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THE ASYMPTOTIC BEHAVIOR OF THE RENEWAL FUNCTION CONSTRUCTED FROM A RANDOM WALK IN MULTIDIMENSIONAL TIME WITH RESTRICTED TIME DOMAIN

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Dedicated to Prof. Karl-Heinz Indlekofer on the occasion of his 60th birthday

Abstract. We consider the renewal function defined for a random walk in multidimensional time with a restriction to a subset of the time space. Our main result provides the asymptotic behavior of the renewal function as $t \to \infty$. This corresponds to the problem for the classical setting in which the asymptotic behavior of the renewal function is studied for a random walk considered only at a subsequence of indices.

1. Introduction

Let N^d be the space of vectors with d positive integer coordinates. Elements of N^d are denoted by $\overline{k}$, $\overline{n}$, etc. Consider a family of independent,

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identically distributed (i.i.d.) random variables $X(\overline{n})$, $\overline{n} \in N^d$, and their (multiple) sums

$$S(\overline{n}) = \sum_{\overline{k} \prec \overline{n}} X(\overline{k}),$$

where "$\prec$" is the coordinate-wise (partial) ordering, meaning $k_1 \le n_1, \dots, k_d \le$ $\le n_d$ for $\overline{k} = (k_1, \dots, k_d)$ and $\overline{n} = (n_1, \dots, n_d)$. In analogy with the case $d = 1$, the family $S(\overline{n})$, $\overline{n} \in N^d$, is called a random walk in multidimensional time.

Some properties of random walks in multidimensional time are immediate consequences of their counterparts for the classical random walk. For example, if the expectation exists, that is,

$$EX(\overline{n}) = \mu \in (-\infty, \infty), \tag{1}$$

then

$$\frac{S(\overline{n})}{|\overline{n}|} \xrightarrow{P} \mu, \quad \text{as} \quad |\overline{n}| \to \infty,$$

where $\overline{n} = (n_1, \dots, n_d)$ and $|\overline{n}| = n_1 \cdots n_d$. The proof is obvious and makes use of the classical law of large numbers.

Some other properties of random walks in multidimensional time are not so clear and immediate. Just to give the reader an impression, we mention the strong law of large numbers, i.e.

$$\frac{S(\overline{n})}{|\overline{n}|} \xrightarrow{a.s.} \mu, \quad \text{as} \quad |\overline{n}| \to \infty,$$

if and only if

$$EX(\overline{n}) = \mu, \qquad E|X(\overline{n})|\mathrm{Log}^{d-1}(|X(\overline{n})|) < \infty,$$

where $\mathrm{Log}(z) = \ln(1 + |z|)$ for real numbers z (see Smythe [15]).

One of the classical problems in case $d = 1$ is to find the asymptotic behavior of the renewal function $U(t)$, $t > 0$, which is defined as the expectation of the renewal process

$$N(t) = \sum_{\overline{n} \in N^d} I\{S(\overline{n}) < t\}, \quad t > 0,$$

where $I\{A\}$ denotes the indicator function of an event A. Note that $N(t)$ is finite almost surely for all $t > 0$, if e.g. the random variables $X(\overline{n})$, $\overline{n} \in N^d$, are (say) nonnegative and nondegenerate. Then the renewal function can be expressed as

$$U(t)=\sum_{\overline{n}\in N^d} P(S(\overline{n})<t), \quad t>0. \tag{2}$$

One can easily show that $U(t)$ is also finite for all $t>0$, if the random variables $X(\overline{n})$, $\overline{n}\in N^d$, are nonnegative and nondegenerate. For $d=1$, the classical renewal theorem yields

$$\frac{U(t)}{t}\to\frac{1}{\mu}, \qquad \text{as} \quad t\to\infty, \tag{3}$$

if, e.g., the $X(\overline{n})$ are nonnegative and condition (1) holds with $\mu>0$. A natural question then is to ask whether this property remains true for $d>1$, too.

Ney and Wainger [14] were the first to study this problem. For further developments see also Maejima and Mori [13], Galambos and Kátai [5]-[6], and Galambos, Indlekofer and Kátai [4]. All these papers deal with the asymptotic behavior of the series of corresponding probability densities instead of the series of distribution functions (2), which gives a sharper result compared to (3). However, the conditions are stronger, too, and there are several complications in the proofs related to the Dirichlet divisors problem in number theory (for a more detailed discussion of relationships between probability theory and the Dirichlet divisors problem see also Indlekofer and Klesov [9]). The series (2) has been investigated in Klesov [11], while the asymptotic properties of the renewal process itself were studied in Klesov and Steinebach [12].

