Annales Univ. Sci. Budapest., Sect. Comp. 21 (2004) 187-194

A CHARACTERIZATION OF SOME UNIMODULAR MULTIPLICATIVE FUNCTIONS II.

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Dedicated to the memory of Professor Péter Kiss

1. Introduction and results

Let $I\!N$ denote the set of all positive integers. Let $\mathcal{M}$ $(\mathcal{M}^*)$ be the set of complex-valued multiplicative (completely multiplicative) functions. A function g is said to be unimodular if g satisfies the condition $|g(n)| = 1$ for all positive integers n. In the following we shall denote by $\mathcal{M}(1)$ and $\mathcal{M}^*(1)$ the class of all unimodular functions $g \in \mathcal{M}$ and $g \in \mathcal{M}^*$, respectively.

More than 15 years ago I.Kátai stated as a conjecture that $f \in \mathcal{M}$, $\Delta f(n) = f(n+1) - f(n) = o(1)$ as $n \to \infty$ imply that either $f(n) = o(1)$ or $f(n) = n^s$ $(n \in I\!N)$, $0 \le \text{Re } s < 1$. This was proved by Wirsing in 1984 and some years later independently by Tang and Shao. The joint paper of Wirsing, Tang and Shao [5] contains two different proofs. It is not hard to deduce from Wirsing's theorem that if $f, g \in \mathcal{M}$, $g(n+1) - f(n) = o(1)$ as $n \to \infty$, then either $f(n) = o(1)$, or $f(n) = g(n)$ $(n \in I\!N)$, and in the last case $f(n) = n^s$ $(n \in I\!N)$, $0 \le \text{Re } s < 1$.

Recently, improving these results we showed in [2] that if $k \in I\!N$ is given and $f, g \in \mathcal{M}$ satisfy the condition

$$g(n+k) - f(n) = o(1) \quad \text{as} \quad n \to \infty,$$

then either $f(n) = o(1)$ as $n \to \infty$ or there are $F, G \in \mathcal{M}$ and a complex constant s such that

$$f(n) = n^s F(n), \quad g(n) = n^s G(n), \quad 0 \le \text{Re } s < 1$$

and $G(n+k) = F(n)$ are satisfied for all $n \in I\!N$. In [1] the equation

$$G(n+k) = F(n)$$

for functions $F,\ G \in \mathcal{M}$ is solved completely.

The general case concerning the characterization of those $f,\ g \in \mathcal{M}$ for which

$$g(an+b) - Cf(An+B) = o(1) \quad \text{as} \quad n \to \infty,$$

where $a > 0,\ b,\ A > 0,\ B$ are fixed integers with $Ab - aB \neq 0$ and C is a non-zero complex constant, seems to be a hard problem and a complete solution is not known.

For unimodular multiplicative functions there are few partial results concerning this problem. In [3] and [4] we obtained a generalization of E.Wirsing's theorem, namely we proved that if functions $g_1 \in \mathcal{M}(1)$ and $g_2 \in \mathcal{M}(1)$ satisfy the condition

$$g_1(an+b) - dg_2(cn) = o(1) \quad \text{as} \quad n \to \infty$$

for some positive integers $a,\ b,\ c$ and non-zero complex number d, then there are a real number τ and functions $G_1, G_2 \in \mathcal{M}(1)$ such that

$$g_1(n) = n^{i\tau}G_1(n),\ g_2(n) = n^{i\tau}G_2(n),\ G_1(an+b) - d\frac{c^{i\tau}}{a^{i\tau}}G_2(cn) = 0$$

for all positive integers n. Furthermore, it is shown in [4] that if $a,\ b,\ c$ are positive integers and d is a non-zero complex number, then the functions $g_1 \in$ $\in \mathcal{M}(1)$ and $g_2 \in \mathcal{M}(1)$ satisfy the condition

$$\sum_{n\le x}\frac{1}{n}|g_1(an+b) - dg_2(cn)| = o(\log x) \quad \text{as} \quad x \to \infty$$

if and only if there are functions $g \in \mathcal{M}^*(1)$ and $G_1, G_2 \in \mathcal{M}(1)$ such that

$$g_1(n) = g(n)G_1(n),\ g_2(n) = g(n)G_2(n),\ G_1(an+b) - d\frac{g(c)}{g(a)}G_2(cn) = 0$$

are satisfied for all positive integers n, furthermore

$$\sum_{n\le x}\frac{1}{n}|g(n+1) - g(n)| = o(\log x) \ \text{ as } \ x \to \infty.$$

