Annales Univ. Sci. Budapest., Sect. Comp. 22 (2003) 269-285

ON ANALYTIC PROBLEMS FOR ADDITIVE ARITHMETICAL SEMIGROUPS

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On the occasion of the 60th jubilee of Professor Karl-Heinz Indlekofer

Abstract. We examine the asymptotic behaviour of the n-th Taylor coefficient of an analytic in the unit disc function having some product representation. It is assumed that the function has a fixed number of zeros and singularities on the unit circumference. The obtained analytic result is applied in some enumerative problem for an additive arithmetical semigroup and to prove a local limit theorem for an additive function defined on such semigroup.

1. Introduction and results

In most cases the generating series appearing in the abstract analytic number theory or combinatorics have a fairly particular form. So, as the favorite object of Professor Karl-Heinz Indlekofer an *additive arithmetical semigroup* $(\mathbf{G}, \delta)$ (for a *multiset*, in the definitions preferred by statisticians [1], [2] or for an *additive number system*, in the terminology of logicians [3]), we deal with the function

$$Z(y) = \sum_{n\geq 0} g(n)y^n = \prod_{j=1}^{\infty} \left(1 - y^j\right)^{-\pi(j)},$$

where $g(n)$ denotes the number of semigroup elements a of degree $\delta(a) = n$, while $\pi(j)$ stands for the number of prime elements p with $\delta(p) = j$. Typically, the function $Z(y)$ is analytic in a nontrivial disk and have singularities and, sometimes, zeros on the convergence circle. An analytic investigation of the mean value problems for mappings defined on such or other structures leads to more complicated generating functions but also having a product representation. In some neighborhood of the convergence circle, these functions are first compared with a relevant function $Z(y)$ or with its complex powers. The Cauchy formula is applied in the next step. As a rule, the desire to obtain a possibly general mean value theorem encounters the main obstacle: there is no information about the functions outside the convergence disk and only mild conditions can be assumed for the generating functions on the convergence circle. The attempts to get over this obstacle can be seen in [9-13], [6], and other papers. Observe that the remark concerning priorities on page 461 in [6] is not true, apart from the classical Tauber type theorems, the history on asymptotic analysis without information beyond the convergence disc goes back to [12] and even to some previous number-theoretic papers. Now, motivated by the combinatorial applications, we do a comparative analysis of the Taylor coefficients of two functions one of which has a finite number of singularities and zeros on the the unit circumference. In contrast to the very influential paper written by Ph. Flajolet and A.M. Odlyzko [4], we do not assume any condition beyond the unit disc.

The aforementioned powers of $Z(y)$ and of many other generating series of combinatorial structures are representable as a function $V(y) = W(y)H_0(y)$, where

$$W(y) := \prod_{l=0}^{m-1} \left(1 - y\xi_l^{-1}\right)^{-\vartheta_l},$$

where ξ_l are different points on the convergence circle and $\vartheta_l \in \mathbf{C}$ are bounded complex valued quantities, $0 \le l \le m-1$ and $m \ge 1$. Substituting, if necessary, the argument we may assume that $\xi_l = \exp\{2\pi i\varphi_l\}$ and $0 = \varphi_0 < \varphi_1 < \ldots <$ $< \varphi_{m-1} < 1$. Typically, the function $H_0(y)$ is analytic in $|y| < 1$, has no zeros in $|y| \le 1$, and has some smoothness properties on $|y| = 1$. Therefore comparing some other function with $V(y)$ we can use only its factor $W(y)$. Set

$$\log W(y) = \sum_{j=1}^{\infty} \frac{\gamma_j}{j} y^j, \qquad \gamma_j = \sum_{l=0}^{m-1} \vartheta_l \xi_l^{-j}, \quad |y| < 1.$$

Let the other generating function have the following expression

$$(1.1) \qquad F(y) = \sum_{n=1}^{\infty} f_n y^n = H(y) \exp\left\{L(y)\right\} =: \sum_{k=1}^{\infty} h_k y^k \exp\left\{\sum_{j=1}^{\infty} \frac{a_j}{j} y^j\right\}$$

with analytic in $|y| < 1$ functions $H(y)$ and $L(y)$. The function $H(y)$ will also satisfy some smoothness conditions on $|y| = 1$. The function $F(y)$ can be a bivariate generating function or depend on other parameters. We will check uniformity with respect to them in our estimates.

