Annales Univ. Sci. Budapest., Sect. Comp. 22 (2003) 193-206

ON BINARY AND FIBONACCI COMPOSITIONS

A. Knopfmacher (Johannesburg, South Africa)
N. Robbins (San Francisco, CA, USA)

Dedicated to Karl-Heinz Indlekofer
on the occasion of his 60th birthday

1. Introduction

If n is a natural number, then a *composition* of n is a representation

$$n = n_1 + n_2 + \ldots + n_k,$$

where the n_i are natural numbers, and representations that differ only in the order of summands are considered distinct. In this note we investigate compositions of n such that all summands are (i) powers of 2; (ii) Fibonacci numbers.

2. Binary compositions

Let $b(n)$ denote the number of binary compositions of n, with $b(0) = 1$.

Theorem 1. *The $b(n)$ satisfy the recurrence relation*

$$b(n) = \sum_{i=0}^{\lfloor \log_2 n \rfloor} b(n - 2^i). \tag{1}$$

Proof. Let $b_k(x)$ denote the ordinary generating function for compositions of n into exactly k powers of 2. Then for all $k \geq 1$,

$$b_k(x) = \left(\sum_{i=0}^{\infty} x^{2^i} \right)^k . \tag{2}$$

Thus, if $B(x)$ is the generating function for all compositions of n into powers of 2, where $|x| < 1$, then

$$B(x) = 1 + \sum_{k=1}^{\infty} b_k(x) = \left(1 - \sum_{i=0}^{\infty} x^{2^i} \right)^{-1} . \tag{3}$$

Now (3) implies

$$\left(1 - \sum_{i=0}^{\infty} x^{2^i} \right) \left(1 + \sum_{k=1}^{\infty} b(k) x^k \right) = 1. \tag{4}$$

If we match coefficients of like powers of x in (4), the conclusion follows.

Remarks. To a given binary partition

$$n = a_0 2^0 + a_1 2^1 + \ldots + a_k 2^k$$

the number of corresponding binary compositions is

$$\left(\frac{(a_0 + a_1 + \ldots + a_k)!}{a_0! a_1! \ldots a_k!} \right) .$$

Some results regarding binary partitions appear in [4] and [5].

Table 1 below lists $b(n)$ for $1 \leq n \leq 50$.

The following recursive algorithm may be used to generate the set of all binary compositions of n, which we denote by *bincomp*(n):

Set *bincomp*$(0) = \emptyset$ (the empty set). Then for $n \geq 1$ *bincomp*(n) is the union taken over all i such that $0 \leq i \leq \lfloor \log_2 n \rfloor$ of the binary compositions of $n - 2^i$, with 2^i added on the right.

Using the algorithm, we obtain:

$$\begin{aligned} &\textit{bincomp}(1) = \{1\}, \\ &\textit{bincomp}(2) = \{1+1,\ 2\}, \\ &\textit{bincomp}(3) = \{1+1+1,\ 2+1,\ 1+2\}, \end{aligned}$$

n	$b(n)$	n	$b(n)$
1	1	26	1558798
2	2	27	2753447
3	3	28	4863696
4	6	29	8591212
5	10	30	15175514
6	18	31	26805983
7	31	32	47350056
8	56	33	83639030
9	98	34	147739848
10	174	35	260967362
11	306	36	460972286
12	542	37	814260544
13	956	38	1438308328
14	1690	39	2540625074
15	2983	40	4487755390
16	5272	41	7927162604
17	9310	42	14002525142
18	16448	43	24734033936
19	29050	44	43690150992
20	51318	45	77174200244
21	90644	46	136320361910
22	160118	47	240796030130
23	282826	48	425341653750
24	499590	49	751322695068
25	882468	50	1327134992166

Table 1. $b(n)$

$$bincomp(4) = \{1+1+1+1,\ 2+1+1,\ 1+2+1,\ 1+1+2,\ 2+2,\ 4\}.$$
$$bincomp(5) = \{1+1+1+1+1,\ 2+1+1+1,\ 1+2+1+1,\ 1+1+2+1,\ 2+2+1,\ 4+1,\ 1+1+1+2,\ 2+1+2, 1+2+2,\ 1+4\}.$$

The following theorem specifies the parity of the $b(k)$:

Theorem 2. *$b(m)$ is odd if and only if $m = 2^r - 1$ for some $r \geq 1$.*

Proof. By inspection, the theorem holds when $m = 0, 1$. We will assume that the statement is true for all $m < 2^n$, and use induction on n. By Theorem 1 and induction hypothesis, we have

$$b(2^n) = \sum_{i=0}^{n} b(2^n - 2^i) = b(2^n - 1) + b(0) + \sum_{i=1}^{n-1} b(2^n - 2^i) \equiv 1+1+0 \equiv 0 \pmod 2.$$

Also

$$b(2^n + 1) = \sum_{i=0}^{n} b(2^n + 1 - 2^i) \equiv b(2^n - 1) + b(1) \equiv 1 + 1 \equiv 0 \pmod 2;$$
$$b(2^n + 2) = \sum_{i=0}^{n} b(2^n + 2 - 2^i) \equiv 0 \pmod 2;$$
$$b(2^n + 3) = \sum_{i=0}^{n} b(2^n + 3 - 2^i) \equiv b(2^n - 1) + b(3) \equiv 1 + 1 \equiv 0 \pmod 2.$$

Continuing in like manner, we see that if $k \neq 2^j - 1$ for some j such that $0 \le j \le n$, then

$$b(2^n + k) = \sum_{i=0}^{n} b(2^n + k - 2^i) \equiv 0 \pmod 2.$$

If $k = 2^j - 1$ for some j such that $0 \le j \le n-1$, then

$$b(2^n + k) = \sum_{i=0}^{n} b(2^n + k - 2^i) \equiv b(2^n - 1) + b(2^j - 1) \equiv 1 + 1 \equiv 0 \pmod 2.$$

Finally, if $k = 2^n - 1$, so that $2^n + k = 2^{n+1} - 1$, then we have

$$b\left(2^{n+1} - 1\right) = \sum_{i=0}^{n} b\left(2^{n+1} - 1 - 2^i\right) \equiv \sum_{i=0}^{n-1} b\left(2^{n+1} - 1 - 2^i\right) + b(2^n - 1) \equiv$$
$$\equiv 0 + 1 \equiv 1 \pmod 2.$$

The following theorem provides good estimates for $b(n)$ when n is large:

Theorem 3. *Let ρ be the unique positive root of the equation*

$$a(x) = \sum_{k=0}^{\infty} x^{2^k} = 1, \tag{5}$$

namely $\rho \approx 0.56612379268455991824$. *Then, as* $x \to \infty$, *we have*

$$b(n) \sim \frac{1}{\rho^{n+1} a'(\rho)}, \tag{6}$$

where $a'(\rho) \approx 3.0102538220931950079$. *In addition, if* $\overline{b}(n)$ *denotes the mean number of summands in a random binary composition of* n, *then as* $n \to \infty$, *we have*

$$\overline{b}(n) \sim \frac{1}{\rho a'(\rho)} n \approx 0.586796n.$$

Proof. The function

$$a(x) = \sum_{k=0}^{\infty} x^{2^k} \tag{7}$$

satisfies the conditions: $a(0) = 0,\ a'(0) \neq 0$, and there exists a number ρ, such that $0 < \rho < 1,\ a(\rho) = 1$, and $a(x)$ is analytic at $x = \rho$. Thus the function $B(x) = 1/(1 - a(x))$ has a simple pole at $x = \rho$, and from the local expansion of $b(x)$ at this dominant pole

$$b(x) \sim \left(\frac{1}{\rho a'(\rho)}\right)\left(\frac{1}{1 - \frac{x}{\rho}}\right),$$

it follows that

$$b(n) = [x^n]B(x) \sim \frac{1}{\rho a'(\rho)} \rho^{-n}$$

as $n \to \infty$.