Trivially, this relation holds for functions of the type $g(n) = n^{i\tau}$, where τ is a real number, and it has been conjectured that these are the only multiplicative functions of modulus 1 that satisfy the last relation. This conjecture remains open.

Our purpose in this paper is to prove the following

Theorem 1. *Let $a>0$, b, $A>0$, B be integers with $(a,A)=1$, $\Delta = Ab-aB\neq 0$ and let C be a non-zero complex number. If $g\in\mathcal{M}^*(1)$ satisfies the condition*

$$\sum_{n\le x}\frac{1}{n}|g(an+b)-Cg(An+B)| = o(\log x) \quad \textit{as} \quad x\to\infty, \tag{1}$$

then

$$\sum_{n\le x}\frac{1}{n}|g(n+1)-g(n)| = o(\log x) \quad \textit{as} \quad x\to\infty. \tag{2}$$

Theorem 2. *Let $a>0$, b, $A>0$, B be integers with $(a,A)=1$, $\Delta = Ab-aB\neq 0$ and let C be a non-zero complex number. If $g\in\mathcal{M}^*(1)$ satisfies the condition*

$$g(an+b)-Cg(An+B) = o(1) \quad \textit{as} \quad n\to\infty, \tag{3}$$

then there is a real-number τ such that

$$g(n) = n^{i\tau} \quad \textit{for all} \quad n\in I\!N. \tag{4}$$

We note that if $h(n)$ is a real-valued completely function, then $f(n) := e^{2\pi i h(n)}$ is a complex-valued completely multiplicative function of modulus 1. By using the fact

$$\| x \| \ll \ | e^{2\pi i x}-1 \ | \ \ll \| x \|,$$

where $\| x \|$ denotes the distance of a real number x to the nearest integer, the following corollary follows directly from Theorem 2

Corollary. *If a real-valued completely additive function $h(n)$ satisfies*

$$\| h(an+b)-h(An+B)-D \| = o(1) \quad \textit{as} \quad n\to\infty$$

with integers $a>0$, b, $A>0$, B, $(a,A)=1$, $\Delta = Ab-aB\neq 0$ and some real number D, then there exists a real number τ for which

$$\| h(n)-\tau\log n \| = 0 \quad \textit{for all} \quad n\in I\!N.$$

2. Proof of Theorem 1

We may suppose without loss of generality that $(A,\ B)=(a,\ b)=1$, $\Delta = Ab - aB > 0$, and so by our assumption we have $(a,A)=(a,\Delta)=(A,\Delta)=1$. Let

$$M := (aA+1)^{\max(A-1,\ a-1)}.$$

First we prove that

$$\sum_{\substack{n\le x\\ (n,\ A)=1}} \frac{1}{n}\left|g\left(Mn+A\Delta\right)-g(M)g(n)\right| = o(\log x) \tag{5}$$

and

$$\sum_{\substack{m\le x\\ (m,\ a)=1}} \frac{1}{m}\left|g\left(Mm-a\Delta\right)-g(M)g(m)\right| = o(\log x) \tag{6}$$

as $x\to\infty$.

