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ON THE DISTRIBUTION OF EXPONENTIAL DIVISORS

I. Kátai (Budapest, Hungary)
M.V. Subbarao (Edmonton, Canada)

Dedicated to Professor Karl-Heinz Indlekofer on his sixtieth birthday

1. Introduction

1.1. Let $\mathcal{P}$ be the whole set of the primes. p, p_i, p_j always denote prime numbers. For some integer n with prime decomposition $n = p_1^{\alpha_1} \dots p_r^{\alpha_r}$ the divisor $d = p_1^{\beta_1} \dots p_r^{\beta_r}$ is said to be an exponential divisor of n, if β_j divides α_j for every $j = 1, \dots, r$. It is obvious, that the number of the exponential divisors of n (it is denoted as $\tau^{(e)}(n)$) is $\tau(\alpha_1) \dots \tau(\alpha_r)$, where $\tau(m)$ is the number of the divisors of m. The notion of exponential divisors was introduced by Subbarao [4]. In [13] Fabrykowski and Subbarao proved that

$$\sum \tau^{(e)}(n) = A_1 x + O\left(x^{1/2}\log x\right).$$

Recently Wu [2] observed that the generating Dirichlet series

$$F(s) := \sum_{n=1}^{\infty} \frac{\tau^{(e)}(n)}{n^s}$$

can be written as $F(s) = \zeta(s)\zeta(2s)U(s)$, where $U(s)$ can be written as an absolute convergent Dirichlet series in the halfplane $\sigma > 1/5$, whence, by using the estimate

$$\sum_{d^2 \le x} \left(\left\{\frac{x}{d^2}\right\} - 1/2\right) \ll x^{2/9}\log x$$

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(see [6]) he was able to deduce that

$$\sum \tau^{(e)}(n) = A_1 x + A_2\sqrt{x} + O\left(x^{2/9}\log x\right).$$

Smati and Wu [3] recently proved that

$$\sum_{p\le x} \tau^{(e)}(p-1) = c \text{ li } x + O_A\left(\frac{x}{(\log x)^A}\right) \tag{1.1}$$

holds for every fixed A.

1.2. Let $\mathcal{B}$ be the set of square full numbers. For some integer n, let $E(n)$ be the square full, and $F(n)$ be the square free part of n. Then $n = E(n)F(n)$, $(E(n), F(n)) = 1$ and $E(n)$ is the largest divisor of n which belongs to $\mathcal{B}$.

For some $b \in \mathcal{B}$ let $\mathcal{R}_b$ be the set of those integers n for which $E(n) = b$.

Let

$$\begin{aligned} \nu_x(b) &:= \frac{1}{x}\#\{n < x,\ n \in \mathcal{R}_b\}, \\ \nu(b) &:= \lim_{x\to\infty} \nu_x(b). \end{aligned} \tag{1.2}$$

By elementary sieve one can deduce that

$$\nu(b) = \frac{1}{\zeta(2)b}\prod_{p\mid b}\frac{1}{1+1/p}. \tag{1.3}$$

Let $m \in \mathbb{N},\ m = p_1^{\alpha_1}\dots p_r^{\alpha_r}$. Let

$$D(m) := \{p_1^{\gamma_1}\dots p_r^{\gamma_r} \mid \gamma_1,\dots,\gamma_r \in \mathbb{N}_0\},$$

where $\mathbb{N}_0$ is the set of nonnegative integers.

Let

$$M(x) = \sum_{n\le x}|\mu(n)|; \quad M(x|b) = \sum_{\substack{n\le x\\(n,b)=1}}|\mu(n)|. \tag{1.4}$$

It is known that

$$M(x) - \frac{x}{\zeta(2)} \ll \sqrt{x}\ \exp\left(-c(\log x)^{3/5}(\log\log x)^{-1/5}\right)$$

(Walfisz [7]). We shall use the somewhat weaker inequality

$$M(x) - \frac{x}{\zeta(2)} = O(\sqrt{x}). \tag{1.5}$$

Theorem 1. *We have, for $b \in \mathcal{B}$*

$$\nu_x(b) = \nu(b) + O\left(\frac{1}{b\sqrt{x}} \prod_{p|b} \left(1 + \frac{1}{\sqrt{p}}\right)\right). \tag{1.6}$$

Remark. A. Ivić [10] proved: if f is a multiplicative function such that $f(p^\alpha) = g(\alpha) \in \mathbb{N},\ g(1) = 1$, then

$$\frac{1}{x} \sum_{\substack{n \le x \\ f(n)=k}} = d_k + O\left(\frac{1}{\sqrt{x}} \exp\left(-c(\log x)^{3/5}(\log\log x)^{-1/5}\right)\right).$$

From our Theorem 1 one can deduce a similar theorem which is weaker than his, if k is small, and stronger than his for large $|k|$.

Theorem 2. *Let f be a multiplicative function for which $f(p^\alpha) = g(\alpha),\ g(1) = 1,\ g(2) > 0$. Assume furthermore that*

$$\frac{|g(2)|}{2^2} + \frac{|g(3)|}{2^3} + \dots$$

is finite.