Let in what follows $r(u)$ be a monotone decreasing function on $[1,\infty)$ such that $r(u)/u$ is integrable. For $1 \le q \le q_0$, we set

$$\rho(u) := \sum_{j \le u} q^j \left(a_j - \gamma_j\right).$$

The main condition which allows us to find an asymptotic formula for f_n as $n \to \infty$ is the bound

$$|\rho(u)| \le q^u r(u), \qquad \int_1^\infty \frac{r(u)}{u} du \le C < \infty. \tag{1.2}$$

Denote

$$G(y) = L(y) - \log W(y), \qquad H_1(y) = H(y) \exp\{G(y)\},$$

and

$$W_k(y) = (1 - y\xi_k^{-1})^{\vartheta_k} W(y) = \prod_{\substack{0 \le l \le m-1 \\ l \ne k}} \left(1 - y\xi_l^{-1}\right)^{-\vartheta_l}.$$

Set

$$R(n) := \max\left\{ r(n),\ \frac{1}{n},\ \frac{1}{n}\int_1^n r(u)du,\ \int_n^\infty \frac{r(u)}{u} du \right\}$$

and $a^+ = \mathbf{1}\{a \ge 0\}a$. In what follows we will assume that

$$\max_{0 \le k \le m-1} |\vartheta_k| \le C_1 \tag{1.3}$$

and

$$\min\left\{ \min_{0 \le k \le m-2} (\varphi_{k+1} - \varphi_k), 1 - \varphi_{m-1} \right\} \ge c_1 > 0. \tag{1.4}$$

We now formulate the main analytic result.

Theorem 1. *Let the function $F(y)$ have the above described form (1.1) with a function $H(y)$, analytic in $|y| < 1$, continuous in $|y| \le 1$, and such that*

$$\sum_{k=1}^\infty k|h_k| \le C_2. \tag{1.5}$$

If conditions (1.2), (1.3), and (1.4) are satisfied, then

$$f_n = \sum_{k=0}^{m-1} \frac{H_1(\xi_k)W_k(\xi_k)}{\Gamma(\vartheta_k)} \xi_k^{-n} n^{\vartheta_k - 1} + BR^*(n), \tag{1.6}$$

where $\Gamma(z)$ *denotes the Euler gamma-function with the agreement that* $\Gamma(z)^{-1} = 0$ *for* $z = 0, 1, \dots$ *and*

$$R^*(n) = R(n) \max_k \Big\{ n^{(\Re\vartheta_k - 1)^+} \min \big\{ \log n, |1 - \Re\vartheta_k|^{-1} \big\} \Big\}.$$

Here and in what follows B *denotes a quantity bounded by a constant depending on* q_0, C, C_1, C_2, c_1, *and* m *only.*

Stressing advantages of our approach, we note that condition (1.2) allows us to avoid the individual requirements put on $|a_k - \gamma_k|$ for each $k \geq 1$ used, for instance, in [6]. Moreover, we can consider the cases when $r(u)$ is a rather slowly decreasing function, say, $r(u) = (\log(u+1))^{-2-\varepsilon}$, $\varepsilon > 0$. If $m = 1$ and $r(u) = u^{-c}$ with $c > 0$, an asymptotic formula for f_n is given on page 465 of Hwang's paper [6]. Nevertheless, seeking for generality, we are failing in the expression of the remainder R^*. Sometimes it swallows the main term.

The condition (1.2) naturally appears if the sequence $\{a_j\}$ in the definition of $F(y)$, it satisfies some arithmetical constraint. For instance, we have the following corollary of Theorem 1.

Theorem 2. *Assume that (1.2) is changed by*

$$\sum_{s=0}^{m-1} \sum_{\substack{1 \leq j \leq u \\ j \equiv s(m)}} q^j \big(a_j - \beta_s\big) =: \tilde{\rho}(u), \qquad |\tilde{\rho}(u)| \leq q^u r(u) \tag{1.7}$$

with some $\beta_s \in \mathbf{C}$ *and the same condition on* $r(u)$. *Let other conditions of Theorem 1 be satisfied. Then formula (1.6) holds with* $\xi_k = \zeta^k := \exp\{2\pi i k/m\}$ *and*

$$\vartheta_k := \frac{1}{m} \sum_{r=0}^{m-1} \beta_r \zeta^{kr}.$$

Theorem 2 generalizes the main result of our remark [13]. It can be applied to solve the converse problems of additive arithmetical semigroups. By this we mean that the number $g(n)$ of semigroup elements of degree n is sought when an asymptotical formula for the number $\pi(j)$ of prime or generating elements is known *a priori*. Such problems are very common for weighted multisets (see

[1], [2], [13]). For this purpose we examine the generating function $Z(y)$. Check that formally

$$Z(y) = \exp\left\{ \sum_{k=1}^{\infty} \frac{\Pi(y^k)}{k} \right\} = e^{\Pi(y)} K(y),$$

where

$$\Pi(y) = \sum_{j=1}^{\infty} \pi(j) y^j, \qquad K(y) = \exp\left\{ \sum_{k=2}^{\infty} \frac{\Pi(y^k)}{k} \right\}.$$