Next, the bivariate generating function with u marking then number of summands is

$$B(x, u) = \frac{1}{1 - ua(x)}.$$

The expectation is

$$\overline{b}(n) = \frac{1}{b(n)} [x^n] \frac{\partial}{\partial u} \left(\frac{1}{1 - ua(x)}\right)\Bigg|_{u=1} = \frac{1}{b(n)} [x^n] \frac{a(x)}{(1 - a(x))^2}.$$

The function

$$h(x) = \frac{a(x)}{(1 - a(x))^2}$$

has a double pole at $x = \rho$. Expanding $h(x)$ near $x = \rho$,

$$h(x) \sim \left(\frac{1}{a'(\rho)^2}\right)\left(\frac{1}{(x-\rho)^2}\right),$$

from which

$$[x^n]h(x) \sim \left(\frac{n}{\rho^2(a'(\rho)^2}\right)\rho^{-n}$$

as $n \to \infty$, and hence

$$\bar{b}(n) \sim \left(\frac{1}{\rho a'(\rho)}\right)n$$

as $n \to \infty$.

We consider next the average number of summands equal to 1 in binary compositions of n. Our results shows that 1's make up more than half of the summands on the average, when n is large.

Theorem 4. *With the same notation as in Theorem 3, the proportion of summands equal to* 1 *in a random binary composition of* n *tends to* $\rho = 0.566123\ldots$ *as* $n \to \infty$.

Proof. Let $b_k(x,u)$ denote the bivariate generating function for compositions of n into exactly k powers of 2, with the variable u marking the number of 1's. Then for $k \geq 1$, we have

$$b_k(x,u) = \left(\sum_{i=0}^{\infty} x^{2^i} + (u-1)x\right)^k.$$

Thus if $b(x,u)$ is the bivariate generating function for all compositions of n into powers of 2, with u marking the number of 1's, we have

$$b(x,u) = \frac{1}{1-(u-1)x-\sum_{i=0}^{\infty} x^{2^i}} = \frac{1}{1-ux-\sum_{i=1}^{\infty} x^{2^i}}.$$

The mean value of the number of 1's is

$$\mu(n) = \frac{1}{b(n)}[x^n]\frac{x}{(1-a(x))^2} = \frac{1}{b(n)}[x^{n-1}]B^2(x).$$

As in the proof of Theorem 3, near $x = \rho$ we have

$$B^2(x) \sim \left(\frac{1}{a'(\rho)^2}\right)\left(\frac{1}{(x-\rho)^2}\right),$$

from which it follows that as $n \to \infty$

$$\left[x^{n-1}\right] B^2(x) \sim \left(\frac{n}{\rho^2(a'(\rho)^2)}\right) \rho^{-n+1}.$$

Hence

$$\mu(n) \sim \frac{1}{a'(\rho)} n$$

as $n \to \infty$. The result now follows from the $\bar{b}(n)$ estimate of Theorem 3.

Now let $b^*(n)$ denote the number of binary compositions of n with distinct summands. The following theorem enables us to compute $b^*(n)$.

Theorem 5. *Let the binary representation of n have m $1's$. Then $b^*(n) = m!$, and every such composition of n has exactly m summands.*

Proof. This follows immediately from the hypothesis and the fact that the binary representation of any natural number is unique. The m $1's$ represent the distinct powers of 2, and $m!$ is the number of permutations of these quantities that yield the corresponding binary compositions with distinct parts.

Let $B^*(x) = \sum_{n \geq 0} b^*(n) x^n$ be the generating function for binary compositions with distinct parts.

Theorem 6.

$$B^*(x) = \int_0^\infty e^{-t} \prod_{j=0}^{\infty} \left(1 + t x^{2^j}\right) dt.$$

Proof. To the unique binary representation of n, which has $m = m(n)$ $1's$, there correspond $m!$ compositions. Now

$$\prod_{j=0}^{\infty} \left(1 + t x^{2^j}\right) = 1 + \sum_{n \geq 1} x^n t^m.$$

Multiplying both sides by e^{-t} and integrating from 0 to infinity gives the required factor of $m!$.

3. Fibonacci compositions

Let F_n denote the n-th Fibonacci number, and let $g(n)$ denote the number of Fibonacci compositions of n, with $g(0) = 1$.

