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NUMERICAL INVESTIGATION OF A CYCLIC–WAITING QUEUEING SYSTEM WITH TWO TYPES OF CUSTOMERS

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Abstract. The paper investigates by means of simulation a queueing system which enter two Poisson processes, the service times are exponentially distributed. The customers of first type may only join the free system and there can be present only one of them, all other ones of this type are refused. There is no restriction on the customers of second type. The service may start at the moment of arrival (in case of free system) or at moments differing from it by the multiples of a given time T. The paper deals with the determination of equilibrium distribution, the simulation shows a good coincidence with the theoretical results from [12].

1. Introduction

In recent decades the examination of certain public-service systems has come into the forefront of probability theory. In this paper we investigate a process in which customers with different priorities arrive at a system, and they are either refused or, after some period of waiting, they are served. The operation conditions of such queueing systems raise several interesting mathematical problems. It is relevant, for example, to examine the distribution of the customers' arriving times and of the service times. These investigations, called the theory of queues, have developed into an important field of probability theory. However, accurate mathematical descriptions can only be provided about very simple systems. In practice more complicated systems occur whose examination necessitates the implementation of computer-simulation methods. In the present paper we concentrate our attention on a question which has yielded remarkable theoretical results. Our goal is to assist the examination of the system's operation with the use of computer simulations, and to support the validity of mathematical calculations.

Depending on the complexity, we have two ways to investigate the queueing systems. The first way is examination by exact mathematical methods, the second one is a model constructed by a computer program. In order to clarify our aim we describe the functioning of our system more accurately and refer to some papers which presented interesting results prior to our investigation of this field.

2. Problems and results

2.1. Description of the problem

In this paper we focus our attention on a model describing the landing of airplanes. While in a conventional queueing system the service process runs continuously, in the system investigated by us the starting moment of a service process depends on the moment of the completion of the previous service and the moment of the arrival of the actual customer. Naturally, by customer we mean an airplane having a landing request. If there is free system, i.e. the entering entity can be serviced at the moment of request, the airplane can start landing. However, if the server is busy, i.e. a formerly arrived plane has not accomplished landing yet or other planes are already queueing for service, then the incoming plane starts a circular maneouvre. The radius of the circle is fixed in a way that it takes the airplane T cycle time to be above the runway again, i.e. the airplane can only put further landing requests to the control tower at every nT moment after arrival, where $n \in \mathbb{N}$. Naturally, the request can only be serviced if there is no airplane queueing before it. The reception and service of the incoming planes follow the FIFO rule, according to which the earlier arriving planes are given landing permission earlier.

In general the object of the investigation is the probabilities in free systems of having $0, 1, 2, \ldots$ airplanes in the queue at the starting moments of services, because this system only operates properly if there are not too many planes cyclic queueing.

2.2. Previous achievements

In [1] we examined by computer simulation some continuous cyclic-waiting systems in which the arriving customers form a Poisson process with parameter λ, the service time distribution is either exponential with parameter μ or uniform on the interval $[c, d]$. In [2] similar results were published for discrete cyclic-waiting systems. In this case the time between two arriving requests has geometrical distribution with parameter r, the service time with parameter $1-q$. [3] presents some computations connected with the convergence to the limit distribution, which support the mathematical results.

The basic theoretical results for such systems were obtained in the papers [5-11], the ergidicity condition for a generalized system was examined in [4].

2.3. The purpose of the present research

In this paper we consider a queueing system which involves customers with two different priorities. The arrival times of the two types of airplanes constitute Poisson processes with parameter λ_1 and parameter λ_2. The distributions of the related service times are exponential. In this system exclusively second-type airplanes are allowed to join the queue, while first-type planes are serviced right upon arrival provided there is free system, otherwise they are refused. Thus, from the aspect of the second-type costumers the system operates in the way described in the previous subsection, but the service can lose some first-type ones.

3. Theoretical results

The exact solution of the investigated system is contained in [12], the theorem below is taken from it.