Then

$$\frac{1}{x} \sum_{n \le x} f(n) = \sum_{b \in \mathcal{B}} f(b)\nu(b) + O\left(\sqrt{x}(\log x)^{g(2)-1}\right), \tag{1.7}$$

$$\sum_{b \in \mathcal{B}} f(b)\nu(b) = \frac{1}{\zeta(2)} \prod_{p \in \mathcal{P}} \left(1 + \sum_{j=2}^{\infty} \frac{g(j)}{(1+1/p)\,p^j}\right). \tag{1.8}$$

Corollary of Theorem 1. We have

$$\frac{1}{x} \sum_{n \le x} \Omega\left(\tau^{(e)}(n)\right) = A + O\left(\frac{\log\log x}{\sqrt{x}}\right), \tag{1.9}$$

where

$$A = \sum_{b \in \mathcal{B}} \Omega\left(\tau^{(e)}(b)\right) \nu(b), \tag{1.10}$$

and

$$\frac{1}{x} \sum_{n \le x} \omega\left(\tau^{(e)}(n)\right) = B + O\left(\frac{\log\log x}{\sqrt{x}}\right), \tag{1.11}$$

where

$$B = \sum_{b \in \mathcal{B}} \omega\left(\tau^{(e)}(b)\right) \nu(b). \tag{1.12}$$

Here $\omega(n)$ is the number of the prime divisors, and $\Omega(n)$ is the number of prime-power divisors of n.

Remark. (1.11) is somewhat stronger than Corollary 1 in [3].

We shall prove

Theorem 3. *We have*

$$\#\{n \in [X, X+H], \quad n \in \mathcal{R}_b\} = \\ = H\nu(b) + O\left(X^{\Theta+\varepsilon} \cdot 2^{\omega(b)}\right) + O\left(H^{1/2} X^{\varepsilon} \prod_{p \mid b} (1 + 1/\sqrt{p})\right)$$

uniformly as $0 < H < x$. *Here* $\Theta = 0,2204$ *and* ε *is an arbitrary positive constant. The implied constants in the order terms may depend on* ε.

1.3 We have

Theorem 4. *Let* $\Theta = 7/12$, A *and* B *be arbitrary positive constants. Assume that* $x^{\Theta+\varepsilon} \le y \le x$. *Let* $b \in \mathcal{B}$ *and* $b < (\log x)^A$. *Then*

$$\#\{p < x \mid p - 1 \in \mathcal{R}_b\} = \rho(b) \text{ li } x + O\left(\frac{x}{(\log x)^B b}\right) \tag{1.13}$$

$$\#\{p \in [x, x+y] \mid p - 1 \in \mathcal{R}_b\} = \\ = \rho(b)\left(\text{li }(x+y) - \text{li } x\right) + O\left(\frac{y}{(\log x)^B b}\right), \tag{1.14}$$

where

$$\rho(b) := \frac{c}{b} \prod_{\substack{\pi \mid b \\ \pi \in \mathcal{P}}} \frac{\pi(\pi-1)}{\pi^2-\pi-1}, \quad C = \prod_{p \in \mathcal{P}} \left(1 - \frac{1}{p(p-1)}\right). \tag{1.15}$$

Corollary of Theorem 4. Let $y \in \left[x^{\Theta+\varepsilon}, x\right]$, $\Theta = 7/12$, $\quad r \in \mathbb{N}$. Then

$$\sum_{p \in [x, x+y]} \tau^{(e)r}(p-1) = D_r(\text{li } (x+y) - \text{li } x) + O\left(\frac{y}{(\log x)^{B_1}}\right), \tag{1.16}$$

where B_1 is an arbitrary constant.

Here

$$D_r = \sum_{b \in \mathcal{B}} \tau^{(e)}(b)^r \cdot \rho(b). \tag{1.17}$$

1.4. Let $a_1, a_2, \ldots, a_k$ be distinct positive integers, $G := \prod_{i<j} (a_i - a_j)$, $G = q_1^{\gamma_1} \ldots q_r^{\gamma_r}, q_1, \ldots, q_r$ be primes.

Let $T = G^{[A \log\log x]}$, and for $l \in [1, T-1]$, $(l, T) = 1$, $1 \leq l \leq T-1$ let $t_1(l), t_2(l), \ldots, t_k(l)$ be defined as

$$t_j(l) = \prod_{p^\alpha \| l + a_j} p^\alpha \qquad (l = 1, \ldots, k).$$

Let furthermore $e_j^{(l)} := E(t_j(l))$, the square full part of $t_j(l)$. Let $c_1, \ldots, c_k \in \mathcal{B}$, such that $(c_i, c_j) = 1$ $(i \neq j)$, and $(c_i, G) = 1$. Assume that $\max c_j < (\log x)^A$.

Theorem 5. *Let* $x^{\Theta+\varepsilon} \leq y \leq x$, $\Theta = 7/12$. *Then*

$$\begin{aligned} &\# \left\{ p \in [x, x+y] \mid p \equiv l \pmod{T}, \quad p + a_j \in \mathcal{R}_{e_j c_j} \quad (j = 1, \ldots, k) \right\} = \\ &= \frac{(\text{li } (x+y) - \text{li } (x))}{\varphi(T)} E(c_1, \ldots, c_k) + O_B\left(\frac{y}{(\log x)^B}\right), \end{aligned} \tag{1.18}$$

where

$$E(c_1, \ldots, c_k) = \frac{1}{c_1, \ldots, c_k} \prod_{\substack{\pi \nmid G c_1 \ldots c_k \\ \pi \in \mathcal{P}}} \left(1 - \frac{k}{\pi(\pi-1)}\right). \tag{1.19}$$

As a consequence, for $h(p) := \tau^{(e)r_1}(p+a_1)\dots\tau^{(e)r_k}(p+a_k)$ we have

$$\sum_{p\in[x,x+y]} h(p) = K_{r_1,\dots,r_k}(a_1,\dots,a_k)(\text{li}\,(x+y)-\text{li}\,x)+O_B\left(\frac{y}{(\log x)^B}\right), \tag{1.20}$$

B is an arbitrary positive constant.

