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ON LINEAR COMBINATIONS OF q–ADDITIVE FUNCTIONS

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1. Let $q \geq 2$, $A = \{0, 1, \ldots, q-1\}$, $\varepsilon(n)(\in A)$ be the sequence of the digits in the q-ary expansion of n,

$$n = \sum \varepsilon_j(n) q^j. \tag{1.1}$$

Let $\mathcal{A}_q$ be the set of real valued q-additive functions. We say that $f : \mathbb{N}_0(= N \cup \{0\}) \to \mathbb{R}$ belongs to $\mathcal{A}_q$, if $f(0) = 0$ and

$$f(n) = \sum_{j=0}^{\infty} f(\varepsilon_j(n) q^j) \tag{1.2}$$

holds for every n.

Let $1 \leq a_1 < a_2 < \ldots < a_k \ (< q)$, $(a_i, q) = 1$, $(a_i, a_j) = 1$ for every i and $j \neq i$. Let furthermore $f_1, \ldots, f_k \in \mathcal{A}_q$,

$$l(n) := f_1(a_1 n) + f_2(a_2 n) + \ldots + f_k(a_k n). \tag{1.3}$$

We say that a function $g : \mathbb{N} \to \mathbb{R}$ is "tough" if there is a sequence E_N such that

$$\begin{cases} \limsup\limits_{N\to\infty} \dfrac{1}{q^N} \# \left\{ n < q^N \,\middle|\, |f(n) - E_N| > K \right\} := c(K), \\ c(K) \to 0 \quad (K \to \infty). \end{cases} \tag{1.4}$$

We say that g is "bounded in mean" if (1.4) holds with $E_N = 0$.

Let $\alpha > 0$. We say that g belongs to the class L^α, if

$$\limsup_{x\to\infty} \frac{1}{x} \sum_{n<x} |g(n)|^\alpha < \infty. \tag{1.5}$$

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The following assertions are quite obvious.

Theorem 1. *The function $f \in \mathcal{A}_q$ is tough if and only if*

$$\sum_{j=0}^{\infty} \sum_{b \in A} f^2(bq^j) < \infty. \tag{1.6}$$

Theorem 2. *Let $f \in \mathcal{A}_q$. f is bounded in mean if and only if (1.6) holds and*

$$E_N^* := \frac{1}{q} \sum_{j=0}^{N-1} \sum_{b \in A} f(bq^j) \tag{1.7}$$

is bounded.

Theorem 3. *Let $f \in \mathcal{A}_q$. If f is bounded in mean, then $f \in L^\alpha$ for each $\alpha > 0$.*

Remark. The opposite assertion is clear. Theorems 1,2 are well-known in probability theory.

Let

$$m_j := \frac{1}{q} \sum_{b=0}^{q-1} f(bq^j), \quad E_N^* = \sum_{j=0}^{N-1} m_j,$$

$X_0, X_1, \ldots$ be a sequence of independent random variables,

$$P(X_j = f(bq^j) - m_j) = 1/q \quad (b \in \mathbb{A}) \quad b = 0, \ldots, q-1$$

and let

$$Y_N = X_0 + \ldots + X_{N-1}.$$

Let us prove Theorem 3. Assume that f is bounded in mean, that is (1.6) holds and (1.7) is bounded.

It is clear that EX_j, the mean value of X_j, is zero, furthermore that $X_j \to \to 0 \quad (j \to \infty)$ in measure.

Let $k \geq 1$. Then

$$Y_N^{2k} = \sum_{\substack{\alpha_1 + \ldots + \alpha_r = 2k \\ \alpha_\nu \geq 1}} c(\alpha_1, \ldots, \alpha_r) \sum X_{i_1}^{\alpha_1} \ldots X_{i_r}^{\alpha_k},$$

where the coefficient $c(\alpha_1, \dots, \alpha_r)$ depends only on k. From the independency of $X_0, \dots, X_{N-1}$ we obtain that $E\left(X_{i_1}^{\alpha_1} \dots X_{i_r}^{\alpha_k}\right) = 0$ if $\alpha_\nu = 1$ for some ν. Furthermore

$$\left|E\left(X_{i_1}^{\alpha_1} \dots X_{i_r}^{\alpha_r}\right)\right| \leq C \cdot E\left(X_{i_1}^2\right) \dots E\left(X_{i_r}^2\right)$$

with some constant C which may depend on k.

