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A DISCRETE RETRIAL SYSTEM WITH UNIFORMLY DISTRIBUTED SERVICE TIME

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Dedicated to Professor Karl-Heinz Indlekofer on his sixtieth birthday

Abstract. We consider a discrete retrial system with geometrically distributed interarrival and uniformly distributed service time. The system is examined by means of the embedded Markov chain technique, we find the transition probabilities, the ergodic distribution and come to a natural condition of existence of equilibrium. The paper is the discrete generalization of [7].

1. Introduction

It is well known that a telephone subsciber who obtains a busy signal usually repeats the call until the required connection is made. Such feature plays important role in several computer and communication networks, other applications include aircrafts waiting to land or retail shoppers. For similar models a class of queueing systems, systems with repeated calls (retrial queues, queues with returning customers) was introduced. To such systems is devoted a book by Falin and Templeton [1]. The attention is concentrated on systems with random time of repetition of requests for service. Some aspects of such systems are considered e.g. in [9-10], where a software tool MOSEL is given to formulate and solve similar problems.

In [6] we introduced a special (so-called cyclic-waiting) retrial system with

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Poisson arrivals and exponentially distributed service time. The feature of this system was that the service of a customer could start at the moment of arrival or at moments differing from it by the multiples of a given cycle time T assuming the FIFO rule. So differing from the models of [1] the repetition time was fixed. The service process did not run continuously, there appeared idle periods from the termination of one customer till the beginning of service of the following one. In [8] similar model was considered in discrete time case assuming geometrically distributed interarrival and service time distributions, and in [7] the continuous model was generalized for the case of uniform service time distribution. The condition of existence of ergodic distribution for some more general cases was examined by Koba [3-5], the simulation investigation of the discrete time model was done by Farkas and Kárász [2].

2. The result

Here our aim is to consider the discrete time model in the case of geometrically distributed interarrival and uniformly distributed service times, i.e. to examine the discrete analogue of [7].

So, let us fix a cycle time T and consider a discrete queueing system where the service time may uniformly change on the interval $[c,d]$ (for the sake of simplicity c and d are assumed to be the multiples of T) and T is divided into n equal parts, i.e. we altogether have $\frac{d-c}{T}n$ time slices. The probability to finish the service of a customer on such a time slice is $q=\frac{T}{n(d-c)}$. For a time slice only one new customer may enter the system, the probability of this event is $1-r$, so there is no appearence with probability r.

The result of the paper we formulate in the following

Theorem. *Let us consider a discrete queueing system in which the interarrival time is geometrically distributed with parameter r, the service time is uniformly distributed on $[c,d]$ (c and d are the multiples of cycle time T), the probability of completion of service for a time slice is $q=\frac{T}{n(d-c)}$. The service discipline is FIFO, the service of a customer may be started upon arrival (if the system is free) or at moments differing from it by the multiples of cycle time T equal to n time slices (if the server is busy or there is a queue). Let us define an embedded Markov chain whose states correspond to the number of customers in* *the system at moments $t_k - 0$, where t_k is the moment of beginning of service of the k-th one. The matrix of transition probabilities for this chain has the form*

$$\begin{bmatrix} a_0 & a_1 & a_2 & a_3 & \dots \\ a_0 & a_1 & a_2 & a_3 & \dots \\ 0 & b_0 & b_1 & b_2 & \dots \\ 0 & 0 & b_0 & b_1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{1}$$

whose elements are determined by the generating functions

$$A(z) = \sum_{i=0}^{\infty} a_i z^i = \frac{qr}{1-r}\left[r^{\frac{c}{T}n} - r^{\frac{d}{T}n}\right] + +zq\left[r^{\frac{c}{T}n} - r^{\frac{d}{T}n}\right] +$$

$$+ zqr^{\frac{c}{T}n}\frac{1}{r}\frac{1-\left(\frac{1}{r}\right)^n}{1-\frac{1}{r}}[r+(1-r)z]^n \frac{1-\left[\frac{r+(1-r)z}{r}\right]^{\frac{c}{T}n}}{1-\left[\frac{r+(1-r)z}{r}\right]^n} +$$

$$+ zqn[r+(1-r)z]^n \frac{[r+(1-r)z]^{\frac{c}{T}n} - [r+(1-r)z]^{\frac{d}{T}n}}{1-[r+(1-r)z]^n} - \tag{2}$$