1.5. We would be able to prove the following theorems.

Theorem A. *Let $f_1, f_2, \dots, f_k \in \mathbb{Z}[x]$ be such that every f_j is a product of distinct irreducible polynomials of degree not higher than three. Then*

$$\sum_{n\le x} \tau^{(e)}\left(f_1(n)\right)\tau^{(e)}\left(f_2(n)\right)\dots\tau^{(e)}(f_k(n)) = Cx + o(x)$$

with some positive constant C.

Theorem B. *If $f_1, \dots, f_k \in \mathbb{Z}[x]$, and every f_j is a product of distinct irreducible polynomials of degree not higher than two, then*

$$\sum_{p\le x} \tau^{(e)}\left(f_1(p)\right)\tau^{(e)}\left(f_2(p)\right)\dots\tau^{(e)}\left(f_k(p)\right) = C^*\ \text{li}\ x + o(\text{li}\ x)$$

with some positive constant C^*.

Theorem A can be proved on a routine way by using the following theorem of C. Hooley: if $f \in \mathbb{Z}[x]$ is irreducible, $\deg f \le 3$, then the number of the integers $n \le x$ for which there is a prime $p > \log x$ such that $p^2|f(n)$ is at most $O\left(x(\log x)^{-1/3}\right)$. See C. Hooley [5], Chapter 4, Theorem 3, or [12] for a better estimate.

Let $g \in \mathbb{Z}[x]$ be an irreducible polynomial of degree 2. Let $\varepsilon > 0$ and $y = x^{1/2+\varepsilon}$. One can prove that the number of the integers $n \in [x, x+y]$ for which there is a prime $q > (\log x)^2$ such that $q^2|g(n)$ is at most $O\left(Y/(\log x)^2\right)$.

Hence we can deduce Theorem B, or even a short interval version of it.

Let $\rho(m)$ be the number of solution of $n^2+1 \equiv 0 (\text{mod}\ m)$. We shall prove

Theorem 6. *Let $Y = x^{2/3+\varepsilon}$, $\varepsilon > 0$ be a small constant. Then, for every fixed $A > 0$,*

$$\sum_{n\in[X,X+Y]} \tau^{(e)}(n^2+1) = CY + O_A\left(Y/(\log x)^A\right),$$

where

$$C = \sum_{b \in \mathcal{B}} \frac{\tau^{(e)}(b)\rho(b)\varphi(b)}{b^2} \prod_{\substack{\pi \nmid b \\ \pi \in \mathcal{P}}} \left(1 - \frac{\rho(\pi^2)}{\pi^2}\right).$$

Theorem 7. *Let $Y = x^{2/3+\varepsilon}$, A be an arbitrary positive constant. Then*

$$\sum_{p \in [X, X+Y]} \tau^{(e)}(p^2+1) = C_1(\text{li }(X+Y) - \text{li } X) + O_A\left(Y/(\log x)^A\right),$$

where

$$C_1 = \sum_{b \in \mathcal{B}} \frac{\tau^{(e)}(b)\rho(b)}{b} \prod_{\substack{\pi \nmid b \\ \pi \in \mathcal{P}}} \left(1 - \frac{\rho(\pi)}{\pi(\pi-1)}\right).$$

2. Proof of Theorems 1, 2

Since

$$\sum_{(n,b)=1} \frac{|\mu(n)|}{n^s} = \prod_{p|b} \left(1 + \frac{1}{p^s}\right)^{-1} \cdot \sum_{n=1}^{\infty} \frac{|\mu(n)|}{n^s}$$

holds for $Re\ s > 1$, therefore

$$M(x|b) = \sum_{v \in D(b)} \lambda(v) M\left(\frac{x}{v}\right),$$

whence

$$M(x|b) = \frac{x}{\zeta(2)} \sum_{v \in D(b)} \frac{\lambda(v)}{v} + O\left(x^{1/2} \sum_{v \in D(b)} \frac{1}{v^{1/2}}\right) + O\left(x \sum_{\substack{v \geq x \\ v \in D(b)}} 1/v\right).$$

Since $v \geq \sqrt{x} \cdot v^{1/2}$ in the last sum, therefore

$$M(x|b) = \frac{x}{\zeta(2)} \prod_{p|b} \frac{1}{1+1/p} + O\left(x^{1/2} \prod_{p|b} \left(1 + \frac{1}{\sqrt{p}}\right)\right).$$

Observing that $\nu_x(b) = x^{-1} M\left(\frac{x}{b} \;\middle|\; b\right)$, Theorem 1 immediately follows.

To prove Theorem 2, we start from the equation

$$\frac{1}{x}\sum_{n\le x} f(n) = \sum_{\substack{b\le x\\ b\in\mathcal{B}}} f(b)\nu_x(b) = \sum\nolimits_1 + \sum\nolimits_2,$$

where

$$\sum\nolimits_1 = \frac{1}{\zeta(2)} \sum_{\substack{b\in\mathcal{B}\\ b<x}} \frac{f(b)}{b\prod_{p|b}(1+1/p)},$$

$$\sum\nolimits_2 = O\left(x^{-1/2}\sum_{b\in\mathcal{B}} \frac{f(b)}{b}\prod_{p|b}\left(1+\frac{1}{\sqrt{p}}\right)\right).$$

Let $t(y) = \left(1+\sqrt{y}\right)\left(|g(2)|\cdot y^2 + |g(3)|\cdot y^3 + \ldots\right)$. Then

$$\sum_{b\in\mathcal{B}} \frac{|f(b)|}{b}\prod_{p|b}\left(1+\frac{1}{\sqrt{p}}\right) \le \exp\left(2\sum_p t(1/p)\right).$$

Since

$$\sum_{p\in\mathcal{P}} \frac{1}{p^s} \le \frac{1}{2^{s-2}}\sum \frac{1}{p^2}, \qquad s = 3, 4, \ldots,$$

therefore

$$\sum_p t(1/p) \le \left(\sum \frac{1}{p^2}\right)\left(|g(2)| + \frac{|g(3)|}{2} + \frac{|g(4)|}{2^2} + \ldots\right)$$

and the right hand side is finite. Thus $\sum_2 = O\left(X^{-1/2}\right)$.