Consequently from (1.6) we have that

$$E\left(Y_N^{2k}\right) \ll \sum_{r \leq k} \left(\sum_{j=0}^{N-1} E^2\left(X_j\right) \right)^r \ll C_k.$$

The proof is completed.

2. In our paper [2] we proved the following theorem which we state here as

Lemma 1. *Assume that $l(n)$ is defined by (1.3), and the conditions, stated there, are satisfied. Then $l(n)$ is "tough" if and only if there exist $\gamma_1, \dots, \gamma_k \in \mathbb{R}$ such that $a_1\gamma_1 + \dots + a_k\gamma_k = 0$, and for $\psi_l(n) = f_l(n) - \gamma_l n$*

$$\sum_j \sum_b \psi_l^2(bq^j) < \infty \qquad (l = 1, \dots, k). \tag{2.1}$$

Let

$$E_N^{(l)} = \frac{1}{q} \sum_{j=0}^{N-1} \sum_{b \in \mathbb{A}} \psi_l(bq^j),$$

$$E_N = E_N^{(1)} + \dots + E_N^{(k)}. \tag{2.2}$$

Furthermore, $l(n)$ is bounded in mean, if (2.1) is satisfied and (2.2) is bounded.

By using the argument which was applied by the proof of Theorem 3, we obtain

Theorem 4. *Let $l(n)$ as is (1.3). If $l(n)$ is bounded in mean, then so are $f_j(n)$ $\quad (j = 1, \dots, k)$, and $l \in L^\alpha$ for every $\alpha > 0$.*

3. The $\log n$ is a very special function among the additive arithmetical functions. Somehow its role is played by cn among the q-additive functions.

The function $f(n) = n$ has a simple distribution, if we normalize appropriately:

$$\lim_{x \to \infty} \frac{1}{x} \# \left\{ n < x \;\middle|\; \frac{n}{x} < y \right\} = G(y), \tag{3.1}$$

$$G(y)=\begin{cases} y & \text{if } y\in[0,1],\\ 1 & \text{if } y\ge 1,\\ 0 & \text{if } y\le 0.\end{cases}$$

Theorem 5. *Assume that $f\in\mathcal{A}_q$, and*

$$\lim_{N\to\infty}\frac{1}{x}\left\{n<q^N \;\middle|\; \frac{f(n)}{x}<y\right\}=G(y). \tag{3.3}$$

Then $f(n)=n+h(n)$, and $\dfrac{h(n)}{n}\to 0\quad(n\to\infty)$.

Proof. Assume that (3.3) holds. Let

$$F_x(y):=\frac{1}{x}\#\left\{n<x \;\middle|\; \frac{f(n)}{x}<y\right\}.$$

Since G is continuous, therefore the convergence of $F_{q^N}(y)\to G(y)$ is uniform in y.

First we prove that

$$\limsup_{N\to\infty}\frac{f(bq^N)}{bq^N}\le 1,\quad b=1,\dots,q-1,$$
$$\liminf_{N\to\infty}\frac{f(bq^N)}{bq^N}\ge -1/b,\quad b=1,\dots,q-1.$$

Let $\delta>0$ be an arbitrary small constant, and $N\to\infty$.

From (3.3) we obtain that for all but $o(x)$ of the integers $\nu\le x$,

$$f(\nu)\in[-\delta x,\ (1+\delta)x].$$

Let us consider the integers $n\in\left[bq^N,\ (b+1)q^N\right]$. A typical integer m can be written as $m=bq^N+\nu,\ \nu<q^N$. Since

$$-\delta q^N<f(m)\le(b+1+\delta)q^N$$

for all but $o(q^N)$ integers, and for $\varepsilon>0$ the number of the integers ν satisfying $\dfrac{f(\nu)}{q^N}\in(1-\varepsilon,\ 1+\varepsilon)$ has a positive proportion, therefore

$$\frac{f(bq^N)}{q^N}+1\in[-\delta-\varepsilon,\ b+1+\delta+\varepsilon],$$

and so

$$\frac{f(bq^N)}{bq^N} < 1 + \frac{\delta + \varepsilon}{b},$$

if N is large enough. Thus

$$\limsup_{N\to\infty} \frac{f(bq^N)}{bq^N} \le 1.$$

Similarly, we can prove that

$$\liminf \frac{f(bq^N)}{bq^N} \ge -1/b.$$

Let $c \in \{1, \dots, q\}$. Count the integers $n < cq^N$ for which $\frac{f(n)}{cq^N} < y$. We subdivide the set of those n according to the leading digit.