As a corollary of Theorem 2 we have

Theorem 3. *Assume that for an additive arithmetical semigroup the number $\pi(j)$ of prime elements of degree j satisfies the relation*

$$\left| \sum_{s=0}^{m-1} \sum_{\substack{1 \le j \le u \\ j \equiv s (m)}} q^j \left(j\pi(j) q^{-j} - \pi_s \right) \right| \le q^u r(u) \tag{1.8}$$

for some $q > 1$, $m \ge 1$, $\pi_s \ge 0$, $0 \le s \le m-1$, and a function $r(u)$ having the above mentioned properties. Then the assertion of Theorem 2 is true for $f_n = q^{-n} g(n)$ if we substitute $\beta_s = \pi_s$, $a_j = j\pi(j)q^{-j}$, $L(y) = \Pi(q^{-1}y)$, and $H(y) = K(q^{-1}y)$.

Considering the most popular cases (see [7], [8]) we have, for instance, the following result.

Corollary. *Let $I(\mathbf{G}) \in \{0, 1\}$ and*

$$\left| \sum_{j \le u} q^j \left(j\pi(j) q^{-j} - \big(1 - I(\mathbf{G})(-1)^j\big) \right) \right| \le C_3 q^u \log^{-3}(u+1) \tag{1.9}$$

for some $q > 1$ and $C_3 > 0$. Then

$$g(n) = Aq^n + Bq^n \log^{-1} n \tag{1.10}$$

with

$$A = \big(1 + I(\mathbf{G})\big) K(q^{-1}) \exp\left\{ \sum_{j=1}^{\infty} j^{-1} \left(j\pi(j)q^{-j} - \big(1 - I(\mathbf{G})(-1)^j\big) \right) \right\} =$$

$$= (1 + I(\mathbf{G})) \prod_{j=1}^{\infty} \left[\left(1 - \frac{1}{q^j}\right)^{-\pi(j)} e^{-\left(1 - I(\mathbf{G})(-1)^j\right)/j} \right] > 0.$$

Conversely, to obtain an asymptotic formula for $\pi(n)$, one needs (1.10) with the remainder $Bq^n n^{-2-\varepsilon}$ (see Theorem 5.4.1 in [8]). In [18] and some earlier papers W.-B. Zhang uses conditions involving asymptotical expressions

$$q^{-n}g(n)=\sum_{r=1}^{s}A_r n^{\eta_r-1}+\text{Remainder},$$

where $A_s\in\mathbf{R}$, $A_s>0$ and $0\le\eta_1<\cdots<\eta_s$ with $\eta_s\ge 1$. Observe that such type of formulas does not exhaust all possibilities. Theorem 3 shows that in the asymptotic formulas for $g(n)$ terms like $a(-1)^n q^n n^{\eta-1}$ with $a\ne 0$, $\eta\ge 1$, and having the alternating factor $(-1)^n$ can appear as well. To see this, take, for instance, $m=2$ together with $\pi_0=3$ and $\pi_1=1$ and calculate the terms in (1.6). Finally, the possible different asymptotic expressions for $\pi(n)$ show also that K.-H.Indlekofer's [7] analytic axioms $\mathcal{A}_1$ or $\mathcal{A}_2$ on the generating series $Z_1(y)$ can be further generalized. One can, for instance, assume that $Z(y)$ has general factors like $W(y)$ defined above and to derive new prime element theorems.

Stressing the uniformity with respect to possible parameters in the estimates of the remainder terms, we now demonstrate that Theorem 1 is applicable to prove limit theorems for additive functions defined on an additive arithmetic semigroup $(\mathbf{G},\delta)$ generated by a countable set $\mathbf{P}$ of prime elements. We now can do this under rather mild conditions. In general the asymptotic formulas are rather complicated therefore we confine ourselves with the semigroups discussed in Corollary above. We examine the asymptotic local value distribution for an additive function $h:\mathbf{G}\to\mathbf{Z}$ having regular behavior on the prime elements.

Definition. We say that an additive function $h:\mathbf{G}\to\mathbf{Z}$ belongs to the class $\mathcal{A}(\mathbf{G})$ if

$$\sum_{\substack{p\in\mathbf{P},\delta(p)=j\\ h(p)=l}}1=:\pi(j)\big(\lambda_l+\rho_l(j)\big),\quad l\in\mathbf{Z},\ j\ge 1,$$

where $\lambda_l\in[0,1]$ are constants and the remainder terms $\rho_l(j)=:C_l(j)\big(\log(j+ +1)\big)^{-3}$ satisfy the condition

$$\sum_l|C_l(j)|<\infty$$

uniformly in $j\ge 1$.