We introduce the following probabilities:

a_{ji} - the probability of appearence of i customers of second type at the service of j-th type (from the beginning of service of a j-th type customer till the beginning of service of the next one) if at the beginning there is only one customer in the system;

b_i - the probability of appearence of i customers of second type at the service of second type (at the beginning of service there are at least two customers in the system);

c_i - the probability of appearence of i customers of second type for a service after free state;

their generating functions are

$$A_j(z) = \sum_{i=0}^{\infty} a_{ji} z^i, \quad B(z) = \sum_{i=0}^{\infty} b_i z^i, \quad C(z) = \sum_{i=0}^{\infty} c_i z^i, \quad j = 1, 2.$$

Theorem. *Let us consider a queueing system with two types of customers forming Poisson processes with parameters λ_1 and λ_2, the service time distributions are exponential with parameters μ_1 and μ_2, respectively. There is no restriction on the customers of second type, the customers of first type may only join the free system and there can only be present one of them, all other ones of this type are refused. The service of a customer may start at the moment of arrival (in case of free system) or at moments differing from it by the multiples of cycle time T according to the FIFO rule. We define an embedded Markov chain whose states correspond to the number of customers in the system at moments just before starting their services. Then the generating function of equilibrium distribution is given by*

$$P(z) = \frac{p_0\left[C(z) - \frac{B(z)}{z}\right] + p_1[A_2(z) - B(z)]}{1 - \frac{B(z)}{z}}, \tag{1}$$

where

$$A_j(z) = \sum_{i=0}^{\infty} a_i z^i = \frac{\mu_j}{\lambda_2 + \mu_j} + \frac{\lambda_2 z}{\lambda_2 + \mu_j} \frac{e^{-\lambda_2(1-z)T}(1 - e^{-\mu_j T})}{1 - e^{-[\lambda_2(1-z)+\mu_j]T}}; \tag{2}$$

$$B(z) = \sum_{i=0}^{\infty} b_i z^i = \frac{\lambda_2}{(1 - e^{-\lambda_2 T})[1 - e^{-[\lambda_2(1-z)+\mu_2]T}]} \times \tag{3}$$

$$\times \left[\frac{1 - e^{-\lambda_2(2-z)T}}{\lambda_2(2-z)} - \frac{1 - e^{-[\lambda_2(2-z)+\mu_2]T}}{\lambda_2(2-z)+\mu_2}\right] + \frac{\lambda_2 e^{-\lambda_2(1-z)T}}{(1 - e^{-\lambda_2 T})[1 - e^{-[\lambda_2(1-z)+\mu_2]T}]} \times$$

$$\times \left[\frac{1 - e^{-[\lambda_2(2-z)+\mu_2]T}}{\lambda_2(2-z)+\mu_2} - e^{-\mu_2 T}\frac{1 - e^{-\lambda_2(2-z)T}}{\lambda_2(2-z)}\right];$$

$$C(z) = \sum_{i=0}^{\infty} c_i z^i = \frac{\lambda_1}{\lambda_1 + \lambda_2} A_1(z) + \frac{\lambda_2}{\lambda_1 + \lambda_2} A_2(z)$$

and p_0 and p_1 are the first two probabilities from the desired distribution. They are connected with the relation

$$p_1 = \frac{1 - c_0}{a_{20}} p_0.$$

p_0 is determined by the condition $P(1) = 1$, it is equal to

$$p_0 = \frac{1 - B'(1)}{1 - B'(1) + C'(1) + k[A_2'(1) - B'(1)]}, \tag{4}$$

where k establishes connection between the customers of two types

$$k = \frac{\lambda_1\lambda_2^2 + \lambda_2^3 + \lambda_2^2\mu_1 + \lambda_1\lambda_2\mu_2}{\mu_2(\lambda_1+\lambda_2)(\lambda_2+\mu_1)}.$$

The condition of existence of ergodic distribution is the fulfilment of inequality

$$\frac{\lambda_2}{\mu_2} < \frac{e^{-\lambda_2 T}(1-e^{-\mu_2 T})}{1-e^{-\lambda_2 T}}.$$