If $n = bq^N + \nu$, then

$$\frac{f(n)}{cq^N} < y \quad \text{if and only if} \quad \frac{f(\nu)}{cq^N} < y - \frac{f(bq^N)}{cq^N},$$

i.e. if

$$\frac{f(\nu)}{q^N} < cy - \frac{f(bq^N)}{q^N}.$$

Hence we obtain that

$$cq^N F_{cq^N}(y) = \sum_{b=0}^{c-1} q^N F_{q^N}\left(cy - \frac{f(bq^N)}{q^N}\right),$$

and from (3.3) we obtain that

$$cG(y) = \sum_{b=0}^{c-1} G\left(cy - \frac{f(bq^N)}{q^N}\right) + O(\varepsilon_N), \tag{3.4}$$

where $\varepsilon_N \to 0$ as $N \to \infty$. The relation (3.4) is valid uniformly in y. Thus

$$\sum_{b=0}^{c-1} G\left(\frac{-f(bq^N)}{q^N}\right) \to 0 \quad (N \to \infty)$$

which proves that

$$\liminf \frac{f(bq^N)}{bq^N} \geq 0.$$

Assume that $\liminf\limits_{N\to\infty} \dfrac{f(bq^N)}{bq^N} < 1$ holds for some b. Let b be the smallest digit with this property. Let

$$\frac{f(bq^{N_j})}{bq^{N_j}} \to 1-\Delta \quad (j\to\infty),$$
$$\frac{f(sq^{N_j})}{sq^{N_j}} \to 1 \quad \text{as} \quad s=1,\dots,b-1, \quad j\to\infty.$$

Choose $c=b+1$. Then

$$cG(y) = G(cy-(c-1)(1-\Delta)) + \sum_{s=0}^{c-2} G(cy-s).$$

Put now $y = \dfrac{c-1}{c}$. Then

$$c-1 = cG\left(\frac{c-1}{c}\right) = G(\Delta(c-1)) + \sum_{s=0}^{c-2} G(c-1-s).$$

The sum on the right is $c-1$, $G(c-1-s)=1$ for $s=0,\dots,c-2$, consequently $G(\Delta(c-1))=0$, and so $\Delta(c-1)\leq 0$, which by $\Delta\geq 0$ implies that $\Delta=0$. The proof is ready.

4. Let $a_1,\dots,a_k$, q be as earlier, $f_1,\dots,f_k$ be integer valued q-additive functions. For some integers $m_1,\dots,m_k$ let

(4.1) $$\delta_{m_1,\dots,m_k}(u_1,\dots,u_k) =$$

$$= \lim_{x\to\infty} \frac{1}{x} \#\left\{n\leq x \mid f_j(a_j n)\equiv u_j \pmod{m_j}, \quad j=1,\dots,k\right\},$$

assuming that the limit exists.

Let $\mathcal{P}$ be the set of "generating elements",

$$\mathcal{P} = \left\{bq^j \mid b\in A, \quad j=0,1,2,\dots\right\}.$$

Theorem 6. *Assume that* $(m_i, m_j) = 1,\ (a_i, m_i) = 1$ *for* $i = 1, \dots, k,\ j \neq \neq i$. *Let* Δ_j *be the largest divisor of* m_j *for which* $\Delta_j \mid f_j(\pi)$ *holds for all but finitely many* $\pi \in \mathcal{P}$.

The relation

$$\delta_{m_1,\dots,m_k}(u_1, \dots, u_k) = \frac{1}{m_1 \dots m_k} \tag{4.2}$$

holds for every $u_j = 0, \dots, m_j - 1 \quad (j = 1, \dots, k)$ *if and only if* $\Delta_1 = \Delta_2 = = \dots = \Delta_k = 1$.

Proof. I. Assume that the limit in (4.1) exists for every $u_1, \dots, u_k$. Let $m_1^* \mid m_1, \dots, m_k^* \mid m_k$. Then $\delta_{m_1^*,\dots,m_k^*}\ (v_1, \dots, v_k)$ exists for every $v_1, \dots, v_k$. If the distribution is uniform for $\{m_1, \dots, m_k\}$, (i.e. if (4.2) holds), then it is uniform for $\{m_1^*, \dots, m_k^*\}$ as well.