The properties of the lattice distribution with the characteristic function

$$\kappa(t)=\sum_l\lambda_l e^{ilt}$$

play an essential role. If the maximal step is not one, apart from $t=0$, in the interval $(-\pi,\pi]$ we have finitely many points t_0 and t_1 such that $\kappa(t_0)=1$ and $\kappa(t_1)=-1$. They are solutions of the equations

$$\sum_l \lambda_l \sin^2(lt/2)=0, \qquad \sum_l \lambda_l \cos^2(l\tau/2)=0, \tag{1.11}$$

respectively. Let further,

$$E=\sum_l l\lambda_l, \qquad \sigma^2=\sum_l l^2\lambda_l, \qquad \beta=\sum_l |l|^3\lambda_l,$$

provided that they exist, $x=(m-E\log n)/\sigma\sqrt{\log n}$ and $\varphi(u)$ be the density of the standard normal law.

Denote $||a||=q^{\delta(a)}$ for $a\in\mathbf{G}$ and

$$S_s(m;h)=\sum_{t_s} e^{-it_s m}\prod_{p\in\mathbf{P}}\left(1-\frac{(-1)^{s\delta(p)}}{||p||}\right)^{(-1)^s}\sum_{\alpha=0}^{\infty}\frac{(-1)^{s\alpha\delta(p)}\exp\{it_s h(p^\alpha)\}}{||p||^\alpha},$$

where $m\in\mathbf{Z}$, $s\in\{0,1\}$, and t_s run through the set of solutions of (1.11), respectively. If $I(\mathbf{G})=1$, we also set

$$A_1=2\prod_{j=1}^{\infty}\left(1-\frac{(-1)^j}{q^j}\right)^{\pi(j)} e^{-\left(1-(-1)^j\right)/j}.$$

Theorem 4. *Let the semigroup* $(\mathbf{G},\delta)$ *satisfy the prime element law* (1.9). *If* $h\in\mathcal{A}(\mathbf{G})$, $\lambda_0<1$, *and the series*

$$\sum_l |l|^3\lambda_l, \quad \sum_{p,j\geq 2}|h(p^j)|q^{-j\delta(p)}, \quad \sum_l |l||C_l(k)| \tag{1.12}$$

converge (the last one uniformly in $k\geq 1$*), then*

$$\nu_n(m):=\frac{1}{Aq^n}\#\{a\in\mathbf{G};\quad \delta(a)=n,\, h(a)=m\}=$$
$$=\frac{\varphi(x)}{\sigma\sqrt{\log n}}\Big(S_1(m;h)+(-1)^n I(\mathbf{G})A_1A^{-1}S_2(m;h)\Big)+\frac{B}{\log n}$$

as $n\to\infty$.

The zone of non-triviality of the asymptotic formula for $\nu_n(m)$ is determined by the function $\varphi(x)$ used in the main term and the remainder. In fact, it agrees with

$$|m - E\log n| \le (1 + o(1))\sigma\sqrt{(\log n)\log\log n}.$$

W.-B. Zhang has announced a large deviation theorem for the functions h with $h(p) \in \{0, 1\}$ under mild conditions on the semigroup $(\mathbf{G}, \delta)$. If $h \in \mathcal{A}(\mathbf{G})$ and the distribution in Definition has exponential moments, the large deviation theorems can also be proved by our approach. Under classical Axiom $A^{\#}$ asserting that $g(n) = Aq^n + Bq^{\nu n}$ with $q > 1$, $A > 0$ and $\nu < 1$, more results can be found in [15], [17]. The second author [16] has started to examine the local distributions for multiplicative functions. K.-H. Hwang's deep investigations [5], [6] concern only the counting functions of prime divisors and require strong conditions on the semigroup.

Corollary. *Let $h(a) = \Omega(a)$ be the number of all prime divisors of $a \in \mathbf{G}$. Then*

$$\nu_n(m) = \frac{\varphi(y)}{\lambda\sigma}\left\{1 + (-1)^{n+m} I(G) \prod_{p\in\mathbf{P}} \left(1 + \frac{1}{||p||}\right)^{-1} \left(1 - \frac{(-1)^{\delta(p)}}{||p||}\right)\right\} + \frac{B}{\lambda^2}.$$

When $I(G) = 1$, by virtue of (1.9) the product over primes is a nonzero constant. It is worthwhile to stress that the zeros of the generating series $Z(y)$ on the convergence circumference have influence to the main terms of the local probabilities.