We shall prove that

$$\sum\nolimits_0 := \sum_{\substack{b\in\mathcal{B}\\ b\ge x}} \frac{f(b)}{b\prod_{p|b}(1+1/p)} \ll \frac{1}{\sqrt{x}}(\log x)^{g(2)-1},$$

and this completes the proof of Theorem 2.

We can write each b as v^2u, where v is square free and u is three full, i.e. $p|u$ implies that $p^3|u$.

Thus

$$\Delta(Y) := \sum_{uv^2 \le Y} |f(uv^2)| \le \sum_{u \le Y} |f(u)| \cdot \sum_{v < \sqrt{Y/u}} \left| f(v^2) \right| .$$

Since

$$\sum_{v < \kappa} |f(v^2)| \ll \kappa (\log \kappa)^{g(2)-1},$$

therefore

$$\Delta(Y) \ll (\log Y)^{g(2)-1} \sqrt{Y} \sum_{u \le Y} \frac{|f(u)|}{u^{1/2}}.$$

Furthermore

$$\sum \frac{|f(u)|}{u^{1/2}} \le \prod_{p \le Y} \left(1 + \frac{|g(3)|}{p^{3/2}} + \frac{|g(4)|}{p^2} + \dots \right). \tag{2.1}$$

Arguing as earlier, we can deduce that

$$\sum_{p \in \mathcal{P}} \left(\frac{|g(3)|}{p^{3/2}} + \frac{|g(4)|}{p^2} + \dots \right)$$

is convergent, thus the right hand side of (2.1) is bounded.

Thus $\Delta(Y) \ll (\log Y)^{g(2)-1} \sqrt{Y}$, and so

$$\sum\nolimits_0 \ll \sum_{j=0}^{\infty} \frac{1}{2^j X} \Delta(2^j X) \ll \frac{(\log X)^{g(2)-1}}{\sqrt{X}}.$$

3. Proof of the Corollary of Theorem 1

We shall prove (1.9) only. The proof of (1.11) is almost the same.
From (1.6) we obtain that

$$\frac{1}{x} \sum_{n \le x} \Omega \left(\tau^{(e)}(n) \right) = A + O\left(\sum\nolimits_1 \right) + O\left(\frac{1}{\sqrt{x}} \sum\nolimits_2 \right),$$

where

$$\sum\nolimits_1 = \sum_{\substack{b>x \\ b\in\mathcal{B}}} \frac{\Omega\left(\tau^{(e)}(b)\right)}{b},$$

$$\sum\nolimits_2 = \sum_{\substack{b<x \\ b\in\mathcal{B}}} \frac{\Omega\left(\tau^{(e)}(b)\right)}{b} \prod_{p|b}\left(1+\frac{1}{\sqrt{p}}\right).$$

Let $h(n) := \Omega\left(\tau^{(e)}(n)\right)$. h is completely additive, therefore

$$\sum\nolimits_2 \le 2\sum_{p^\nu} \frac{h(p^\nu)}{p^\nu}\left(\sum_{c\in\mathcal{B}} \frac{1}{c}\prod_{p|c}(1+1/\sqrt{p})\right).$$

The inner sum is convergent,

$$= \prod_p \left(1+\left(1+\frac{1}{\sqrt{p}}\right)\left(\frac{1}{p^2}+\frac{1}{p^3}+\dots\right)\right) \le C\prod_p\left(1+\frac{2}{p^2}\right).$$

Furthermore

$$\sum_{\substack{p\in\mathcal{P} \\ \nu\ge 2}} \frac{h(p^\nu)}{p^\nu} = O(1),$$

thus $\sum_2 = O(1)$.

Since

$$E(Y) : = \sum_{\substack{b<Y \\ b\in\mathcal{B}}} h(b) = \sum_{\substack{p^\nu\le Y \\ \nu\ge 2}} h(p^\nu) \sum_{\substack{c\le Y/p^\nu \\ c\in\mathcal{B}}} 1 \le$$

$$\le \sqrt{Y}\sum_{\substack{p\le\sqrt{Y} \\ p\in\mathcal{P}}} \frac{h(p^2)}{p} + \sqrt{Y}\sum_{\substack{p\in\mathcal{P} \\ \nu\ge 3}} \frac{h(p^\nu)}{p^{\nu/2}},$$

thus

$$E(Y) \ll \sqrt{Y}\log\log Y,$$

and so

$$\sum\nolimits_1 \ll \sum_{j=0}^{\infty} \frac{1}{2^j x} E\left(2^{j+1}x\right) \ll \frac{\log\log x}{\sqrt{x}}.$$

The proof is completed.

4. Proof of Theorem 3

We shall use the next Lemma 1 due to P. Varbanec [8].