II. If the distribution of $\{f_1(a_1 n) \pmod{m_1}, \dots, f_k(a_k n) \pmod{m_k}\}$ is uniform, and $\{i_1, \dots, i_h\}$ is a subset of $\{1, \dots, k\}$, then the distribution of

$$\{f_{i_l}(a_{i_l} n) \bmod m_{i_l} \mid l = 1, \dots, h\}$$

is uniform. Especially

$$\{f_i(a_i n) \bmod m_i\}$$

is distributed uniformly.

III. Let $m > 1,\ a \in \mathbb{N},\ (a, m) = (m, q) = (a, q) = 1,\ f \in \mathcal{A}_q,\ f(n) \in \in \mathbb{Z} \quad (n \in \mathbb{N})$. Assume that for some $l_0 \in \mathbb{N}_0,\ f(nq^{l_0}) \equiv 0 \pmod{m}$. Then the limit distribution of $f(an) \pmod{m}$ exists and it is non-uniform.

Indeed, let us write n as $n = n_0 + q^{l_0} n_1,\ s(n) = n_1,\ T(n) = n_1 = \left[\frac{n}{q^{l_0}}\right]$. Then $an = s(an) + q^{l_0} T(an),\ an_0 = s(an_0) + q^{l_0} T(an_0)$, consequently $s(an) \equiv \equiv s(an_0) \bmod q^{l_0}$, and

$$f(an) \equiv f(s(an_0)) \pmod{m}. \tag{4.3}$$

The density of the integers $n \equiv u \pmod{q^{l_0}}$ is $1/q^{l_0}$, for every $u \pmod{q^{l_0}}$. Therefore

$$\delta_m(v) := \lim \frac{1}{x} \sum_{\substack{f(an) \equiv v \pmod{m} \\ n \leq x}} 1 = \sum_{h=0}^{q^{l_0}-1} \tau_h(v),$$

where

$$\tau_h(v) = \lim_{x \to \infty} \frac{1}{x} \sum_{\substack{n \leq x \\ n \equiv h \pmod{q^{l_0}} \\ f(an) \equiv v \pmod{m}}} 1.$$

Since the fulfilment of $f(an) \equiv v \pmod{m}$ does depend only on $n \pmod{q^{l_0}}$, therefore $\tau_h(v)$ equals 0 or $1/q^{l_0}$. Consequently $\delta_m(v) = \dfrac{\text{integer}}{q^{l_0}}$, so $\delta_m(v) = 1/m$ cannot hold.

IV. If (4.2) holds, then $\Delta_1 = \ldots = \Delta_k = 1$. This is a direct consequence of III. Assume that $\Delta_1 > 1$. If (4.2) holds, then by II., $\{f_1(a_1 n) \bmod m_1\}$ is distributed uniformly, and from I., $\{f_1(a_1 n) \pmod{\Delta_1}\}$ is distributed uniformly. But this is impossible due to III.

V. From now we assume that $\Delta_1 = \ldots = \Delta_k = 1$. The fulfilment of (4.2) is equivalent to

$$\frac{1}{x}\sum_{n<x} e\left(\frac{h_1}{m_1} f_1(a_1 n) + \ldots + \frac{h_k}{m_k} f_k(a_k n)\right) \to 0 \tag{4.4}$$

for each choice of $h_j \pmod{m_j}$ $(j = 1, \ldots, k)$ except the case when $h_j \equiv \equiv 0 \pmod{m_j}$ for every j. Assume that (4.2) does not hold. Then there exists $h_1, \ldots, h_k, \quad h_j \in \{0, \ldots, m_j - 1\}$, $(h_1, \ldots, h_k) \neq (0, \ldots, 0)$ such that

$$\limsup \frac{1}{x} \left| \sum e\left(\frac{h_1 f_1(a_1 n)}{m_1} + \ldots + \frac{h_k f_k(a_k n)}{m_k}\right)\right| > 0. \tag{4.5}$$

From our theorem [2] we have: there exist $\gamma_1, \ldots, \gamma_k \in \mathbb{R}$ such that

$$a_1\gamma_1 + \ldots + a_k\gamma_k = \frac{E}{q^{l_0}},$$

$l_0 \geq 0$, $E \in \mathbb{Z}$, and

$$\sum_{\pi\in\mathcal{P}} \left\| \frac{h_j f_j(\pi)}{m_j} - \gamma_j \pi \right\|^2 < \infty. \tag{4.6}$$

It implies that

$$\sum_{\pi\in\mathcal{P}} \|m_j\gamma_j\pi\|^2 < \infty. \tag{4.7}$$

Hence we obtain that $m_j\gamma_j$ is rational. Indeed, assume that $\lambda = m_j\gamma_j$ is irrational. We shall show that for every c there is an $l > c$ for which $\|\lambda q^l\| > > 1/q^2$, and this is enough.