Indlekofer and Klesov [10] obtained some limit properties of random walks in multidimensional time in the case when the "time" parameter $\overline{n}$ is restricted to a subset of the space N^d. Our aim here is to extend the investigations started in [10] to the case of the renewal process in multidimensional time.

2. Main result

In what follows we assume that the random variables $X(\overline{n}), \overline{n}\in N^d$, are nonnegative and nondegenerate, and that condition (1) holds with $\mu>0$.

Let D be a subset of N^d and consider the process

$$N(t)=\sum_{\overline{n}\in D} I\{S(\overline{n})<t\}, \quad t>0,$$

and its expectation

$$U(t)=\sum_{\overline{n}\in D} P(S(\overline{n})<t), \quad t>0.$$

Put $C(x,y)=\{\overline{n}\in D\colon x\le|\overline{n}|<y\}$ and $A(x,y)=\operatorname{card}C(x,y)$ for $x<y$. Then, for all $0<\varepsilon<1$, one has

$$U(t)=\sum_{\overline{n}\in C_1(\varepsilon,t)} P(S(\overline{n})<t)+\sum_{\overline{n}\in C_2(\varepsilon,t)} P(S(\overline{n})<t)+\sum_{\overline{n}\in C_3(\varepsilon,t)} P(S(\overline{n})<t),$$

where

$$C_1(\varepsilon,t)=C(0,\ (1-\varepsilon)t/\mu), \quad C_2(\varepsilon,t)=C((1-\varepsilon)t/\mu,\ (1+\varepsilon)t/\mu), \quad \text{and}$$

$$C_3(\varepsilon,t)=C((1+\varepsilon)t/\mu,\ \infty).$$

We consider the three terms separately.

Term $C_3(\varepsilon,t)$. Let $r>0$ be such that

$$\mu^r=\int\limits_{[0,r)} x\,dF(x)>\frac{\mu}{1+\varepsilon},$$

where F is the common distribution function of the random variables $X(\overline{n}),\overline{n}\in\in N^d$. We apply a truncation procedure (at level r) leading to new i.i.d. random variables $X^r(\overline{n})=X(\overline{n})I\{X(\overline{n})<r\}$, and set $S^r(\overline{n})=\sum_{\overline{k}\le\overline{n}} X^r(\overline{n})$. The random variables $X^r(\overline{n})$ are bounded almost surely by r, and $EX^r(\overline{n})=\mu^r$. Moreover

$$\sum_{\overline{n}\in C_3(\varepsilon,t)} P(S(\overline{n})<t)\le\sum_{\overline{n}\in C_3(\varepsilon,t)} P(S^r(\overline{n})<t).$$

Since $t-\mu^r|\overline{n}|<0$, for $\overline{n}\in C_3(\varepsilon,t)$, we have

$$\sum_{\overline{n}\in C_3(\varepsilon,t)} P(S(\overline{n})<t)\le\sum_{\overline{n}\in N^d} P\left(|S^r(\overline{n})-|\overline{n}|\mu^r|\ge\delta|\overline{n}|\right),$$

with $\delta=\mu^r-\frac{\mu}{1+\varepsilon}>0$. The series on the right hand side converges for all $\delta>0$ by Smythe's [16] d-dimensional generalization of the Hsu-Robbins-Erdős concept of complete convergence (see Hsu and Robbins [7] and Erdős [2], [3]).

In fact, the necessary and sufficient condition for the convergence of that series is

$$EX^r(\overline{n}) = \mu^r, \qquad E\left(X^r(\overline{n})\right)^2 \mathrm{Log}^{d-1}(|X^r(\overline{n})|) < \infty,$$

which is satisfied, since the random variables are bounded almost surely.

Term $C_2(\varepsilon, t)$. Obviously,

$$\sum_{\overline{n}\in C_2(\varepsilon,t)} P(S(\overline{n}) < t) \leq A(0,(1+\varepsilon)t/\mu) - A(0,(1-\varepsilon)t/\mu).$$

In what follows we assume that $A(0,t)$ is an Avakumović function, i.e. that $A(0,t) \to \infty$, as $t \to \infty$, and

$$\lim_{c\to 1} \limsup_{t\to\infty} \frac{A(0,ct)}{A(0,t)} = 1.$$

For a detailed discussion of Avakumović functions we refer to Buldygin, Klesov and Steinebach [1]. Note that an important subset of Avakumović functions is given by the class of regularly varying functions.