$$- zqr^{\frac{d}{T}n}\frac{1}{r}\frac{1-\left(\frac{1}{r}\right)^n}{1-\frac{1}{r}}[r+(1-r)z]^n \frac{1-\left[\frac{r+(1-r)z}{r}\right]^{\frac{d}{T}n}}{1-\left[\frac{r+(1-r)z}{r}\right]^n},$$

$$B(z) = \sum_{i=0}^{\infty} b_i z^i = q\frac{1-r}{1-r^n}\frac{[r+(1-r)z]^{\frac{c}{T}n} - [r+(1-r)z]^{\frac{d}{T}n}}{1-[r+(1-r)z]^n} \times \tag{3}$$

$$\times \left\{ [r+(1-r)z] \cdot \right.$$

$$\cdot \frac{1 - r^n[r+(1-r)z]^n - nr^n[r+(1-r)z]^n\{1-r[r+(1-r)z]\}}{\{1-r[r+(1-r)z]\}^2} +$$

$$+ n[r+(1-r)z]^{n+1}\frac{1-r^n[r+(1-r)z]^n}{1-r[r+(1-r)z]} -$$

$$- [r+(1-r)z]^{n+1} \cdot$$

$$\left. \cdot \frac{1 - r^n[r+(1-r)z]^n - nr^n[r+(1-r)z]^n\{1-r[r+(1-r)z]\}}{\{1-r[r+(1-r)z]\}^2} \right\}.$$

The generating function of ergodic distribution $P(z)=\sum_{i=0}^{\infty}p_i z^i$ *for this chain has the form*

$$P(z)=p_0\frac{zA(z)+[a_0z-a_0-z]B(z)}{a_0[z-B(z)]}, \tag{4}$$

where

$$p_0=\frac{a_0[1-B'(1)]}{a_0+A'(1)-B'(1)}. \tag{5}$$

The condition of existence of ergodic distribution is

$$\frac{1-r}{T/n}\frac{c+d+T}{2}<1. \tag{6}$$

3. Proof of the theorem

We replace our original system with idle periods by another one in which the service process is not interrupted, in the modified system the service of a request consists of two parts: the first part means the real service, the second part covers time from its completion till moment when the following one reaches the starting position. We will consider the number of requests present there at moments t_k-0, i.e. at moments just before starting the service of the k-th one. By using the fact that the interarrival time has geometrical distribution, and the intervals between two neighbouring starting moments of services are determined by a geometrically distributed random variable and the service time of the request, one can easily show that the numbers of present at these moments requests form a Markov chain. We find the transition probabilities for this chain. It is necessary to distinguish two cases: at moment when the service of a request begins the next one is present or not. Let us consider the second possibility, it takes place in cases of states zero and one.

In this case in the system there is only one customer. We denote its service time by u and the next customer enters v time after the beginning of its service. The probability of event $\{u-v=\ell\}$ is

$$P\{u-v=\ell\}=\sum_{k=\frac{c}{T}n+1}^{\frac{d}{T}n}qr^{k-\ell-1}(1-r)=q\left[r^{\frac{c}{T}n-\ell}-r^{\frac{d}{T}n-\ell}\right]$$

if $0 < \ell \leq \dfrac{c}{T}n$, and

$$P\{u - v = \ell\} = \sum_{k=\ell+1}^{\frac{d}{T}n} qr^{k-\ell-1}(1-r) = q\left[1 - r^{\frac{d}{T}n-\ell}\right]$$

if $\dfrac{c}{T}n < \ell \leq \dfrac{d}{T}n$. We have to take into account two more possibilities: either for the service no new customer appears, its probability is

$$\sum_{i=\frac{c}{T}n+1}^{\frac{d}{T}n} qr^i = \frac{q}{1-r}\left[r^{\frac{c}{T}n+1} - r^{\frac{d}{T}n+1}\right],$$

or the waiting time is equal to zero (the second customer enters for the last time slice of service of the first one), the corresponding probability equals

$$\sum_{i=\frac{c}{T}n+1}^{\frac{d}{T}n} qr^{i-1}(1-r) = q\left[r^{\frac{c}{T}n} - r^{\frac{d}{T}n}\right].$$