Assume that $q^{-T} < \|\lambda q^h\| < q^{-T+1}$ (for $j = 0, \ldots, T-2$), and so $\|\lambda q^{h+(T-2)}\| > 1/q^2$. Let $\lambda = \dfrac{U_j}{V_j}$, $(U_j, V_j) = 1$. From (4.7) we obtain that

$V_j \mid q^{l_0}$ for some suitable integer l_0. If $V_j = P_j Q_j,\ (Q_j, q) = 1,\ P_j \mid q^{l_0},\ Q_j > 1$, then $\|\lambda q^l\| \geq \frac{1}{Q_j}$, which contradicts to (4.7).

Thus we can write

$$\gamma_j = \frac{U_j}{V_j m_j}, \quad V_j \mid q^{l_0} \quad (j = 1, \ldots, k)$$

and from (4.6), that

$$\frac{h_j f_j(\pi \cdot q^{l_0})}{m_j} - \frac{U_j \left(\frac{q^{l_0}}{V_j}\right) \pi}{m_j} \equiv 0 \pmod 1,$$

whence

$$h_j f_j \left(\pi q^{l_0}\right) \equiv D_j \pi \pmod{m_j}$$

$$D_j = U_j \cdot \left(\frac{q^{l_0}}{V_j}\right),$$

$$\frac{a_1 U_1}{m_1 \cdot V_1} + \ldots + \frac{a_k U_k}{m_k V_k} = \frac{E}{q^{l_1}}.$$

We may assume that $l_1 \geq l_0$. Then

$$\text{(4.8)} \qquad \begin{gathered} \frac{a_1 U_1}{m_1} W_1 + \ldots + \frac{a_k U_k}{m_k} W_k = E, \\ W_j = \frac{q^{l_1}}{V_j} \in \mathbb{Z}. \end{gathered}$$

Let us multiply (4.8) with $m_1, \ldots, m_k$. Then m_j is a divisor of

$$a_j U_j W_j \prod_{\substack{\nu=1 \\ \nu \neq j}}^{k} m_\nu.$$

Since $(m_j, m_\nu) = 1$, $(m_j, q) = 1$ and $W_j | q^{l_1}$, therefore $(m_j, W_j) = 1$, $(m_j, a_j) = 1$ holds by the assumptions, therefore $m_j \mid U_j \quad (j = 1, \ldots, k)$. Consequently $\gamma_j \pi = \frac{(U_j \mid m_j)}{V_j} \pi =$ integer if $q^{l_0} \mid \pi$, i.e. for all but finitely many $\pi \in \mathcal{P}$. Therefore

$$\text{(4.9)} \qquad \sum_{\pi \in \mathcal{P}} \left\| \frac{h_j f_j(\pi)}{m_j} \right\|^2 < \infty \quad (j = 1, \ldots, k).$$

Let j be such an index for which $h_j \not\equiv 0 \pmod{m_j}$. From (4.9) we obtain that

$$h_j f_j(\pi) \equiv 0 \pmod{m_j} \tag{4.10}$$

for all but finitely many $\pi \in \mathcal{P}$. Then $(h_j, m_j) < m_j$, $\Delta_j := \dfrac{m_j}{(m_j, h_j)} > 1$. (4.10) implies that

$$f_j(\pi) \equiv 0 \pmod{\Delta_j}$$

for all but finitely many $\pi \in \mathcal{P}$.

This contradicts to our assumption. The theorem is proved.

5. Kym proved that for $f \in \mathcal{A}_q$ $f(n)$ mod 1 is distributed uniformly if and only if for every nonzero integer k either

$$\sum_{b=0}^{q-1} e\left(kf\left(bq^{l_k}\right)\right) = 0$$

for some l_k, or

$$\sum_{\pi \in \mathcal{P}} \|kf(\pi)\|^2 = \infty. \tag{5.1}$$

Let $s_n^{(h)} := f(n \cdot q^h)$ mod 1. From Kym's result we have: the sequences $\left\{s_n^{(h)}\right\}_{n=0}^{\infty}$ are distributed uniformly for every $h = 0, 1, 2, \ldots$ if and only if (5.1) holds.