Lemma 1. *Let $\phi(d)$ be a multiplicative function, such that $\phi(d) = O(d^\varepsilon)$ for $\varepsilon > 0$. Let*

$$f(n) = \sum_{d^2|n} \phi(d).$$

Then

$$\sum_{x\le n\le x+h} f(n) = h\sum_{d=1}^{\infty} \frac{\phi(d)}{d^2} + O\left(h^{1/2}x^\varepsilon\right) + O\left(x^{\Theta+\varepsilon}\right),$$

uniformly in h, $h < x$, where ε is an arbitrary positive constant and $\Theta = 0,2204$.

The exponent $\Theta < 2/9$, thus Lemma 1 is somewhat stronger than that of Graham and Kolesnik in [9].

Let $b \in \mathcal{B}$. Then

$$\begin{aligned}
&\#\left\{n \in [X, X+H] \;\mid n \in \mathcal{R}_b\right\} = \\
&\quad = \#\left\{m \in \left[\frac{X}{b}, \frac{X+H}{b}\right] \;\middle|\; (m,b)=1,\ |\mu(m)|=1\right\} = \\
&\quad = \sum_{\delta|b} \mu(\delta)\#\left\{\nu \in \left[\frac{X}{b\delta}, \frac{X+H}{b\delta}\right] \;\middle|\; p^2 \nmid \nu \quad \text{if } p \nmid b\right\}.
\end{aligned}$$

Let us apply Lemma 1 with $x = \dfrac{X}{b\delta}$, $h = \dfrac{H}{b\delta}$,

$$\phi(p) = \begin{cases} 0 & \text{if} \quad p|b, \\ 1 & \text{if} \quad (p,b)=1. \end{cases}$$

We have

$$\begin{aligned}
&\#\left\{n \in [X, X+H],\ n \in \mathcal{R}_b\right\} = \\
&\quad = H\nu(b) + O\left(X^{\Theta+\varepsilon}\cdot 2^{\omega(b)}\right) + O\left(H^{1/2}X^\varepsilon \prod_{p|b}\left(1+\frac{1}{\sqrt{p}}\right)\right),
\end{aligned}$$

which proves the theorem.

5. Proof of Theorem 4 and that of the Corollary

Let $\prod(x|b) = \#\{p < x \mid p-1 \in \mathcal{R}_b\}$. We shall start from the identity

$$\prod(x|b) = \sum_{\substack{\delta|b \\ (\kappa,b)=1}} \mu(\delta)\mu(\kappa)\pi(x,\delta\kappa^2 b,1), \tag{5.1}$$

which can be proved similarly as we argued at the proof of Theorem 1.

Let A and B be arbitrary constants. Assume that $b \le (\log x)^A$. Let

$$\prod(x|b) = \prod\nolimits_1(x|b) + \prod\nolimits_2(x|b), \tag{5.2}$$

where in $\prod_1$ we sum over $\kappa \le (\log x)^B$ (=: L), and in $\prod_2$ over $\kappa > L$.

By the Siegel-Walfisz theorem we obtain that

$$\prod\nolimits_1(x|b) = (\mathrm{li}\ x) \sum_{\substack{\delta|b \\ (\kappa,b)=1 \\ \kappa<L}} \frac{\mu(\delta)\mu(\kappa)}{\varphi(b\delta)\varphi(\kappa^2)} + O\left((\mathrm{li}\ x)\log x)^{-B} \sum_{\substack{\delta|b \\ (\kappa,b)=1 \\ \kappa\ge L}} \frac{1}{\varphi(b\delta\kappa^2)}\right).$$

Since

$$\sum_{\kappa\ge L} \frac{1}{\varphi(\kappa^2)} \ll \frac{1}{L},$$

we have

$$\prod\nolimits_1(x|b) = \nu(b)\ \mathrm{li}\ x + O\left(\frac{x}{(\log x)^B b}\right),$$

where

$$\rho(b) = \frac{c}{b} \prod_{\substack{\pi|b \\ \pi\in\mathcal{P}}} \frac{\pi(\pi-1)}{\pi^2-\pi-1}, \tag{5.3}$$

$$C = \prod_{p\in\mathcal{P}} \left(1 - \frac{1}{p(p-1)}\right). \tag{5.4}$$

To estimate $\prod_2(x|b)$, we observe that $\pi(x,D,1) \le \dfrac{c \text{ li } x}{\varphi(D)}$ if $D \le x^{3/4}$, and $\pi(x,D,1) \le \dfrac{x}{D}$ if $D \ge x^{1/2}$. Thus

$$\prod\nolimits_2(x|b) \ll (\text{li } x) \sum_{\substack{\delta | b \\ L<\kappa\le\sqrt{x}}} \frac{1}{\varphi(b\delta\kappa^2)} + x \sum_{\substack{\delta|b \\ \kappa>\sqrt{x}}} \frac{1}{\varphi(b\delta\kappa^2)},$$

and the right hand side is less than

$$\ll \frac{\text{li } x}{(\log x)^B} \frac{1}{\varphi(b)} \prod_{p|b} (1+1/p) \ll \frac{1}{b} \frac{x}{(\log x)^B}.$$

Thus

$$\prod(x|b) = \rho(b) \text{ li } x + O\left(\frac{x}{b(\log x)^B}\right) \tag{5.5}$$

if $b \le (\log x)^A$.