Term $C_1(\varepsilon, t)$. It is clear that

$$1 - P(S(\overline{n}) < t) \leq P(|S(\overline{n}) - |\overline{n}|\mu| \geq t - |\overline{n}|\mu) \leq P(|S(\overline{n}) - |\overline{n}|\mu| \geq |\overline{n}|\varepsilon/(1-\varepsilon)),$$

for $\overline{n} \in C_1(\varepsilon, t)$. Since $P(|S(\overline{n}) - |\overline{n}|\mu| \geq |\overline{n}|\varepsilon/(1-\varepsilon)) \to 0$, as $|\overline{n}| \to \infty$, by the law of large numbers, we have

$$\lim_{t\to\infty} \frac{1}{A(0,(1-\varepsilon)t/\mu)} \sum_{\overline{n}\in C(0,(1-\varepsilon)t/\mu)} P(S(\overline{n}) < t) = 0.$$

On the other hand, let $n_0 = n_0(\varepsilon)$ be such that $P(|S(\overline{n}) - |\overline{n}|\mu| < |\overline{n}|\varepsilon) \geq$ $\geq 1 - \varepsilon \quad \forall\, \overline{n} : |\overline{n}| \geq n_0$, and set $C_0(\varepsilon, t) = C(n_0, (1+\varepsilon)t/\mu)$. Then

$$U(t) \geq (1-\varepsilon) A(n_0, (1+\varepsilon)t/\mu).$$

On combining the above relations we arrive at the following result.

Theorem 1. *Assume that* $X(\overline{n}), \overline{n} \in N^d$, *are i.i.d. random variables with* $0 < EX(\overline{n}) = \mu < \infty$. *If* D *is a subset of* N^d *such that* $A(0,t)$ *as defined above is an Avakumović function, then*

$$\lim_{t\to\infty} \frac{U(t)}{A(0,t/\mu)} = 1.$$

3. Examples

Example 1. Let $D = N^d$. Then $A(0,t) \sim t\mathcal{P}(\ln t)$ as $t \to \infty$, where $\mathcal{P}$ is a polynomial of degree $d-1$ with leading coefficient $\frac{1}{(d-1)!}$. This fact is well known, nevertheless we provide an elementary proof below by means of analytical arguments. So, in view of Theorem 1,

$$\lim_{t\to\infty} \frac{U(t)}{t \ln^{d-1} t} = \frac{1}{\mu(d-1)!}$$

(see also Klesov [11]).

Example 2. Let $d = 2$ and $\theta \geq 1$. Consider the domain

$$D = \left\{(m,n)\colon\ \theta^{-1} m \leq n \leq \theta m\right\}.$$

There are two different cases: $\theta = 1$ and $\theta > 1$. If $\theta = 1$, then $A(0,t) \sim \sqrt{t}$, and Theorem 1 implies

$$\lim_{t\to\infty} \frac{U(t)}{\sqrt{t}} = \frac{1}{\sqrt{\mu}}. \tag{4}$$

If $\theta > 1$, then $A(0,t) \sim t \ln \theta$, and therefore

$$\lim_{t\to\infty} \frac{U(t)}{t} = \frac{\ln \theta}{\mu}. \tag{5}$$

Just to see the whole spectrum of the possible asymptotics of the function $U(t)$ in this case, we recall the result of Example 1 corresponding to the case "$\theta \to \infty$", i.e.

$$\lim_{t\to\infty} \frac{U(t)}{t \ln t} = \frac{1}{\mu}.$$

Example 3. Let $d = 2$ and f be an increasing function such that

$$f(x) \leq x, \qquad \text{for all } \ x \geq 0,$$

and let the domain D be defined as

$$D = \{(m,n)\colon\ f(m) \leq n \leq m\}.$$

If f is smooth enough, then $A(0,t)$ is an Avakumović function. Consider e.g. the case of