Let $Q=aA+1$ and $k\in I\!N$. We start from the relation

$$Q^k(an+b) = a\left(Q^k n + b\frac{Q^k-1}{a}\right)+b,$$

whence by (1), we have

$$\sum_{n\le x}\frac{1}{n}\left|g\left(A\left(Q^k n+b\frac{Q^k-1}{a}\right)+B\right)-g(Q)^k g(An+B)\right| = o(\log x).$$

Consequently

$$\sum_{n\le x}\frac{1}{n}\left|g\left(Q^k(An+B)+A\Delta\frac{Q^k-1}{Q-1}\right)-g(Q)^k g(An+B)\right| = o(\log x). \tag{7}$$

Repeating the argument that was used above, we also have

$$\sum_{m\le x}\frac{1}{m}\left|g\left(Q^k(am+b)-a\Delta\frac{Q^k-1}{Q-1}\right)-g(Q)^k g(am+b)\right| = o(\log x) \tag{8}$$

as $x\to\infty$.

Since $Q = aA + 1$, therefore $\frac{Q^k-1}{Q-1} \equiv k \pmod{A}$. This shows that for each positive integer l, $(l,\ A) = 1$, there is a positive integer $k = k(l) < A$ such that $lk(l) \equiv B \pmod{A}$, therefore

$$\frac{Q^{k(l)}-1}{Q-1} l \equiv B \pmod{A}.$$

Hence, if we write $k(l)$ in place of k and

$$\frac{Q^{k(l)}-1}{Q-1} t + \frac{1}{A}\left(\frac{Q^{k(l)}-1}{Q-1} l - B\right)$$

in place of n, then

$$An + B = \frac{Q^{k(l)}-1}{Q-1}(At + l),$$

and so we can write the condition (7) in the form

$$\sum_{t \le x} \frac{1}{t} \left| g\left[Q^{k(l)}(At+l) + A\Delta\right] - g(Q)^{k(l)} g(At+l) \right| = o(\log x).$$

Since

$$M = Q^{k(l)} Q^{\max(A-1,\ a-1)-k(l)} \quad \text{and} \quad Q^{\max(A-1,\ a-1)-k(l)} \equiv 1 \pmod{A},$$

we infer from the above relation that

$$\sum_{t \le x} \frac{1}{t} \left| g\left[M(At+l) + A\Delta\right] - g(M) g(At+l) \right| = o(\log x)$$

holds for each positive integer l, $(l,\ A) = 1$. Hence, (5) immediately follows.

By using (8), the proof of (6) is similar and we omit it. Thus, we have proved (5) and (6).

In the following, we denote by $\mathcal{S}(A)$ the set of those positive integers which are products of the prime factors of A. Next, we prove that there is a positive integer $P = P(a, A)$ such that

$$\sum_{\substack{n \le x \\ (n,\ A)=1}} \frac{1}{n} \left| g\left(aPsn + \Delta\right) - g(aPs) g(n) \right| = o(\log x) \tag{9}$$

for all $s \in \mathcal{S}(A)$.

Let $s\in\mathcal{S}(A)$. Since $(A,\ M)=(a,A)=1$, there are positive integers $x_s,\ y_s$, such that

$$Asx_s+a=My_s,\quad (x_sy_s,\ aA)=1\ \text{ and }\ x_s<aAM.$$

Since $x_s<aAM$ and $(x_s,\ aA)=1$ for all $s\in\mathcal{S}(A)$, we can define P as

$$P:=\mathrm{LCM}_{s\in\mathcal{S}(A)}[x_s].$$

Hence, if $m=(asx_sn+\Delta)y_s$, then we have

$$(aMy_sn+A\Delta)sx_s=Mm-a\Delta$$

and

$$(m,\ a)=((asx_sn+\Delta)y_s,\ a)=1\ \text{ for all }\quad n\in I\!N.$$

Consequently, (5) and (6) imply

$$\sum_{\substack{n\le x\\(n,\ A)=1}}\frac{1}{n}\left|g(asx_sn+\Delta)-g(asx_s)g(n)\right|=o(\log x),$$

which, using the definition of P and the fact $(P,\ aA)=1$, proves (9). Thus, the relation (9) is proved.