We can prove (1.14) similarly. We have to use the short interval version of the Siegel-Walfisz theorem (i.e. the theorem of Hoheisel and Tatuzawa, see K. Prachar [11], Theorem 3.2 in Chapter IX) and that

$$\pi(x+y,D,1) - \pi(x,D,1) \ll \frac{y}{\varphi(D)\log x} \quad \text{for } D < y^{1-\varepsilon},$$

and $\pi(x+y,D,1) - \pi(x,D,1) \ll (y/D+1)$ if $y < D$. We omit the details.

Now we prove the Corollary.

From sieve theorems we know that

$$\begin{aligned} \prod(x+y|b) - \prod(x|b) &\ll \frac{y}{\varphi(b)\log x} && \text{if } \ b \le \sqrt{x}, \\ &\ll \frac{y}{b} && \text{if } \ \sqrt{x} < b. \end{aligned}$$

Thus

$$\begin{aligned} \sum\nolimits_0 := \sum_{p\in[x,x+y]} \left(\tau^{(e)}(p-1)\right)^r = \sum_{\substack{b<x+y \\ b\in\mathcal{B}}} \tau^{(e)r}(b) \left(\prod(x+y|b) - \prod(x|b)\right) = \\ = \sum\nolimits_1 + \sum\nolimits_2, \end{aligned}$$

where in $\sum_1$ we sum over $b < (\log x)^A$ and in $\sum_2$ over the others.

One can prove simply that

$$\sum_{\substack{b \in \mathcal{B} \\ b < z}} \tau^{(e)r}(b) \ll z,$$

whence one gets that

$$\sum\nolimits_2 \ll \frac{y}{\log x} \left(\sum_{\substack{b > (\log x)^A \\ b \in \mathcal{B}}} \frac{\tau^{(e)r}(b)}{b} + (\log x) \sum_{b > \sqrt{x}} \frac{\tau^{(e)r}(b)}{b} \right) \ll$$
$$\ll \frac{y}{(\log x)^{A/2+1}}.$$

Furthermore,

$$\sum\nolimits_1 = (\text{li } (x+y) - \text{li } x) \sum_{b \le (\log x)^A} \tau^{(e)r}(b)\rho(b) + O\left(\sum\nolimits_3\right),$$

$$\sum\nolimits_3 \ll \frac{y}{(\log x)^B} \sum_{b \in \mathcal{B}} \frac{\tau^{(e)r}(b)}{b} \ll \frac{y}{(\log x)^B}.$$

Finally, we observe that

$$\sum_{\substack{b > (\log x)^A \\ b \in \mathcal{B}}} \tau^{(e)r}(b)\rho(b) \ll \frac{1}{(\log x)^{A/2}},$$

whence

$$\sum\nolimits_0 = (\text{li } (x+y) - \text{li } x) D_r + O\left(\frac{y}{(\log x)^{B_1}}\right),$$

D_r is defined in (1.17).

Since $A = 2B_1$ can be chosen, the Corollary is true.

6. Proof of Theorem 5

Let $p \equiv l \pmod{T}$. Then $p + a_j \in \mathcal{R}_{e_j c_j}$ holds, if $\left(\frac{p+a_j}{c_j}, c_j\right) = 1$, and $\kappa^2 \nmid \frac{p+a_j}{c_j}$ if $(\kappa, Gc_j) = 1$.

Thus $p+a_j \in \mathcal{R}_{e_j c_j}$, if

$$\sum_{\substack{\delta|c_j \\ \kappa^2|p+a_j \\ (\kappa,Gc_j)=1}} \mu(\delta)\mu(\kappa) = 1$$

and the above sum is zero if $p+a_j \notin \mathcal{R}_{e_j c_j}$.

Thus the left hand side of (1.18) is

$$\text{(6.1)} \qquad \sum \mu(\delta_1)\dots\mu(\delta_k)\mu(\kappa_1)\dots\mu(\kappa_k)(\pi((x+y),\Delta,r)-\pi(x,\Delta,r)),$$

where Δ is the least common multiple of T and $\delta_j c_j \kappa_j^2 \quad (j=1,\dots,k)$, i.e.

$$\Delta = \left[T, \delta_1 c_1 \kappa_1^2, \dots, \delta_k c_k \kappa_k^2\right],$$

and r mod Δ is such a residue, for which $r \equiv l \pmod T$ and $r+a_j \equiv \equiv 0 \pmod{\delta_j c_j \kappa_j^2}$ $(j=1,\dots,k)$ hold true simultaneously.

The sum is extended over all $\delta_1,\dots,\delta_k,\ \kappa_1,\dots,\kappa_k$ such that

$$\delta_j | c_j,\ (\kappa_j, Gc_j) = 1.$$

Since $(\delta_{i_1} c_{i_1} \kappa_{i_1}^2,\ \delta_{i_2} c_{i_2} \kappa_{i_2}^2) \mid a_{i_2} - a_{i_1}$ if $i_1 \neq i_2$, and r satisfies the above equations, therefore $(\delta_{i_1} c_{i_1} \kappa_{i_1}^2,\ \delta_{i_2} c_{i_2} \kappa_{i_2}^2) = 1$ for every couple $i_1 \neq i_2$. Thus

$$\begin{aligned}\varphi(\Delta) &= \varphi(T)\cdot\varphi(\delta_1 c_1 \kappa_1^2)\dots\varphi(\delta_k c_k \kappa_k^2) = \\ &= \varphi(T)\delta_1\dots\delta_k\varphi(c_1)\dots\varphi(c_k)\varphi((\kappa_1\dots\kappa_k)^2).\end{aligned}$$