$$f(x) = cx + o(1), \quad x \to \infty,$$

with $0 \le c \le 1$. A direct computation (cf. Indlekofer [8]) shows that

$$A(0,t) = \frac{t}{2} \ln \frac{1}{c} + o(t), \quad t \to \infty,$$

if $0 < c < 1$, and (5) retains with $\theta = c^{-1/2}$. The case $c = 0$, that is

$$f(x) = o(x), \quad x \to \infty,$$

follows also from the asymptotics obtained by Indlekofer [8]. First, define an integer number m_x as the unique solution of the inequalities

$$m_x[f(m_x)] \le x < (m_x + 1)[f(m_x + 1)],$$

where $[z]$ denotes the integer part of a real number z. Second, introduce the function δ as

$$\delta(x) = \frac{1}{x} f^2(m_x).$$

Then,

$$A(0,t) = t \ln \frac{1}{\delta(t)} + O(t), \quad t \to \infty,$$

and therefore

$$\lim_{t \to \infty} \frac{U(t)}{t \ln \delta^{-1}(t/\mu)} = \frac{1}{\mu}.$$

The asymptotic behavior of $A(0,t)$ for the case $c = 1$ is still an open problem. Just for the sake of demonstration, we mention the function $f(x) = x/\ln(x)$. Then $m_x \asymp \sqrt{x \ln(x)}$ and $\delta^{-1}(x) \asymp \ln(x)$, as $x \to \infty$, so that

$$\lim_{t \to \infty} \frac{U(t)}{t \ln \ln t} = \frac{1}{\mu}.$$

4. Calculation of the leading coefficient in the Dirichlet polynomial

For $x \ge 1$ put $T^{(d)}(x) = \operatorname{card}\{\overline{n} : |\overline{n}| \le x\}$. Our aim in this section is to prove that, for $d \ge 2$ and $x \ge 1$,

$$T^{(d)}(x) = \frac{1}{(d-1)!} x \ln^{d-1} x + g_d(x) x \ln^{d-2} x, \tag{6}$$

where the function g_d is such that $\sup\limits_{x \ge 1} |g_d(x)| < \infty$.

We need two auxiliary results.

Lemma 1. *For* $\alpha \ge 0$,

$$\sum_{i=1}^{[x]} \frac{1}{i} \ln^\alpha i = \frac{1}{\alpha+1} \ln^{\alpha+1}[x] + h_\alpha(x),$$

where

$$h_\alpha(x) = \psi(x) + \int_1^x \rho(t) \frac{\ln^\alpha t - \alpha \ln^{\alpha-1} t}{t^2}\, dt \ ,$$

and

$$\rho(x) = [x] - x - \frac{1}{2}, \qquad \psi(x) = \begin{cases} \dfrac{\ln^\alpha [x]}{2\,[x]}, & \text{for } \alpha > 0, \\ \dfrac{1}{2}\left(\dfrac{1}{[x]} + 1\right), & \text{for } \alpha = 0. \end{cases} \tag{7}$$

Note that

$$\sup_{x \ge 1} |h_\alpha(x)| < \infty \quad \text{for all } \alpha \ge 0,$$

$$h_\alpha(x) \to \int_1^\infty \rho(t) \frac{\ln^\alpha t - \alpha \ln^{\alpha-1} t}{t^2}\, dt, \qquad x \to \infty.$$

Lemma 2. *Let* C_n^m *denote the binomial coefficient* $\binom{n}{m}$. *Then*

$$\sum_{l=0}^{d-2} C_{d-2}^l (-1)^{d-2-l} \frac{1}{d-1-l} = \frac{1}{d-1}.$$

Proof of (6). Put $|\overline{n}|_{2,\dots,d} = n_2 \cdots n_d$. Then

$$T^{(d)}(x) = \operatorname{card}\left\{\overline{n} : n_1 \le x, |\overline{n}|_{2,\dots,d} \le \frac{x}{n_1}\right\}.$$

Thus

$$T^{(d)}(x) = \sum_{n_1=1}^{[x]} T^{(d-1)}(x/n_1). \tag{8}$$

We use (8) to obtain (6) by induction. First note that, for $d = 1$,

$$T^{(1)}(x) = \sum_{k_1=1}^{[x]} \sharp_{k_1} = [x] = x + g_1(x),$$

with $g_1(x) = [x] - x$. Next, we treat the case $d = 2$. Lemma 1, with $\alpha = 0$, implies

$$T^{(2)}(x) = \sum_{n_1=1}^{[x]} T^{(1)}\left(\frac{x}{n_1}\right) = \sum_{n_1=1}^{[x]} \left(\frac{x}{n_1} + g_1\left(\frac{x}{n_1}\right)\right) = x \ln x + g_2(x)x,$$

where

$$g_2(x) = \ln[x] - \ln x + h_0(x) + \frac{1}{x}\sum_{n_1=1}^{[x]} g_1\left(\frac{x}{n_1}\right).$$