2. Proofs of Theorems 1, 2, and 3

We will use the Cauchy integral formula

$$f_n = \frac{1}{2\pi i} \int\limits_{|y|=r} \frac{F(y)}{y^{n+1}} dy, \tag{2.1}$$

where $r = \max\{q, e\}^{-1/n}$. All neighborhoods of the points ξ_k, $0 \le k \le m - 1$ will contribute to the main term of the integral. Let $\tau = (\arg y)/2\pi$, $0 < \varepsilon <$ $< c_1/2$, $\Delta_k = \big\{y : |y| = r,\ \ |\tau - \varphi_k| \le \varepsilon\big\}$ for $k = 1, \ldots, m - 1$,

$$\Delta_0 = \big\{y : |y| = r,\quad 0 \le \tau \le \varepsilon\big\} \cup \big\{y : |y| = r,\ \ 1 - \varepsilon \le \tau \le 1\big\},$$

and

$$\Delta = \{y : |y| = r\} \setminus \bigcup_{k=0}^{m-1} \Delta_k.$$

We first investigate the asymptotic behavior of the integrand $F(y)$ in (2.1).

Lemma 1. *If condition (1.2) is satisfied, then*

(i) the function $G(y)$ is analytic in $|y| < 1$;

(ii) $G(y) = B$ in $|y| \le 1$ and $G'(y) = BnR(n)$ for $|y| = r$;

(iii) $\exp\{G(y)\} = \exp\{G(\xi_k)\}\big(1 + BnR(n)|y - \xi_k|\big)$ for $y \in \Delta_k$ and $0 \le \le k \le m-1$;

(iv) if, in addition, (1.5) holds, then $H_1(y) = H_1(\xi_k) + BnR(n)|y - \xi_k|$ for $y \in \Delta_k$ and $0 \le k \le m-1$.

Proof. Summation by parts yields

$$G(y) = L(y) - \log W(y) = \left(\log \frac{q}{y}\right) \int\limits_1^\infty \rho(u) \frac{y^u du}{q^u u} + \int\limits_1^\infty \rho(u) \frac{y^u du}{q^u u^2}$$

if $|y| \le 1$. Condition (1.2) shows that the integrals converge uniformly in $|y| \le 1$. This proves (i) and the first part of (ii). Further we introduce

$$I_{1n}(y) = \int\limits_1^n \rho(u) \frac{y^u du}{q^u u}, \qquad I_{2n}(y) = \int\limits_1^n \rho(u) \frac{y^u du}{q^u u^2},$$

and $G_n(y) := \log(q/y) I_{1n}(y) + I_{2n}(y)$. Now by (1.2),

$$G(y) = G_n(y) + B \int\limits_n^\infty \frac{r(u)}{u} e^{-u/n} du = G_n(u) + Br(n) \tag{2.2}$$

if $|y| \le 1$ and

$$G'(y) = G_n'(y) + B \int\limits_n^\infty r(u) e^{-u/n} du = G_n'(u) + Bnr(n) =$$

$$= B \int\limits_1^n r(u) du + Bnr(n) = BnR(n)$$

if $|y| = r$. If $y \in \Delta_k$, then
(2.3)

$$G_n(y) - G_n(\xi_k) =$$
$$= \left(\log \frac{\xi_k}{y}\right) I_{1n}(y) + \left(\log \frac{q}{\xi_k}\right) \big(I_{1n}(y) - I_{1n}(\xi_k)\big) + \big(I_{2n}(y) - I_{2n}(\xi_k)\big) =$$
$$= B|y - \xi_k| + B|y - \xi_k| \int\limits_1^n r(u)du = BnR(n)|y - \xi_k|.$$

Here we have used the estimate $1 - z^u = Bu|1 - z|$ if $|1 - z| \le 1/2$ and the inequality $n|y - \xi_k| \ge 1$ for $y \in \Delta_k$. Estimates (2.2) and (2.3) imply

$$\begin{aligned}\exp\{G(y)\} &= \exp\{G_n(y)\}\big(1 + Br(n)\big) = \\ &= \exp\{G_n(\xi_k)\}\big(1 + B|G_n(y) - G_n(\xi_k)|\big)\big(1 + Br(n)\big) = \\ &= \exp\{G_n(\xi_k)\}\big(1 + BnR(n)|y - \xi_k|\big)\end{aligned}$$

if $y \in \Delta_k$ and $k = 0, \dots, m-1$. Since by (2.2) we have $G(\xi_k) - G_n(\xi_k) = Br(n)$, the last estimate can be rewritten so as formulated in (iii). The estimate (iv) follows from (iii) and condition (1.5).