Let $l(n)$ be defined by (1.3). We are interested in the following question. Under what condition is true that $\left\{l(nq^j) \pmod 1\right\}_{n=0}^{\infty}$ is distributed uniformly for every j.

Theorem 7. *The sequences* $\left\{l(n \cdot q^j) \bmod 1\right\}_{n=0}^{\infty}$ *are* mod 1 *uniformly distributed for every j in each case, except when there is an integer $m \neq 0$, $\gamma_1, \ldots, \gamma_k \in \mathbb{R}$, $l \in \mathbb{N}_0$ such that*

$$q^l(a_1\gamma_1 + \ldots + a_k\gamma_k) \equiv 0 \pmod 1,$$

and

$$\sum_{\pi \in \mathcal{P}} \|mf_j(\pi) - \gamma_j \pi\|^2 < \infty \quad (j = 1, \ldots, k) \tag{5.2}$$

hold.

The assertion is a direct consequence of Weyl's theorem and our result in [2] which we quote now as Lemma 1.

Let $t(n) = g_1(a_1 n) \dots g_k(a_k n)$, $t_j(n) = t(n \cdot q^j)$, where $g_\nu(n)$ are q-multiplicative functions, $|g_\nu(n)| = 1$ $(n \in \mathbb{N}_0,\ \nu = 1, \dots, k)$. Assume furthermore that $a_1, \dots, a_k$ satisfies the conditions stated in Section 1. Let

$$M_j(x) = \sum_{n<x} t_j(n), \quad m_j(N) = \frac{1}{q^N} M_j(q^N),$$
$$\alpha_j = \liminf |m_j(N)|, \quad \beta_j = \limsup |m_j(N)|.$$

Lemma 2. *Assume that $\beta_j > 0$ for some j. Then $\alpha_l = \beta_l \to 1$ as $l \to \infty$, furthermore there exist suitable real numbers $\gamma_1, \dots, \gamma_k$ and some $j_0 \geq 0$ such that*

$$q^{j_0}(\gamma_1 a_1 + \dots + \gamma_k a_k) \equiv 0 \pmod 1,$$

and in the notation $h_j(n) := e\,(-\gamma_j n) g_j(n)$,

$$\sum_{j=0}^{\infty} \sum_{c \in \mathcal{A}_q} Re\left(1 - h_l(cq^j)\right) < \infty \quad (l = 1, \dots, k).$$

Proof of Theorem 7. Let $m \neq 0$, $g_\nu(n) = e\,(mf_\nu(n))$, $t(n) = g_1(a_1 n) \dots g_k(a_k n)$.

By Weyl's famous theorem $\left\{l(nq^j) \bmod 1\right\}$ is distributed uniformly mod 1 for every j, if and only if $\dfrac{M_j(x)}{x} \to 0$ $(x \to \infty)$ for every j, and every $m \neq 0$. If $\dfrac{M_j(x)}{x} \to 0$ $(x \to \infty)$, then $\alpha_l = \beta_l = 0$ for every l and m. This proves the necessity of the conditions. Let us assume now that for some $m (\neq 0)$, $\beta_j = 0$ $(j = 0, 1, \dots)$.

Let $q^N < x \leq q^{N+1}$. Consider the sequences $A^M = \{(\varepsilon_0, \varepsilon_1, \dots, \varepsilon_{M-1}),\ \varepsilon_\nu \in \mathbb{A}\}$. We classify them according to the following rule. We say that $(\varepsilon_0, \varepsilon_1, \dots, \varepsilon_{M-1}) \in \mathcal{B}_0$, if $\varepsilon_{M-1} = 0$, and that $(\varepsilon_0, \dots, \varepsilon_{M-1}) \in \mathcal{B}_h$, if $\varepsilon_{M-h-1} = 0$, and $\varepsilon_{M-h}, \varepsilon_{M-h+1}, \dots, \varepsilon_{M-1}$ are nonzero. Let finally $\mathcal{B}^*$ be the set of those elements for which $\varepsilon_\nu \neq 0$ $(\nu = M-1, M-2, \dots, 0)$.