The limit distribution $P^*(z)$ as $T \to 0$ is given by the formula

$$P^*(z) = p_0^* \times$$

$$\times \frac{\frac{\lambda_1}{\lambda_1+\lambda_2}\frac{\mu_1}{\lambda_2(1-z)+\mu_1} + \frac{\lambda_2}{\lambda_1+\lambda_2}\frac{\mu_2}{\lambda_2(1-z)+\mu_2} - \frac{\mu_2}{z[\lambda_2(1-z)+\mu_2]}}{1 - \frac{\mu_2}{z[\lambda_2(1-z)+\mu_2]}}, \tag{5}$$

where

$$p_0^* = \frac{1-\rho_2}{1-\rho_2+\frac{\lambda_2}{\lambda_1+\lambda_2}(\rho_1+\rho_2)}, \qquad \rho_j = \frac{\lambda_j}{\mu_j}, \quad j = 1,2.$$

4. Computed results

The numerical investigation was implemented with different values of the parameters. For every fixed $T, \lambda_1, \lambda_2, \mu_1, \mu_2$ we carried out 1000 independent experiments with different computer generated arrival and service times. In order to support the above mentioned theoretical computations we examined the values $p_0, p_1, p_2, ...$, where p_i denotes the probability in free system of having i airplanes in the queue at the starting moments of services and $0 \le i \le 30$.

We included our results in tables where the values p_i are given in columns. The index number of each row shows the number of the incoming airplanes. We present a table of probabilities for the values of parameters $T = 0.1, \lambda_1 = 2, \lambda_2 = 3, \mu_1 = 10, \mu_2 = 6$ in Figure 1. For the sake of clarity we give the values p_i multiplied by 1000.

The diagrams in Figure 2-3 present the calculated results where the values of probabilities p_0, p_1, p_2, p_3 are found on the Y axis and the number of incoming

Probability-table (the values are given multiplied by 1000)
$T = 0.1, \lambda_1 = 2, \lambda_2 = 3, \mu_1 = 10, \mu_2 = 6.$

