 \log x / \log_2 x\}$ we may use for the sum in (34) the asymptotics (4). Hence,

$$\frac{1}{\#\mathcal{F}_x} S(f,x) = \exp\{i(A(x) - A(y,x))\} \prod_{p \le y} \left(1 - \frac{2}{p+1}\right) \sum_{|\nu| \ge 0} \frac{f(p^\nu)}{p^{|\nu|}} + o(1).$$

We are going to replace the factor $\exp\{i(A(x) - A(y,x))\}$ by the appropriate product over primes. We do this as follows:

$$\begin{aligned}
\exp\{i(A(x) - A(y,x))\} &= \exp\left\{ i \sum_{\substack{p^\nu \in \Pi^* \\ y \le p}} \frac{\theta(p^\nu)}{p} \right\} = \\
&= \prod_{\substack{p^\nu \in \Pi^* \\ y \le p}} \left(1 - \frac{1}{p}\right) \left(1 + \frac{f(p^\nu)}{p}\right) L(y,x), \\
L(y,x) &= \exp\left\{ \sum_{\substack{p^\nu \in \Pi^* \\ y \le p < x}} \frac{1 - f(p^\nu) + i\theta(p^\nu)}{p} + O\left(\sum_{p > y} \frac{1}{p^2}\right) \right\}.
\end{aligned}$$

We want to show, that $L(y,x) = 1 + o(1)$. It suffices to prove, that the sum under exponent vanishes, as x grows unboundedly. It does indeed, as the following relations show:

$$\begin{aligned}
\sum_{\substack{p^\nu \in \Pi^* \\ y \le p}} \frac{1 - f(p^\nu) + i\theta(p^\nu)}{p} &= \sum_{\substack{p^\nu \in \Pi^* \\ y \le p}} \frac{1 - r(p^\nu)\mathrm{e}^{i\theta(p^\nu)} + i\theta(p^\nu)}{p} = \\
= \sum_{\substack{p^\nu \in \Pi^* \\ y \le p}} \frac{1 - r(p^\nu)}{p} &+ i \sum_{\substack{p^\nu \in \Pi^* \\ y \le p}} \theta(p^\nu) \frac{1 - r(p^\nu)}{p} + O\left(\sum_{p > y} \frac{\theta^2(p^\nu)}{p}\right).
\end{aligned}$$

The convergence of the series (2) and the inequality $\theta^2(p^\nu) \ll 1 - \mathrm{Re}\, f(p^\nu)$ ensures, that the sum tends to zero, as $x \to \infty$. Now we have

(35)
$$\frac{1}{\#\mathcal{F}_x} S(f,x) = \prod_{\substack{p^\nu \in \Pi^* \\ y \le p}} \left(1 - \frac{1}{p}\right) \left(1 + \frac{f(p^\nu)}{p}\right) \prod_{p \le y} \left(1 - \frac{2}{p+1}\right) \sum_{|\nu| \ge 0} \frac{f(p^\nu)}{p^{|\nu|}} + o(1).$$

For to come to the asymptotics (3) required by the Theorem 1 we need to consider, which primes p or their reciprocals p^{-1} belong to Π^*. Let us show first, that for all $p \le x$ $p^{-1} \in \Pi^*$.

If $m/np \in \mathcal{F}_x$ with $(p, mn) = 1$, then $m/n \in (\alpha p, \beta p)$ and $n \le x/p$. For to count the number of such m/n we use the Lemma 3 again.

According to the Lemma 3 the number of $m/np \in \mathcal{F}_x$ with $(p, mn) = 1$ equals to

$$\frac{3}{\pi^2}(\beta-\alpha)p\frac{x^2}{p^2}\left(1-\frac{2}{p+1}\right)\left\{1+O\left(\frac{\log(x/p)}{(x/p)}+\frac{1}{x(\beta-\alpha)}\right)\right\}.$$

Due to $(\beta-\alpha)x \to \infty$, the value of this expression is positive if $x/p > c_{15}$ with c_{15} large enough. If $x/p \le c_{15}$, then $p \ge x/c_{15}$ and there exists $u/p \in (\alpha, \beta)$, supposed that $(\beta-\alpha)p > 2$. This is evidently true for $x \ge x_0$ because of

$$(\beta-\alpha)p \ge \frac{(\beta-\alpha)x}{c_{15}}, \quad (\beta-\alpha)x \to \infty.$$

We proved then that $p^{-1} \in P^*$ for all $p \le x$.

Let us investigate now which primes p belong to $P^* \subset \Pi^*$. If there exists some $mp/n \in \mathcal{F}_x, (p, mn) = 1$, then $m/n \in (\alpha/p, \beta/p)$ and $n \le x$. By the Lemma 3 we get, that the number of such rationals equals

$$\frac{3}{\pi^2}\frac{\beta-\alpha}{p}x^2\left(1-\frac{2}{p+1}\right)\left\{1+O\left(\frac{\log x}{x}+\frac{p}{x(\beta-\alpha)}\right)\right\}.$$

It is evident, that this term is positive as $p \le c_{16}x(\beta-\alpha)$ with an appropriate constant c_{16}. Hence, factors corresponding to primes $p \le c_{16}x(\beta-\alpha)$ all appear in (35). If we add the factors corresponding to primes in the range $c_{16}x(\beta-\alpha) \le$ $\le p \le x(\beta-\alpha)$, the changes will affect only the remainder term $o(1)$. Then for all p such that $y < p \le \min((\beta-\alpha)x, x)$ the product

$$\left(1-\frac{1}{p}\right)\left(1+\frac{f(p)}{p}\right)\left(1-\frac{1}{p}\right)\left(1+\frac{f(p^{-1})}{p}\right)$$

appears in (35). If we replace this quantity by

$$\left(1-\frac{2}{p+1}\right)\sum_{|\nu|\ge 0}\frac{f(p^\nu)}{p^\nu},$$

only the remainder term changes. This completes the proof of the theorem.

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V. Stakėnas
Department of Mathematics and Informatics
Vilnius University
Naugarduko 24
2006 Vilnius, Lithuania