We begin by letting $g_k(x)$ denote the generating function for compositions into exactly k Fibonacci numbers. Then for $k \geq 1$, we have

$$g_k(x) = \left(\sum_{i=2}^{\infty} x^{F_i} \right)^k. \tag{8}$$

(The lower limit of summation is $i = 2$ because $F_1 = F_2 = 1$, and the summand 1 must not be used twice.) If we let $G(x)$ denote the generating function for all compositions of n with Fibonacci number summands, then we have

$$G(x) = 1 + \sum_{k=1}^{\infty} g_k(x) = \left(1 - \sum_{i=2}^{\infty} x^{F_i} \right)^{-1}. \tag{9}$$

If also

$$G(x) = \sum_{n=0}^{\infty} g(n) x^n$$

with $g(0) = 1$, then (10) implies

$$\left(1 - \left(\sum_{i=2}^{\infty} x^{F_i} \right) \left(\sum_{n=0}^{\infty} g(n) x^n \right) \right) = 1.$$

Equating coefficients of like powers of x for $n \geq 1$, we obtain

$$g(n) = \sum_{i=2}^{k_n} g(n - F_i), \tag{10}$$

where $k_n = \max\{i : F_i \leq n\}$. Since

$$F_n \leq \frac{\phi^n}{\sqrt{5}},$$

where $\phi = \dfrac{1+\sqrt{5}}{2}$, we can take $k_n = \lfloor \log_\phi(n\sqrt{5} \rfloor$. Using (11), we generate Table 2 below, which lists $g(n)$ for $1 \leq n \leq 50$.

n	$g(n)$	n	$g(n)$
1	1	26	9315811
2	2	27	17656534
3	4	28	33464955
4	7	29	63427148
5	14	30	120215370
6	26	31	227847814
7	49	32	431846824
8	94	33	818492263
9	177	34	1551313038
10	336	35	2940250271
11	637	36	5572744810
12	1206	37	10562190960
13	2288	38	20018838331
14	4335	39	37942306721
15	8216	40	71913195697
16	15574	41	136299243785
17	29515	42	258332058332
18	55943	43	489624523869
19	106030	44	928000093918
20	200959	45	1758866503528
21	380889	46	3333632612035
22	721906	47	6318334205671
23	1368251	48	11975328951940
24	2593291	49	22697201325330
25	4915135	50	43018688678483

Table 2. $g(n)$

The following theorem may be used to estimate $g(n)$ when n is large.

Theorem 7. *Let* $e(x) = \sum_{i=2}^{\infty} x^{F_i}$. *As* $n \to \infty$, *we have*

$$g(n) \sim \frac{1}{e'(\alpha)} \alpha^{-n-1},$$

where α *is the unique positive root of*

$$e(x) = 1,$$

namely $\alpha \approx 0.52761258008208832339$*, so that* $e'(\alpha) \approx 3.3749752101093828772$*. In addition, if* $\overline{g}(n)$ *denotes the mean number of summands in a random Fibonacci composition of* n*, then as* $n \to \infty$*, we have*

$$\overline{g}(n) \sim \frac{1}{\rho e'(\rho)} n = 0.561583n.$$

Proof. The proof of Theorem 7 is similar to the proof of Theorem 3, and is therefore omitted.

Corresponding to Theorem 4, we have

Theorem 8. *The proportion of summands equal to* 1 *in a random Fibonacci omposition of* n *tends to* $\alpha = 0.527613\ldots$*, as* $n \to \infty$*.*

Proof. We omit the proof, which is similar to that of Theorem 4.

Remarks. Let $A \subset N$, where N denotes the set of all natural numbers. Let $c_A(n)$ denote the number of compositions of n, all whose summands belong to A. In [3] V.E. Hoggatt and D. Lind obtained formulas for $c_A(n)$ that involve Fibonacci numbers for certain special choices of A. For example, if $A = \{1, 2\}$, then $c_A(n) = F_{n+1}$; also, if $A = N - 2N$ (the set of all odd natural numbers), then $c_A(n) = F_n$.

Theorem 9. *The function* $g(n)$ *changes parity infinitely often as* $n \to \infty$*.*

Proof. First, suppose there exists a positive integer, m, such that $g(n) \equiv$ $\equiv 0 \pmod 2 \quad \forall n \geq m$. Then (10) implies

$$g(n) \equiv \sum_{i\geq 2} \{g(n + F_i) : n - F_i < m\} \pmod 2,$$

that is,

$$g(n) \equiv \sum_{i\geq 2} \{g(n - F_i) : n - m < F_i \leq n\} \pmod 2.$$

Therefore, if $F_j \geq m$, then we have

$$g(F_j) \equiv \sum_{i\geq 2} \{g(F_j - F_i) : F_j - m < F_i \leq F_j\} \pmod 2.$$

Choose the least index, j, such that $F_{j-2} \geq m$, that is, $F_j - F_{j-1} \geq m$, or $F_j -$ $-m \geq F_{j-1}$. This implies

$$g(F_j) \equiv g(F_j - F_j) \equiv g(0) \equiv 1 \pmod 2,$$

an impossibility.