Assume that representation (6) holds for all dimensions less than d. We then prove (6) for dimension d. By relation (8) and induction,

$$T^{(d)}(x) = \frac{1}{(d-2)!}\sum_{i=1}^{[x]} \frac{x}{i} \ln^{d-2}\frac{x}{i} + g_d^{(1)}(x) x \ln^{d-2} x,$$

where

$$g_d^{(1)}(x) = \frac{1}{\ln^{d-2} x}\sum_{i=1}^{[x]} g_{d-1}\left(\frac{x}{i}\right)\frac{1}{i}\ln^{d-3}\frac{x}{i}.$$

We have

$$\sum_{i=1}^{[x]} \frac{1}{i}\ln^{d-2}\frac{x}{i} = \sum_{i=1}^{[x]} \frac{1}{i}(\ln x - \ln i)^{d-2} =$$
$$= \sum_{l=0}^{d-2} C_{d-2}^{l}(-1)^{d-2-l}\ln^{l} x \sum_{i=1}^{[x]} \frac{1}{i}\ln^{d-2-l} i.$$

On using Lemma 1 again, with $\alpha = d-2-l$,

$$\sum_{i=1}^{[x]} \frac{1}{i} \ln^{d-2-l} i = \frac{1}{d-1-l} \ln^{d-1-l} x + h_{d-2-l}(x).$$

Hence, by Lemma 2,

$$\frac{1}{(d-2)!} \sum_{i=1}^{[x]} \frac{1}{i} \ln^{d-2} \frac{x}{i} = \frac{1}{(d-1)!} \ln^{d-1} x + g_d^{(2)}(x) x \ln^{d-2} x,$$

where

$$g_d^{(2)}(x) = \frac{1}{x \ln^{d-2} x} \frac{1}{(d-2)!} \sum_{l=0}^{d-2} \binom{d-2}{l} (-1)^{d-2-l} \ln^l(x) \kappa_{d-2-l}(x),$$

$$\kappa_j(x) = h_j(x) + \frac{1}{j+1} \left(\ln^{j+1}[x] - \ln^{j+1} x \right).$$

This implies (6) with $g_d(x) = g_d^{(1)}(x) + g_d^{(2)}(x)$.

Proof of Lemma 1. We use the Euler-Maclaurin summation formula with $\ell_1 = 1$, $\ell_2 = [x]$, and $f(x) = (1/x) \ln^\alpha x$. Then

$$f'(x) = x^{-2} \left(\alpha \ln^{\alpha-1} x - \ln^\alpha x \right)$$

and thus

$$\sum_{i=1}^{[x]} \frac{\ln^\alpha i}{i} = \int_1^{[x]} \frac{\ln^\alpha t}{t} \, dt + \psi(x) + \int_1^{[x]} \rho(t) \frac{\alpha \ln^{\alpha-1} t - \ln^\alpha t}{t^2} \, dt, \tag{9}$$

where the functions ψ and ρ are defined by (7). This proves Lemma 1, since $|\rho(t)| \le \frac{1}{2}$ and the integral $\int_1^\infty \frac{\ln^\alpha t - \alpha \ln^{\alpha-1} t}{t^2} \, dt$ converges absolutely.

Proof of Lemma 2. It is clear that

$$\begin{aligned}
\sum_{l=0}^{d-2} C_{d-2}^l (-1)^{d-2-l} \frac{1}{d-1-l} &= \sum_{k=1}^{d-1} C_{d-2}^{d-1-k} (-1)^{k-1} \frac{1}{k} = \\
&= \frac{1}{d-1} \sum_{k=1}^{d-1} C_{d-1}^k (-1)^{k-1} = \\
&= \frac{1}{d-1} \left(1 - \sum_{k=0}^{d-1} C_{d-1}^k (-1)^k \right) = \\
&= \frac{1}{d-1} \left(1 - (1-1)^{d-1} \right) = \frac{1}{d-1},
\end{aligned}$$

which proves Lemma 2.

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