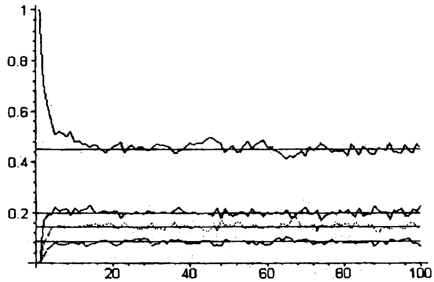
	P_0	P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}	P_{12}	P_{13}	P_{14}	P_{15}	P_{16}
0.	1000	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1.	714	168	78	28	6	3	1	2	0	0	0	0	0	0	0	0	0
2.	625	190	104	54	20	5	2	0	0	0	0	0	0	0	0	0	0
3.	563	191	143	69	21	7	5	1	0	0	0	0	0	0	0	0	0
4.	507	222	144	81	26	14	2	1	0	1	2	0	0	0	0	0	0
5.	521	207	148	72	31	10	7	1	1	1	1	0	0	0	0	0	0
6.	512	214	148	72	30	14	6	1	2	1	0	0	0	0	0	0	0
7.	499	217	143	77	41	12	5	4	2	0	0	0	0	0	0	0	0
8.	522	191	144	82	32	18	6	2	2	0	1	0	0	0	0	0	0
9.	480	218	142	83	41	17	13	5	0	0	1	0	0	0	0	0	0
10.	484	206	148	74	44	23	14	4	2	1	0	0	0	0	0	0	0
11.	472	211	145	84	39	29	14	5	1	0	0	0	0	0	0	0	0
12.	471	214	155	65	49	24	15	5	1	1	0	0	0	0	0	0	0
13.	459	231	150	75	41	24	12	3	3	2	0	0	0	0	0	0	0
14.	467	205	156	78	41	31	13	3	4	1	0	1	0	0	0	0	0
15.	464	196	153	82	53	26	12	7	5	0	1	0	0	1	0	0	0
16.	449	205	148	90	49	29	16	8	2	3	0	0	1	0	0	0	0
17.	440	206	155	89	52	28	14	4	7	3	0	1	1	0	0	0	0
18.	447	200	159	84	49	27	15	8	7	1	1	0	1	0	1	0	0
19.	460	203	153	73	47	34	13	9	4	2	0	0	1	1	0	0	0
20.	467	209	136	70	51	30	20	13	1	1	0	1	1	0	0	0	0
21.	480	179	146	76	51	30	23	7	5	1	1	0	1	0	0	0	0
22.	437	196	162	93	46	40	9	8	6	1	1	0	0	1	0	0	0
23.	461	197	144	88	44	39	8	10	5	2	1	0	1	0	0	0	0
24.	466	196	142	91	52	19	15	8	9	1	0	0	1	0	0	0	0
25.	452	205	135	98	50	26	17	9	5	2	0	1	0	0	0	0	0
26.	461	199	135	94	46	30	20	8	4	0	2	1	0	0	0	0	0
27.	457	195	137	100	53	25	12	10	4	3	3	1	0	0	0	0	0
28.	445	207	142	97	39	35	17	9	4	3	1	1	0	0	0	0	0
29.	452	203	163	68	45	33	20	5	7	1	3	0	0	0	0	0	0
30.	469	200	134	77	54	30	17	8	5	3	2	1	0	0	0	0	0
31.	469	196	140	79	45	33	19	11	5	1	1	1	0	0	0	0	0
32.	466	203	127	91	44	31	21	12	3	0	0	2	0	0	0	0	0
33.	463	198	147	77	51	33	12	13	3	0	0	3	0	0	0	0	0
34.	456	198	143	95	52	22	20	7	2	1	1	2	1	0	0	0	0
35.	433	210	166	88	43	32	12	8	2	1	3	1	1	0	0	0	0
36.	443	222	138	93	46	26	12	9	5	1	3	1	1	0	0	0	0
37.	451	207	145	85	45	33	14	9	4	3	2	1	0	0	1	0	0
38.	463	199	150	78	50	24	19	5	4	3	3	0	1	1	0	0	0
39.	478	198	138	76	54	31	6	7	2	6	2	0	2	0	0	0	0
40.	466	203	145	82	52	22	11	8	4	4	0	3	0	0	0	0	0
41.	477	195	151	76	41	22	17	8	4	4	4	1	0	0	0	0	0
42.	474	200	134	89	48	21	13	8	7	5	1	0	0	0	0	0	0
43.	487	200	128	90	40	20	12	15	4	2	1	1	0	0	0	0	0
44.	497	193	130	89	37	20	17	8	4	3	1	1	0	0	0	0	0

To be continued on the next page!