The generating function of entering customers for a time slice is

$$r + (1-r)z.$$

We find how much time passes till the beginning of service of the second customer from the moment of its arrival. It is equal to zero if the second customer appears on the last time slice servicing the first one, n if $u - v$ is contained in the interval $[1, n]$, $2n$ if $u - v \in [n+1, 2n]$, …, and, generally, in if $u - v \in [(i-1)n + 1, in]$. The probability of this event in the first case is

$$\sum_{\ell=(i-1)n+1}^{in} q\left[r^{\frac{c}{T}n-\ell} - r^{\frac{d}{T}n-\ell}\right] = q\left[r^{\frac{c}{T}n} - r^{\frac{d}{T}n}\right]\frac{1}{r}\frac{1-\left(\frac{1}{r}\right)^n}{1-\frac{1}{r}}\left(\frac{1}{r}\right)^{(i-1)n}.$$

The generating function of entering customers

$$\sum_{i=1}^{\frac{c}{T}} q\left[r^{\frac{c}{T}n} - r^{\frac{d}{T}n}\right]\frac{1}{r}\frac{1-\left(\frac{1}{r}\right)^n}{1-\frac{1}{r}}\left(\frac{1}{r}\right)^{(i-1)n}[r + (1-r)z]^{in} =$$

$$= q\left[r^{\frac{c}{T}n} - r^{\frac{d}{T}n}\right]\frac{1}{r}\frac{1-\left(\frac{1}{r}\right)^n}{1-\frac{1}{r}}[r+(1-r)z]^n\frac{1-\left[\frac{r+(1-r)z}{r}\right]^{\frac{c}{T}n}}{1-\left[\frac{r+(1-r)z}{r}\right]^n}.$$

The probability of $u-v\in[(i-1)n+1,in]$ in the second case is

$$\sum_{\ell=(i-1)n+1}^{in} q\left[1-r^{\frac{d}{T}n-\ell}\right]=qn-qr^{\frac{d}{T}n}\frac{1}{r}\frac{1-\left(\frac{1}{r}\right)^n}{1-\frac{1}{r}}\left(\frac{1}{r}\right)^{(i-1)n}.$$

In this case i varies from $\frac{c}{T}+1$ till $\frac{d}{T}$, consequently the generating function of entering customers will be

$$\sum_{i=\frac{c}{T}+1}^{\frac{d}{T}}\left[qn-qr^{\frac{d}{T}n}\frac{1}{r}\frac{1-\left(\frac{1}{r}\right)^n}{1-\frac{1}{r}}\left(\frac{1}{r}\right)^{(i-1)n}\right][r+(1-r)z]^{in}=$$

$$=qn[r+(1-r)z]^n\frac{[r+(1-r)z]^{\frac{c}{T}n}-[r+(1-r)z]^{\frac{d}{T}n}}{1-[r+(1-r)z]^n}-$$

$$-qr^{\frac{d}{T}n}\frac{1}{r}\frac{1-\left(\frac{1}{r}\right)^n}{1-\frac{1}{r}}[r+(1-r)z]^n\left[\frac{1-\left[\frac{r+(1-r)z}{r}\right]^{\frac{d}{T}n}}{1-\left[\frac{r+(1-r)z}{r}\right]^n}-\frac{1-\left[\frac{r+(1-r)z}{r}\right]^{\frac{c}{T}n}}{1-\left[\frac{r+(1-r)z}{r}\right]^n}\right].$$

Taking into account that the first customer obligatorily enters and the waiting time may be equal to zero, after some simplification we obtain (2).

Now we are going to find the transition probabilities for all other states. In this case at the beginning of service of the first customer the second one is already present in the system. Let y denote the *mod* T interarrival time and consider the interval $[in+1,(i+1)n]$. If the service of the first customer is completed till $in+y$ (including y) the waiting time from the beginning of service of the first customer till the beginning of service of the following one is $in+y$, the probability of this event is qy, and the generating function of entering customers is $[r+(1-r)z]^{in+y}$. If the service is completed on $[in+y+1,(i+1)n]$, then the waiting time will be $(i+1)n+y$, the corresponding probability $(n-y)q$. Consequently, at fixed y the generating function of entering customers will be

$$\sum_{i=\frac{c}{T}}^{\frac{d}{T}-1}\left[qy[r+(1-r)z]^{in+y}+(n-y)q[r+(1-r)z]^{(i+1)n+y}\right]=$$

$$=\left\{qy[r+(1-r)z]^y+(n-y)q[r+(1-r)z]^{n+y}\right\}\cdot$$

$$\cdot\frac{[r+(1-r)z]^{\frac{c}{T}n}-[r+(1-r)z]^{\frac{d}{T}n}}{1-[r+(1-r)z]^n}.$$

y has truncated geometrical distribution with probabilities $\dfrac{r^{k-1}(1-r)}{1-r^n}$, $k = 1, 2, \ldots, n$, so

$$B(z) = q\frac{1-r}{1-r^n}\frac{[r+(1-r)z]^{\frac{c}{T}n} - [r+(1-r)z]^{\frac{d}{T}n}}{1-[r+(1-r)z]^n}\times$$

$$\times\sum_{k=1}^{n}\left\{kr^{k-1}[r+(1-r)z]^k + (n-k)r^{k-1}[r+(1-r)z]^{n+k}\right\},$$

from which we obtain (3).