Next, suppose there exists m such that $g(n) \equiv 1 \pmod 2 \quad \forall n \geq m$. Using (10), we have

$$g(n) = \sum_{i\geq 2} \{g(n - F_i) : n - F_i < m\} + \sum_{i\geq 2} \{g(n - F_i) : n - F_i \geq m\}.$$

Let j be the least index such that $F_{2j-1} \geq m$, that is, $F_{2j+1} - F_{2j} \geq m$. Then

$$g(F_{2j+1}) \equiv g(0) + \sum_{i=2}^{2j} \{g(F_{2j+1} - F_i)\} \equiv 1 + 1 \equiv 0 \pmod 2,$$

an impossibility.

4. Compositions of n with distinct Fibonacci summands

We will need the identity

(11) $$\sum_{j=1}^{k} F_j = F_{k+1} - 1.$$

Let $c(n)$ denote the number of such compositions. By direct evaluation, we have

$$c(1) = c(2) = 1,\ c(3) = 3,\ c(4) = 2,\ c(5) = 3,$$
$$c(6) = 8,\ c(7) = 2,\ c(8) = 9,\ c(9) = c(10) = 8.$$

The following theorem concerns the parity of $c(n)$.

Theorem 10.

$$c(n) \equiv \begin{cases} 1 \pmod 2, & \textit{if } n \textit{ is a Fibonacci number,} \\ 0 \pmod 2, & \textit{otherwise.} \end{cases}$$

Proof. A partition of n into k distinct parts gives rise to $k!$ corresponding compositions of n into distinct parts. If $k \geq 2$, then $k!$ is even. Since $c(n)$ is a sum of factorials, $c(n)$ will be even unless $1!$ occurs oddly many times in the sum. But this can happen only if n has a representation as a single Fibonacci number, i.e. if $n = F_r$ for some $r \geq 2$.

Theorem 11. $c(F_n) \equiv 0 \pmod 3 \quad \forall n \geq 3.$

Proof. Again, any partition of n into k distinct parts gives rise to $k!$ corresponding compositions of n into distinct parts. If $k \geq 3$, then $3|k!$. There is only one partition of F_n into one part (and hence a single corresponding composition). There is only one partition of F_n into two parts, namely: $F_n = F_{n-1} + F_{n-2}$ (and hence two corresponding compositions). Therefore $c(F_n)$, the sum of these numbers, is divisible by 3.

We can improve on Theorem 11 to obtain

Theorem 12. *If $m \geq 2$, then*

$$c(F_m) = \sum_{k=1}^{\lfloor m/2 \rfloor} k!.$$

Proof. Let $r(n)$ denote the number of partitions of n into distinct Fibonacci parts. In Theorem 1 of [6] the second author proved that if $m \geq 2$, then $r(F_m) = \left\lfloor \frac{m}{2} \right\rfloor$. But we can say more, namely, if $m \geq 2$ and $1 \leq k \leq \left\lfloor \frac{m}{2} \right\rfloor$, then there is exactly one partition of F_m into distinct Fibonacci parts, namely

$$F_m = F_{m-1} + F_{m-3} + \ldots + F_{m+3-2k} + F_{m+2-2k}$$

if $m \geq 3$, and $F_2 = 1$ if $m = 2$. The above displayed equation shows that at least one such partition of F_m exists; Theorem 1 of [6] implies that at most one such partition of F_m exists. Finally, since each such partition of F_m of length k gives rises to $k!$ corresponding compositions, the conclusion now follows.

Theorem 13. *If $m \geq 3$, then $c(F_m - 1) = \left\lfloor \frac{m-1}{2} \right\rfloor !$.*

Proof. $F_m - 1$ has a unique representation as a sum of $\left\lfloor \frac{m-1}{2} \right\rfloor$ distinct Fibonacci numbers, namely

$$F_m - 1 = F_{m-1} + F_{m-3} + \ldots = \sum_{j=0}^{\lfloor \frac{m-3}{2} \rfloor} F_{m-1-2j}$$

from which the conclusion follows. (See [6], Theorem 8.)