45.	495	192	151	71	30	27	18	4	9	2	0	1	0	0	0	0	0
46.	480	219	126	72	43	27	11	12	6	3	0	1	0	0	0	0	0
47.	475	177	171	68	49	19	23	10	3	1	3	1	0	0	0	0	0
48.	436	218	147	95	37	35	17	5	3	4	1	2	0	0	0	0	0
49.	451	192	157	88	55	24	16	7	6	2	2	0	0	0	0	0	0
50.	465	188	145	84	53	27	16	13	4	1	2	2	0	0	0	0	0
51.	441	191	164	85	51	27	17	8	7	6	3	0	0	0	0	0	0
52.	442	202	157	85	48	26	13	11	9	2	4	0	1	0	0	0	0
53.	466	200	143	90	38	25	13	10	8	1	3	1	1	0	1	0	0
54.	483	185	136	78	54	22	16	14	4	0	5	1	1	1	0	0	0
55.	453	217	148	85	41	22	12	9	1	6	2	3	0	1	0	0	0
56.	470	202	153	72	47	20	14	3	6	6	4	2	0	0	1	0	0
57.	478	202	136	86	40	19	13	9	8	3	2	2	1	1	0	0	0
58.	487	187	142	76	42	27	13	11	7	3	1	2	0	1	1	0	0
59.	460	202	147	73	56	27	11	9	7	4	0	1	1	1	0	1	0
60.	461	193	151	76	57	25	14	7	8	2	2	0	0	2	2	0	0
61.	447	201	142	93	43	34	18	7	7	2	1	1	0	2	0	2	0
62.	439	200	136	101	59	30	11	13	3	2	2	0	1	0	2	1	0
63.	428	193	163	89	60	23	21	11	4	2	2	0	1	1	2	0	0
64.	413	211	146	104	48	38	13	12	7	3	1	0	2	2	0	0	0
65.	424	184	162	100	55	34	15	14	4	2	0	3	1	1	1	0	0
66.	418	183	188	82	59	28	18	13	3	2	3	1	1	1	0	0	0
67.	431	199	168	91	44	30	14	11	4	5	1	0	2	0	0	0	0
68.	444	207	143	91	53	27	13	10	3	5	1	2	1	0	0	0	0
69.	423	222	149	94	50	26	15	7	4	5	2	2	1	0	0	0	0
70.	459	194	138	88	53	29	16	9	7	2	2	3	0	0	0	0	0
71.	446	196	156	74	54	29	22	10	5	4	3	0	1	0	0	0	0
72.	456	210	127	82	52	30	17	12	2	9	2	1	0	0	0	0	0
73.	473	171	155	83	40	36	11	15	10	4	1	1	0	0	0	0	0
74.	465	187	147	74	53	34	14	14	9	1	1	1	0	0	0	0	0
75.	450	193	145	87	49	34	24	10	5	2	1	0	0	0	0	0	0
76.	439	203	146	92	48	34	19	6	7	4	1	1	0	0	0	0	0
77.	446	194	156	80	55	31	16	12	4	1	4	0	0	1	0	0	0
78.	442	210	149	86	46	27	11	13	8	3	2	2	0	1	0	0	0
79.	443	197	153	82	54	25	21	11	5	5	3	0	0	0	0	1	0
80.	424	212	160	81	46	36	14	11	6	6	1	1	1	0	0	1	0
81.	470	194	136	82	39	36	18	7	10	2	4	1	0	0	1	0	0
82.	450	184	164	79	47	30	21	9	8	2	5	0	0	0	1	0	0
83.	426	218	147	81	45	41	15	11	7	3	3	1	0	1	1	0	0
84.	461	187	142	80	60	29	14	12	3	5	4	0	1	1	0	1	0
85.	450	193	148	83	52	30	18	12	4	6	1	0	0	2	1	0	0
86.	431	196	157	85	53	33	22	8	7	2	2	0	2	1	0	1	0
87.	430	228	133	69	57	35	25	7	6	4	1	3	0	1	1	0	0
88.	455	210	120	83	53	37	12	12	9	3	2	2	0	0	1	1	0
89.	452	200	139	87	44	30	19	12	6	6	3	0	0	0	2	0	0
90.	448	202	143	73	54	32	20	8	7	6	4	0	1	0	0	2	0

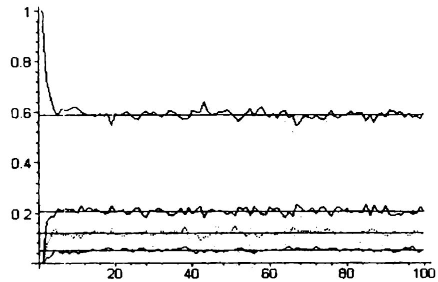
Figure 1

T=0.1



$\lambda_1=2$, $\lambda_2=3$, $\mu_1=10$, $\mu_2=6$

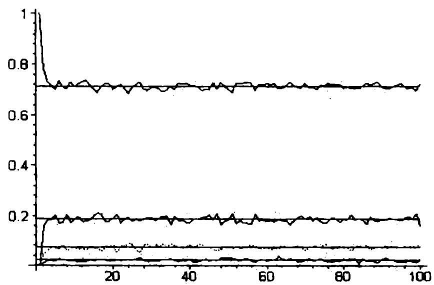
p_0	p_1	p_2	p_3
0.45021050	0.19739998	0.14386830	0.08514379