Lemma 1 is proved.

Denote

$$D_k(y) = F(y) - \frac{W_k(\xi_k)H_1(\xi_k)}{(1 - y\xi_k^{-1})^{\vartheta_k}} = \frac{1}{(1 - y\xi_k^{-1})^{\vartheta_k}} \big(W_k(y)H_1(y) - W_k(\xi_k)H_1(\xi_k)\big).$$

Lemma 2. *Let conditions (1.2) and (1.5) be satisfied and* $0 \le k \le m-1$. *For* $y \in \Delta_k$, *we have*

$$D_k(y) = BnR(n)|y - \xi_k|^{1 - \Re\vartheta_k} = BnR(n)n^{(\Re\vartheta - 1)^+}$$

and

$$D_k'(y) = BnR(n)|y - \xi_k|^{-\Re\vartheta_k}.$$

For $y \in \Delta$ *we have* $D_k(y) = B$ *and* $D_k'(y) = BnR(n)$.

Proof. Since $W_k(y)$ is analytic at the point $y = \xi_k$ and $|y - \xi_k| \ge 1/n$ if $y \in \Delta_k$, the first assertion follows from (iv) of Lemma 1. The same argument applies for the derivative of $D_k(y)$ if $y \in \Delta_k$. The last assertions of Lemma 2 follow from condition (1.5) and (ii) of Lemma 1.

Integration of integral (2.1). We split it into a few parts. Using the estimates of $D_k(y)$ obtained in Lemma 2 for the region Δ, we have

$$\int\limits_{y\in\Delta} \frac{F(y)dy}{y^{n+1}} = \frac{B}{n}\max_{y\in\Delta}|F(y)| + \frac{B}{n}\int\limits_{y\in\Delta}|F'(y)||dy| = \frac{B}{n} + \frac{B}{n}nR(n) = BR(n). \tag{2.4}$$

By the definition of $D_k(y)$,

$$\begin{aligned} J_k := \int\limits_{y\in\Delta_k} \frac{F(y)}{y^{n+1}}dy = W_k(\xi_k)H_1(\xi_k)\int\limits_{y\in\Delta_k}\frac{dy}{y^{n+1}(1-y\xi_k^{-1})^{\vartheta_k}} + \\ + \int\limits_{y\in\Delta_k}\frac{D_k(y)dy}{y^{n+1}} =: W_k(\xi_k)H_1(\xi_k)J_{1k} + J_{2k}. \end{aligned} \tag{2.5}$$

Using also Lemma 2 and summing by parts, we obtain

$$\begin{aligned} J_{2k} &= \frac{B}{n}\max_{y\in\Delta_k}|D_k(y)| + \frac{B}{n}\int\limits_{y\in\Delta_k}|D_k'(y)||dy| = BR(n)n^{(\Re\vartheta-1)^+} + \\ &+ BR(n)\int\limits_{y\in\Delta_k}|y-\xi_k|^{-\Re\vartheta_k}|dy| = BR(n)n^{(\Re\vartheta-1)^+} + \\ &+ BR(n)n^{\Re\vartheta_k-1}\int\limits_1^{\varepsilon n}\frac{dt}{(\log^2\max\{q,e\}+t^2)^{\Re\vartheta_k/2}} = \\ &= BR(n)n^{(\Re\vartheta-1)^+}\min\{\log n, |1-\Re\vartheta_k|^{-1}\}. \end{aligned}$$

By the well known estimates

$$\begin{aligned} J_{1k} &= 2\pi i\xi_k^{-n}\binom{n+\vartheta_k-1}{n} - \int\limits_{|y|=r,\, y\notin\Delta_k}\frac{dy}{y^{n+1}(1-y\xi_k^{-1})^{\vartheta_k}} = \\ &= 2\pi i\xi_k^{-n}\frac{n^{\vartheta_k-1}}{\Gamma(\vartheta_k)} + Bn^{\Re\vartheta_k-2} + Bn^{-1}. \end{aligned}$$

Inserting the estimates of J_{1k} and J_{2k} into (2.5), we obtain the desired asymptotic formula for J_k. Further, (2.5), (2.4), and (2.1) imply the statement of Theorem 1.

Proof of Theorem 2. If ϑ_k is defined in the formulation of Theorem 2 and γ_k is defined as previously, then, for $k \equiv s$ (mod m),

$$\gamma_k = \sum_{l=0}^{m-1} \zeta^{-kl} \left(\frac{1}{m} \sum_{r=0}^{m-1} \beta_r \zeta^{lr} \right) = \sum_{r=0}^{m-1} \beta_r \left(\frac{1}{m} \sum_{l=0}^{m-1} \exp\left\{ 2\pi i \frac{l(r-s)}{m} \right\} \right) = \beta_s.$$

Hence $\rho(u) = \tilde{\rho}(u)$ and Theorem 2 follows from Theorem 1.