Observe that $\#\mathcal{B} = q^{M-1}$, $\#\mathcal{B}_1 = q^{M-2}(1 - 1/q)$, $\#\mathcal{B}_2 = q^{M-1} \cdot (1 - 1/q)^2, \dots, \#\mathcal{B}_h = q^{M-1}(1 - 1/q)^h$, and $\#\mathcal{B}^* = q^M (1 - 1/q)^M$.

Let us write each $n < x$ as $n_0 + q^M u = n$, where $n_0 \in \{0, 1, \dots, q^M - 1\}$, $u \in \left\{0, \dots, \left[\dfrac{x}{q^M}\right]\right\}$. If $u < \left[\dfrac{x}{q^M}\right]$, then $n < x$ holds for each $n_0 \in \left\{0, 1, \dots, q^M - 1\right\}$. $u = \left[\dfrac{x}{q^M}\right]$ occurs for $O(q^M)$ distinct integers n.

Thus

$$M_j(x) = \sum_{u=0}^{\left[\frac{x}{q^M}\right]-1} \sum_{n_0=0}^{q^M-1} t_j\left(n_0 + q^M u\right) + O(q^M) =$$
$$= \sum_h \sum\nolimits_h + O(q^M) + O\left(x\left(1-1/q\right)^M\right).$$

Here

$$\sum\nolimits_h = \sum_{u=0}^{\left[x/q^M\right]-1} \sum_{n_0\in\tilde{\mathcal{B}}_h} t_j\left(n_0 + q^M u\right),$$

where $\tilde{\mathcal{B}}_h$ is the set of those nonnegative integers $n_0 < q^M$ for which $(\varepsilon_0(n_0), \dots, \varepsilon_{M-1}(n_0))$ belongs to $\mathcal{B}_h$.

Let $n_0 \in \tilde{\mathcal{B}}_h$. Then $n_0 = \nu + q^{M-h}\mu$, where $0 \le \nu < q^{M-h-1}$, $0 \le \mu < q^h$, and each digit of μ differs from zero. The opposite assertion is true as well. Let $\nu \in \left[0, q^{M-h-1}-1\right]$, and $\mu \in \left[0, q^h - 1\right]$ such that $\varepsilon_\nu(\mu) \ne 0$ $(\nu = 0, \dots, h-1)$. Let us observe furthermore that

$$t_j\left(n_0 + q^M u\right) = t_j(\nu)\ t_j\left(\mu q^{M-h} + q^M u\right).$$

Thus

$$\sum_h = M_j\left(q^{M-h-1}\right) \sum_{\mu,u} t_j\left(\mu q^{M-h} + q^M u\right),$$

and so

$$\left|\sum\nolimits_h\right| \le \left|M_j\left(q^{M-h-1}\right)\right| \frac{x}{q^M}(q-1)^h. \tag{5.3}$$

Thus

$$\left|\frac{1}{x}\sum\nolimits_h\right| \le m_j(M-h-1)\left(1-1/q\right)^h. \tag{5.4}$$

By using (5.4) for $h = 0, \dots, K$, and the trivial inequality

$$\left|\frac{1}{x}\sum\nolimits_h\right| \le \left(1-1/q\right)^h$$

for $h > K$, we obtain that

$$\left|\frac{M_j(x)}{x}\right| \le \sum_{h=0}^{K} m_j(M-h-1)\left(1-\frac{1}{q}\right)^h + \\ + \sum_{k>K+1}(1-1/q)^h + O\left(\frac{q^M}{x}\right) + O\left(\left(1-\frac{1}{q}\right)^M\right).$$

We shall prove that $\limsup\left|\frac{M_j(x)}{x}\right| = 0$. Indeed, let $K > 0$ be fixed, $M = N-K$, $N = \left[\frac{\log x}{\log q}\right]$. Then for $x\to\infty$, $M\to\infty$, thus $m_j(M-h-1)\to 0$ as $x\to\infty$. Therefore

$$\limsup\left|\frac{M_j(x)}{x}\right| = O\left(q^{-K}\right) + O\left(\left(1-\frac{1}{q}\right)^K\right).$$

Since the inequality holds for every K, therefore it holds for $K\to\infty$, consequently

$$\limsup\frac{|M_j(x)|}{x} = 0.$$

References

[1-2] **Indlekofer K.-H. and Kátai I.,** Investigations in the theory of q-additive and q-multiplicative functions I-II., *Acta Math. Hung.* (accepted)

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