$\lambda_1=2$, $\lambda_2=3$, $\mu_1=6$, $\mu_2=10$

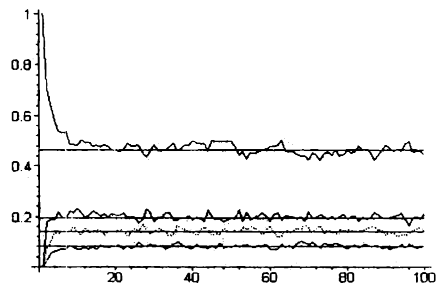
p_0	p_1	p_2	p_3
0.58674888	0.20731793	0.12059318	0.051384

T=0.1



$\lambda_1=3$, $\lambda_2=2$, $\mu_1 6$, $\mu_2=10$

p_0	p_1	p_2	p_3
0.71189701	0.18509322	0.073429	0.02146621

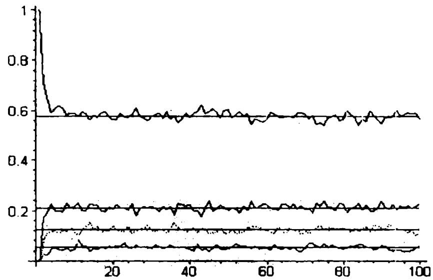


$\lambda_1=3$, $\lambda_2=3$, $\mu_1=10$, $\mu_2=6$

p_0	p_1	p_2	p_3
0.46210166	0.19550455	0.14078797	0.08259192

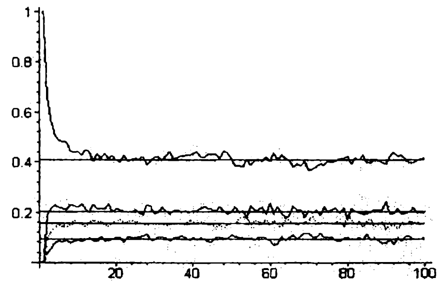
Figure 2

T=0.1



$\lambda_1=3$, $\lambda_2=3$, $\mu_1=6$, $\mu_2=10$

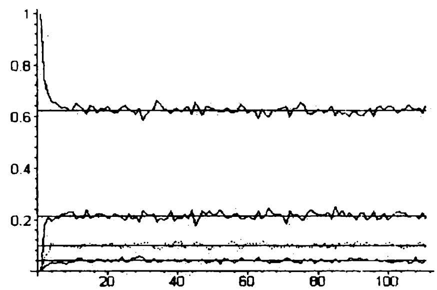
p_0	p_1	p_2	p_3
0.57448943	0.21064612	0.12449755	0.05396457



$\lambda_1=3$, $\lambda_2=3$, $\mu_1=6$, $\mu_2=6$

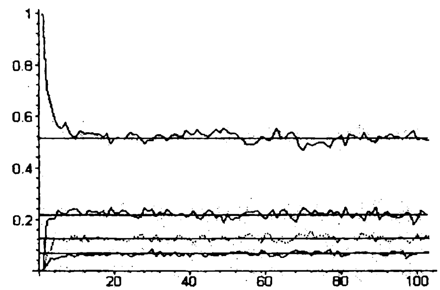
p_0	p_1	p_2	p_3
0.40819456	0.20409728	0.15475224	0.09416051

T=0.05



$\lambda_1=4$, $\lambda_2=3$, $\mu_1=10$, $\mu_2=8$

p_0	p_1	p_2	p_3
0.62216432	0.21280071	0.09915118	0.03946241



$\lambda_1=3$, $\lambda_2=3$, $\mu_1=10$, $\mu_2=6$

p_0	p_1	p_2	p_3
0.51398675	0.21745593	0.12611460	0.06605377

Figure 3

airplanes on the X axis. Also the exact values of p_i can be read here, which appear as horizontal lines in the coordinate system.

Since every experiment starts from an empty state and the computations include the initial values, therefore the average value for p_0 is a bit increased and the other probabilities a bit decreased.

Inspite of this fact we can observe that considering 1000 independent experiments the cases 100 first-type and 100 second-type arriving airplanes show that the computed results clearly approximate the exact values.

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