Proof of Theorem 3. We apply Theorem 2 for $F(y) = Z(q^{-1}y)$ and use the above notation. For $|y| \le 1$, by condition (1.8), we have

$$|K'(y)| = \left| \sum_{k=2}^{\infty} \sum_{j=1}^{\infty} j\pi(j) q^{-jk} y^{jk-1} \right| \le \sum_{j=1}^{\infty} j\pi(j) q^{-2j} \le$$

$$\le \log q \int_1^{\infty} q^{-u} \sum_{j \le u} j\pi(j) q^{-j}\, du = B \int_1^{\infty} (u^2 + r(u)) q^{-u}\, du = B.$$

Thus the conditions of Theorem 2 are satisfied and the desired result follows.

3. Proof of Theorem 4

First, we verify that Theorem 1 yields an asymptotic formula for the characteristic function

$$\psi_n(t) = \frac{1}{Aq^n} \sum_{a \in \mathbf{G}, \delta(a)=n} e^{ith(a)}, \quad t \in \mathbf{R}.$$

Indeed, set $g(a) = e^{ith(a)}$ and, for $|y| < 1$,

$$F(y,t) = A \sum_{n=0}^{\infty} \psi_n(t) y^n =$$

$$= \sum_{a \in \mathbf{G}} g(a)(q^{-1}y)^{\delta(a)} = \exp\left\{ \sum_{j=1}^{\infty} \left(q^{-j} \sum_{\delta(p)=j} g(p) \right) y^j \right\} H(y,t),$$

where

$$H(y,t) = \prod_{p \in \mathbf{P}} \left(1 + \sum_{\alpha=1}^{\infty} \frac{g(p^\alpha)}{||p||^\alpha} y^{\alpha\delta(p)} \right) \exp\left\{ -\frac{g(p)}{||p||} y^{\delta(p)} \right\}.$$

Since $q>1$, in a routine way we can verify that $H(y)=H(y,t)$ is defined in $|y|\le 1$ where it is continuous and satisfies condition (1.5) uniformly in $t\in\mathbf{R}$. Now using Definition of $\mathcal{A}(\mathbf{G})$, we have the following estimate for

(3.1)
$$\frac{a(j)}{j}:=q^{-j}\sum_{\delta(p)=j}g(p)=q^{-j}\sum_{l\in\mathbf{Z}}e^{itl}\sum_{\substack{\delta(p)=j\\ h(p)=l}}1=q^{-j}\pi(j)\sum_{l\in\mathbf{Z}}e^{itl}\big(\lambda_l+\rho_l(j)\big)=$$
$$=q^{-j}\pi(j)\big(\kappa(t)+B\log^{-3}(j+1)\big).$$

This and (1.9) imply (1.2) with $r(u)=\log^{-3}(u+1)$ uniformly in $t\in\mathbf{R}$. Thus,

$$F(y,t)=\frac{(1+y)^{I(\mathbf{G})\kappa(t)}}{(1-y)^{\kappa(t)}}e^{G(y,t)}H(y,t)$$

with

$$G(y,t)=\sum_{j=1}^{\infty}\frac{1}{j}\bigg(jq^{-j}\sum_{\delta(p)=j}e^{ith(p)}-\kappa(t)\big(1-I(\mathbf{G})(-1)^j\big)\bigg)y^j,\quad |y|\le 1,$$

and Theorem 1 is applicable. It yields

$$A\psi_n(t)=2^{I(\mathbf{G})\kappa(t)}(\Gamma(\kappa(t)))^{-1}e^{G(1,t)}H(1,t)n^{\kappa(t)-1}+$$
$$+I(\mathbf{G})2^{-\kappa(t)}(\Gamma(-\kappa(t)))^{-1}e^{G(-1,t)}H(-1,t)(-1)^n n^{-\kappa(t)-1}+\frac{B}{\log n}=:$$
$$=:f_0(t)n^{\kappa(t)-1}+I(\mathbf{G})f_1(t)(-1)^n n^{-\kappa(t)-1}+\frac{B}{\log n}$$

uniformly in $t\in\mathbf{R}$. Hence

(3.2)
$$\nu_n(m)=$$
$$=\frac{1}{2\pi A}\int_{-\pi}^{\pi}e^{-itm}n^{\kappa(t)-1}f_0(t)\,dt+\frac{1}{2\pi A}\int_{-\pi}^{\pi}e^{-itm}n^{-\kappa(t)-1}f_1(t)\,dt+\frac{B}{\log n}=:$$
$$=:J_0+J_1+\frac{B}{\log n}.$$

Such rather typical integral appeared already in [14]. We can use the same argument.