Furthermore,

$$\begin{aligned}\sum \frac{\mu(\delta_1)\dots\mu(\delta_k)\mu(\kappa_1)\dots\mu(\kappa_k)}{\varphi(\Delta)} &= \frac{1}{\varphi(T)\varphi(c_1)\dots\varphi(c_k)}\times \\ \times \prod_{p|c_1}\left(1-\frac{1}{p}\right)\cdot\ldots\cdot\prod_{p|c_k}\left(1-\frac{1}{p}\right)&\prod_{\pi\nmid Gc_1\dots c_k}\left(1-\frac{k}{\pi(\pi-1)}\right) = \\ = \frac{1}{\varphi(T)c_1\dots c_k}\prod_{\pi|G(c_1\dots c_k)}&\left(1-\frac{k}{\pi(\pi-1)}\right) = \frac{1}{\varphi(T)}E(c_1,\dots,c_k).\end{aligned}$$

By the prime number theorem for short intervals we obtain that

$$\varepsilon(\Delta) := \pi(x+y,\Delta,r)-\pi(x,\Delta,r) = \frac{(\text{li }(x+y)-\text{li }x)}{\varphi(\Delta)} + O\left(\frac{y}{(\log x)^A}\right)$$

whenever $\kappa_j \ll (\log x)^{A_1}$, A_1 is large. For larger values of κ_j we can use the upper bounds

$$\varepsilon(\Delta) \ll \frac{y}{\varphi(\Delta)\log x}, \qquad \Delta < y^{1-\varepsilon}, \quad \text{and}$$
$$\varepsilon(\Delta) \ll \frac{y}{\varphi(\Delta)} \qquad \text{for } \Delta > y^{1-\varepsilon}.$$

Substituting this estimates into (6.1), we obtain (1.19) easily. The relation (1.20) is a simple consequence of (1.18).

7. Proof of Theorems 6, 7

We shall use the following

Lemma 2. *For every fixed $A > 0$ the number of solutions of $n^2 - Am^2 = -1$, $0 < n < x$ is at most $O(\log x)$, where the implied constant is absolute.*

Proof. If (n,m) is a solution, then $\left|\frac{n}{m} - \sqrt{A}\right| \le \frac{1}{2m^2}$, thus $\frac{n}{m}$ is an approximant of the continous fraction of $\sqrt{A}$, therefore the assertion is true.

Lemma 3. *Let $y = x^{2/3+\varepsilon}$, and $E(x,y)$ be the number of those integers $n \in [x, x+y]$ for which there is a prime q such that $q^2 \ge y$ and $q^2 | n^2+1$. Then*

$$E(x, x+y) \ll x^{2/3}(\log x) = y\frac{\log x}{x^{\varepsilon}}.$$

Proof. If n_1, n_2 are such integers for which $n_j^2 + 1 \equiv 0 \pmod{q^2}$, $(j = 1,2)$, then $n_2^2 - n_1^2 \equiv 0 \pmod{q^2}$. Since $n_1 - n_2 \equiv 0 \pmod{q}$, $n_1 + n_2 \equiv 0 \pmod{q}$ cannot hold, therefore either $n_1 + n_2 \equiv 0 \pmod{q^2}$, or $n_1 - n_2 \equiv 0 \pmod{q^2}$. It implies that for every q, $q^2 \ge y$ no more than two n exist in $[x, x+y]$ for which $q^2 | n^2 + 1$.

If $q^2 | n^2 + 1$, $n \in [X, X+Y]$, and $q > X^{\lambda}$, then $n^2 + 1 = Aq^2$ and $A < 2X^{2-2\lambda}$. From Lemma 1 we obtain that for fixed A no more than $\log x$ such n exists. Thus the whole contribution of these q is less than $O\left(X^{2-2\lambda}\log X\right)$.

Thus no more than $O\left(X^{\lambda}/\log x\right) + O\left(X^{2-2\lambda}\log x\right)$ integers $n \in [X, X+Y]$ exists for which $q^2 | n^2 + 1$ for some $q \ge \sqrt{Y}$. By $\lambda = \frac{2}{3}$ the inequality follows.

Let $\rho(m)$ be the number of solutions of the congruence $n^2+1 \equiv 0 \pmod m$. As it is known, $\rho(m)$ is multiplicative, $\rho(2) = 1$, $\rho(2^\alpha) = 0$ $(\alpha \geq 2)$, $\rho(p^\alpha) = \rho(p) = 2$ or 0 according to $p \equiv 1$ or $p \equiv -1 \pmod 4$.

Lemma 4. *Let A be an arbitrary constant, $B = 2A$. Then*

$$\sum_{b>(\log x)^B} \sum_{\substack{n\in[x,x+y] \\ n^2+1\in\mathcal{R}_b}} \tau^{(e)}(n^2+1) \ll \frac{y}{(\log x)^A}.$$

Proof. Let $\varepsilon_1 > 0$ be a small constant. Let us consider first those integers $n \in [x, x+y]$ for which $q^2 | n^2+1$, $q > x^{\varepsilon_1}$. The sum of $\tau^{(e)}(n^2+1)$ for those n for which $q^2|n^2+1$, $q > \sqrt{y}$ is less than $y \cdot \dfrac{(\log x)}{x^{\varepsilon/2}}$. (See Lemma 3, and that $\tau^{(e)}(n^2+1) \ll x^{\varepsilon/2}$).