Let us consider the embedded Markov chain describing the functioning of the system. The matrix of its transition probabilities has the form (1). Denoting its ergodic distribution by p_i $(i = 0, 1, 2, \ldots)$ and introducing the generating function $P(z) = \sum\limits_{i=0}^{\infty} p_i z^i$, we have

$$p_j = p_0 a_j + p_1 a_j + \sum_{i=2}^{j+1} p_i b_{j-i+1},$$

from which

$$\sum_{j=0}^{\infty} p_j z^j = p_0 A(z) + p_1 A(z) + \sum_{j=0}^{\infty}\sum_{i=2}^{j+1} p_i b_{j-i+1} z^j =$$

$$= \frac{1}{z}P(z)B(z) - \frac{1}{z}p_0 B(z) + p_0 A(z) + p_1 A(z) - p_1 B(z)$$

or

$$P(z) = \frac{p_0[zA(z) - B(z)] + p_1 z[A(z) - B(z)]}{z - B(z)}. \tag{7}$$

This expression contains two unknown probabilities p_0 and p_1. But

$$p_0 = p_0 a_0 + p_1 a_0,$$

i.e.

$$p_1 = \frac{1-a_0}{a_0}p_0.$$

The unknown p_0 we find from the condition $P(1) = 1$. From (7)

$$p_0 = \frac{a_0[1-B'(1)]}{a_0 + A'(1) - B'(1)}.$$

We have

$$\begin{aligned} A'(1) &= 1 - a_0 + \frac{T}{d-c}\left[r^{\frac{d}{T}n} - r^{\frac{c}{T}n}\right] - \\ &\qquad - 1 + \frac{T}{d-c}\frac{r^{\frac{d}{T}n} - r^{\frac{c}{T}n}}{r^n - 1} + n(1-r)\frac{c+d+T}{2T} = \\ &= -a_0 + \frac{T}{d-c}\frac{r^n}{r^n-1}\left[r^{\frac{d}{T}n} - r^{\frac{c}{T}n}\right] + n(1-r)\frac{c+d+T}{2T}, \\ B'(1) &= n(1-r)\frac{c+d+T}{2T}, \end{aligned}$$

so

$$a_0 + A'(1) - B'(1) = \frac{T}{d-c}\left[r^{\frac{d}{T}n} - r^{\frac{c}{T}n}\right]\frac{r^n}{r^n-1} > 0,$$

consequently, $1 - B'(1) > 0$ must be fulfilled. This leads to the inequality

$$1 - n(1-r)\frac{c+d+T}{2T} > 0,$$

from which we obtain the condition of existence of ergodic distribution (6).

4. Conclusions

It is interesting to consider the condition of existence of equilibrium in the different queueing systems. Usually the fulfilment of $\lambda\tau < 1$ (λ is the arrival rate and τ is the mean value of service time of a customer) is required, it expresses the simple and natural idea that the intensity of service must be higher than the intensity of arrivals. For Poisson arrivals and exponentially distributed service time instead of $\lambda/\mu < 1$ (μ is the service rate) we had

$$\frac{\lambda}{\mu} < \frac{e^{-\lambda T}(1 - e^{-\mu T})}{1 - e^{-\lambda T}},$$

for geometrically distributed interarrival and service times

$$\frac{(1-r)q}{1-q^n}\frac{1-q^n r^n}{1-qr} < r^n$$

(for a time slice the service is terminated with probability $1-q$, and there is an entry with probability $1-r$). These formulae have no clear probabilistic meaning. There is a better situation in case of uniform service time distribution. For the continuous system we obtained

$$\frac{\lambda(c+d+T)}{2} < 1,$$

for the discrete one

$$\frac{1-r}{T/n}\frac{c+d+T}{2} < 1.$$

The last two inequalities may be interpreted on a clear way: to the mean value of service time it is necessary to add a half-cycle $T/2$, it is required on average to reach the starting position. For the completeness we mention that in the second inequality $\frac{T}{n(1-r)}$ is the mean value of time between two arrivals.

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