Now we prove Theorem 1. We shall deduce from (9) that

$$\text{(10)}\qquad \frac{1}{\log x}\sum_{n\le x}\frac{1}{n}|g(aPn+\Delta)-g(aPn)|=o(1)\quad\text{as}\quad x\to\infty.$$

Let $A=\pi_1^{\alpha_1}\cdot\ldots\cdot\pi_r^{\alpha_r},\ r\ge1,\ A_1=\pi_2^{\alpha_2}\cdot\ldots\cdot\pi_r^{\alpha_r}$. To show (10), we shall prove that we can reduce A to A_1 in (9), i.e.

$$\text{(11)}\qquad \frac{1}{\log x}\sum_{\substack{n\le x\\(n,\ A_1)=1}}\frac{1}{n}\left|g\left(aPs_1n+\Delta\right)-g(aPs_1)g(n)\right|=o(1)$$

for all $s_1\in\mathcal{S}(A_1)$. Repeating this argument we conclude that (10) holds.

Let $s_1\in\mathcal{S}(A_1)$ and $\pi=\pi_1$. For each integer $\gamma\ge0$ let

$$S_\gamma(x):=\frac{1}{\log x}\sum_{\substack{n\le x,\ (n,A_1)=1\\ \pi^\gamma\|n}}\frac{1}{n}|g(aPs_1n+\Delta)-g(aPs_1n)|,$$

where $\pi^\gamma \parallel n$ denotes that $\pi^\gamma | n$ and $\left(\frac{n}{\pi^\gamma},\ \pi\right) = 1$. One can get from (9) that $S_\gamma(x) = o(1)$ as $x \to \infty$ is satisfied for each $\gamma \geq 0$. This relation together with $|g| \equiv 1$ implies that for each positive integer μ, we have

$$\frac{1}{\log x} \sum_{\substack{n \leq x \\ (n,\ A_1)=1}} \frac{1}{n} \left|g\left(aPs_1 n + \Delta\right) - g(aPs_1)g(n)\right| \leq$$

$$\leq \sum_{0 \leq j \leq \mu - 1} S_j(x) + \frac{1}{\log x} \sum_{\substack{n \leq x \\ \pi^\mu | n}} \frac{2}{n} \ll o(\mu) + \frac{2}{\pi^\mu},$$

and so

$$\limsup_{x \to \infty} \frac{1}{\log x} \sum_{\substack{n \leq x \\ (n,\ A_1)=1}} \frac{1}{n} \left|g\left(aPs_1 n + \Delta\right) - g(aPs_1)g(n)\right| \ll \pi^{-\mu}.$$

This with $\mu \to \infty$ shows that

$$\frac{1}{\log x} \sum_{\substack{n \leq x \\ (n,\ A_1)=1}} \frac{1}{n} \left|g\left(aPs_1 n + \Delta\right) - g(aPs_1)g(n)\right| = o(1) \quad \text{as} \quad x \to \infty,$$

as asserted in (11). Thus, the proof of (10) is complete.

Finally, it is shown in [4, Lemma 2] that if $g \in \mathcal{M}^*(1)$ satisfies the condition (10), then (2) holds, and so the proof of Theorem 1 is finished.

3. Proof of Theorem 2

Assume that the function $g \in \mathcal{M}^*(1)$ satisfies the condition (3). We will go along a thought line similar to the proof of (10) to deduce that there is a positive integer $P = P(a, A)$ such that

$$g(aPn + \Delta) - g(aPn) = o(1) \quad \text{as} \quad n \to \infty. \tag{12}$$

Finally, it is shown in the proof of Theorem 3 [4] that if the function $g \in \mathcal{M}^*(1)$ satisfies the condition (12), then

$$g(n+1) - g(n) = o(1) \quad \text{as} \quad n \to \infty.$$

This with Wirsing's theorem [5] implies that there is a real number τ such that (4) is true. Thus, Theorem 2 is proved.

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(Received June 8, 2002)

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