It is obvious that

$$\sum_{(\log x)^B \leq b<y} \sum_{\substack{n\in[x,x+y] \\ n^2+1\in\mathcal{R}_b}} \tau^{(e)}(n^2+1) \ll y \sum_{(\log x)^B \leq b<y} \frac{\tau^{(e)}(b)\rho(b)}{b}. \tag{7.1}$$

Since

$$\sum \frac{\tau^{(e)}(b)\rho(b)}{b^s} = \prod_{p\equiv 1 \pmod 4} \left(1 + \frac{2\cdot 2}{p^{2s}} + \frac{2\cdot 2}{p^{3s}} + \dots\right),$$

we can get that

$$\sum_{\substack{b\in\mathcal{B} \\ b<z}} \tau^{(e)}(b)\rho(b) \ll \sqrt{z} \qquad (z\to\infty), \tag{7.2}$$

and so the right hand side of (5.1) is less than $y/(\log x)^{B/2}$.

Finally we consider those $b \in \mathcal{B}$ for which $b > y$ and for each prime divisor q of b, $q < x^{\varepsilon_1}$. If $n^2+1 \in \mathcal{R}_b$, then there exists some $b_1|b$, such that $b_1 \in \mathcal{B}$, and $y \cdot x^{-3\varepsilon_1} < b_1 < x$. For such an n, $\tau^{(e)}(n^2+1) \ll x^\varepsilon$, and the remaining part of the left hand side of (7.1) is less than

$$x^\varepsilon \sum_{\substack{Y\cdot x^{-3\varepsilon_1}<b_1<y \\ b_1\in\mathcal{B}}} \sum_{\substack{n^2+1\equiv 0 \pmod{b_1} \\ n\in[x,x+y]}} \tau^{(e)}(n^2+1) \ll yx^\varepsilon \sum \frac{\rho(b_1)}{b_1} \ll y^{3/4},$$

say. The proof is completed.

Let $H = (\log x)^B$, $B = 2A$. For some $b \in \mathcal{B}$ let $S(b)$ be the number of those $n \in [x, x+y]$ for which $\frac{n^2+1}{b}$ is squarefree, and $S^*(b)$ the number of those n, for which $\frac{n^2+1}{b}$ does not have prime square divisor κ^2, if $\kappa < H$. From Lemma 4 we obtain that

$$\sum_{b\in\mathcal{B}} \tau^{(e)}(b)\left(S^*(b) - S(b)\right) \ll y/(\log x)^A.$$

We have

$$S^*(b) = \sum_{\delta,\kappa} \mu(\delta)\mu(\kappa)\#\left\{n \in [x, x+y] \mid n^2+1 \equiv 0 \ (b\delta\kappa^2)\right\},$$

where $\delta | b$, $(\kappa, b) = 1$, and the largest prime factor of κ is less than H. Let $b < H$. Thus

$$S^*(b) = Y \sum_{\delta,\kappa} \frac{\mu(\delta)\mu(\kappa)\rho(b\delta\kappa^2)}{b\delta\kappa^2} +$$
$$+ O\left(\sum_{b\delta\kappa^2<4x^2} \rho(b\delta\kappa^2)\right) + O\left(Y \sum_{b\delta\kappa^2>y} \frac{\rho(b\delta\kappa^2)}{b\delta\kappa^2}\right).$$

The error terms are clearly less than x^ε. Thus the first sum

$$= \frac{\rho(b)}{b} \prod_{p|b}\left(1 - \frac{1}{p}\right) \prod_{\substack{\pi\nmid b\\ \pi<H}} \left(1 - \frac{\rho(\pi^2)}{\pi^2}\right) =$$
$$= \frac{\rho(b)\varphi(b)}{b^2} \prod_{\pi\nmid b}\left(1 - \frac{\rho(\pi^2)}{\pi^2}\right)\left(1 + O\left(\frac{1}{H}\right)\right).$$

We can continue on a routine way, and deduce Theorem 6.

The proof of Theorem 7 is similar. Doing the same as earlier, we reduce the proof to estimate $\#\left\{p \in [x, x+y] \mid p^2+1 \equiv 0 \ (b\delta\kappa^2)\right\}$ for $b < H$, $\delta|b$, $\kappa^2 < < H$, for which we can use the short interval version of Siegel-Walfisz theorem, according to it equals

$$\frac{\rho(b\delta\kappa^2)}{\varphi(b\delta\kappa^2)}\left(\text{li }(x+y) - \text{li }x\right) + O\left(y/(\log x)^A\right).$$

Hence one can deduce that the number of primes $p \in [x, x+y]$, for which $\left(\frac{p^2+1}{b}, b\right) = 1$, and $\pi^2 \nmid \frac{p^2+1}{b}$ for the primes $\pi < H$, equals

$$(\text{li }(x+y) - \text{li }x)\frac{\rho(b)}{\varphi(b)}\prod_{p|b}\left(1-\frac{1}{p}\right)\cdot\prod_{\substack{\pi\nmid b\\ \pi<H}}\left(1-\frac{\rho(\pi)}{\pi(\pi-1)}\right)+$$
$$+O\left(\frac{y}{(\log x)^A}\frac{1}{\varphi(b)}\right),$$

whence one can deduce Theorem 7 on a routine way. We have

$$C_1 = \sum_{b\in\mathcal{B}}\frac{\tau^{(e)}(b)\rho(b)}{b}\prod_{\pi\nmid b}\left(1-\frac{\rho(\pi)}{\pi(\pi-1)}\right).$$

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I. Kátai
Department of Computer Algebra
Eötvös Loránd University
Pázmány Péter sét. 1/C
H-1117 Budapest, Hungary
katai@compalg.inf.elte.hu

M.V. Subbarao
University of Alberta
Edmonton, Alberta
Canada T6G 2G1
m.v.subbarao@ualberta.ca