Theorem 14. *If $n \geq 3$, then*

$$c(1 + F_{2n}) = c(2 + F_{2n}) = \sum_{k=2}^{n} k! = c(F_{2n}) - 1.$$

Proof. The last equality follows from Theorem 11. It follows from the proof of Theorem 11 that F_{2n} has exactly n representations as sums of distinct Fibonacci summands, namely

$$F_{2n} = F_{2n}, \quad F_{2n} = \sum_{j=1}^{k} F_{2n+1-2j} + F_{2n-2k},$$

where $1 \leq k \leq n-1$. If $1 \leq k \leq n-2$, then the least term in each such representation of F_{2n} exceeds 3, so $1 + F_{2n}$ and $2 + F_{2n}$ have corresponding representations

$$1 + F_{2n} = \sum_{j=1}^{k} F_{2n+1-2j} + F_{2n-2k} + F_2,$$

$$2 + F_{2n} = \sum_{j=1}^{k} F_{2n+1-2j} + F_{2n-2k} + F_3.$$

We also have $1 + F_{2n} = F_{2n} + F_2$, $2 + F_{2n} = F_{2n} + F_3$. Now (12) implies that the largest term in any such representation of $j + F_{2n}$, where $j \in \{1, 2\}$ must be at least F_{2n-1}. Therefore, all such representations of $j + F_{2n}$, where $j \in \{1, 2\}$ have been accounted for. There are $n-1$ such representations, one each for each integer from 2 to n. The conclusion now follows.

Finally, let $C(x) = \sum_{n \geq 0} c(n)x^n$ be the generating function for Fibonacci compositions with distinct parts.

Theorem 15.

$$C(x) = \int_0^{\infty} e^{-t} \prod_{j=2}^{\infty} \left(1 + tx^{F_j}\right) dt.$$

Proof. Let $d(n, m)$ count the number of Fibonacci partitions with m parts, all distinct. We have

$$c(n) = \sum_{m \geq 1} m! d(n, m).$$

Now

$$\prod_{j=2}^{\infty} \left(1 + x^{F_j}\right) = 1 + \sum_{n \geq 1} x^n \sum_{m \geq 1} d(n, m)t^m.$$

Multiplying both sides by e^{-t} and integrating from $t=0$ to $t=\infty$ gives

$$\int_0^\infty e^{-t}\prod_{j=2}^{i} nfty\left(1+tx^{F_j}\right)dt = 1+\sum_{n\geq 1}x^n\sum_{m\geq 1}m!d(n,m)$$

as required.

Remarks. Binary and Fibonacci compositions are special cases of combinatorial structures that are related by a sequence construction. Results analogous to Theorem 3 and 7 appear in [1] and [2], under this more general context. In addition, by applying general theorems concerning sequence constructions in [2, Chapter 9] to binary or Fibonacci compositions, it also follows that the number of summands obeys a Gaussian limiting distribution in each case.

References

[1] **Bender E.A.,** Central and local limit theorems applied to asymptotic enumeration, *J. Comb. Theory (Ser. A),* **15** (1973), 91-101.

[2] **Flajolet P. and Sedgewick R.,** *Analytic combinatorics* (book in preparation)

[3] **Hoggatt V.E.Jr. and Lind D.A.,** Compositions and Fibonacci numbers, *Fibonacci Quart.,* **7** (1969), 253-266.

[4] **Rodseth O.J. and Sellers J.A.,** On m-ary partition function congruences: a fresh look at a past problem, *J. Number Theory,* **87** (2001), 270-281.

[5] **Richmond L.B.,** Mahler's partition problem, *Ars Combinatorica,* **2** (1976), 169-189.

[6] **Robbins N.,** Fibonacci partitions, *Fibonacci Quart.,* **34** (1996), 306-313.

A. Knopfmacher
The John Knopfmacher Centre
for Appl. Analysis and Number Theory
University of the Witwatersrand
Private Bag 3
Wits 2050, South Africa

N. Robbins
Mathematics Department
San Francisco State University
San Francisco, CA 94132
USA