We further consider two cases corresponding to $s=0$ and $s=1$. Let $0<\varepsilon<1$ be a small number such that the intervals $D(t_s):=\{t\in(-\pi,\pi]:$

$|t-t_s|<\varepsilon\}$, as t_s runs through the solutions of the corresponding equation of (11), do not have common points. Here, in the case $t_s=\pi$, the corresponding interval is understood as $(-\pi,-\pi+\varepsilon)\cup(\pi-\varepsilon,\pi]$. As in (3.1), the convergence of the third series of (1.12) assures $G(1,t)=G(1,t_s)+B|t-t_s|$ for $t\in D(t_s)$. More complicated but routine calculations based upon the second series of (1.12) yield $H((-1)^s,t)=H((-1)^s,t_s)+B|t-t_s|$ for $t\in D(t_s)$. Similarly, in the same interval,

$$2^{(-1)^s I(\mathbf{G})}\big(\Gamma((-1)^s\kappa(t))\big)^{-1}=2^{(-1)^s I(\mathbf{G})}+B|t-t_s|.$$

So, we obtain

$$f_s(t)=f_s(t_s)+B|t-t_s|,\quad t\in D(t_s). \tag{3.3}$$

Moreover, if

$$t\in D_s:=(-\pi,\pi]\setminus\bigcup_{t_s}D(t_s),$$

then $f_s(t)=B$.

From the properties of $\kappa(t)$ we have

$$n^{\kappa(t+t_0)-1}=n^{\kappa(t)-1}=\big(1+B|t|^3\log n\big)n^{iEt-\sigma^2t^2/2}$$

if $|t|\le\eta:=C_3\big((\log\log n)/\log n\big)^{1/2}$ and C_3 is a positive constant (later to be chosen sufficiently large). If ε is sufficiently small and $|t|\le\varepsilon$, then $n^{\kappa(t+t_0)-1}= =Bn^{-c_2t^2}$ with $c_2=c_2(\varepsilon,\sigma,\beta)>0$. In $t\in D_0$, we even have $n^{\kappa(t)-1}=Bn^{-c_3}$ with some $c_3>0$. These estimates and (3.3) yield

$$J_0=\frac{1}{2\pi A}\left(\sum_{t_0}\int\limits_{D(t_0)}+\int\limits_{D_0}\right)e^{-itm}n^{\kappa(t)-1}f_0(t)dt==\frac{1}{2\pi A\sigma\sqrt{\log n}}\times$$
$$\times\sum_{t_0}e^{-it_0m}\int\limits_{|v|\le\eta\sigma\sqrt{\log n}}e^{-ivx-v^2/2}\left(f_0(t_o)+\frac{B(|v|^3+|v|)}{\sqrt{\log n}}\right)dv+$$
$$+B\int\limits_{\eta\le|t|\le\varepsilon}n^{-c_2t^2}\,dt+Bn^{-c_3}=\frac{\varphi(x)}{\sigma\sqrt{\log n}}\sum_{t_0}e^{-it_0m}f_0(t_0)+\frac{B}{\log n}. \tag{3.4}$$

Similarly, integrating J_1, we observe that $-\kappa(t+t_1) = \kappa(t)$ therefore the typical integral in the corresponding sum over t_1 equals

$$\frac{1}{2\pi A}\int\limits_{D(t_1)} e^{-itm}n^{-\kappa(t)-1}f_1(t)\,dt =$$

$$=\frac{e^{-it_1m}}{2\pi}\int\limits_{|t|\le\varepsilon} e^{-itm}n^{\kappa(t)-1}(f_1(t_1)+B|t|)\,dt =$$

$$=\frac{\varphi(x)}{\sigma\sqrt{\log n}}e^{-it_1m}f_1(t_1)+\frac{B}{\log n}.$$

The contribution of the integral over D_1 is also Bn^{-c_2}. The sum of these estimates, (3.4) and (3.2) yield

$$\nu_n(m) = \frac{\varphi(x)}{A\sigma\sqrt{\log n}}\left(\sum_{t_0} e^{-it_0m}f_0(t_0) + I(\mathbf{G})(-1)^n\sum_{t_1} e^{-it_1m}f_1(t_1)\right) + \frac{B}{\log n}.$$

The expressions of the constants A and A_1 above show that this is just another form of the desired result.

Theorem 4 is proved.

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