Annales Univ. Sci. Budapest., Sect. Comp. 21 (2004) 119-126

FREGE'S PRINCIPLE IS NOT TRUE IN BLUM'S SKY

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Abstract. Under consideration of Frege's principle of compositionality one can define abstract measures of computational complexity (or Blum measures) with the property of compositionality (or conservation measures). It is shown that it is not true that this property is always satisfied for sufficiently large functions.

1. Some history

An important question from the beginning of theoretical and practical computer science was the question of how to measure the complexity of computational processes. The most common measures today are time and space, usually considered for Turing machine computations. But there are a lot of other computational complexity measures. And all these measures have some simple common properties which caused in 1967 Manuel Blum to make the following definition [1].

Let $\{\varphi_i : i \in \mathbb{N}\}$ be a Gödel numbering of all unary partial recursive functions over the set $\mathbb{N}$ of natural numbers, and let Φ be a binary partial recursive function over $\mathbb{N}$. Set $\Phi_i(x) = \Phi(i, x)$ for all i and x.

The pair (φ, Φ) is said to be an abstract measure of computational complexity or a Blum measure iff (φ, Φ) satisfies:

(A1) for all i and x, $\Phi_i(x)$ is defined if and only if $\varphi_i(x)$ is defined, and

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This article was subject of a lecture given by the author at the Eötvös Loránd University, Budapest in September 1998. It is also a contribution to the 150th anniversary of birth of Friedrich Ludwig Gottlob Frege who was born on 8 November 1848 in Wismar in Germany.

(A2) $\{(i,x,m) : \Phi_i(x) = m\}$ is a recursive set.

The functions Φ_i are also called *run-times* or *step counting functions.*

In this setting, φ is understood as the universal function of the Gödel numbering, $\varphi(i,x) = \varphi_i(x)$ for all i and x. The reference to φ can be omitted if this Gödel numbering is understood.

It is well-known in the complexity community that the two axioms of Blum have farreaching consequences and are powerful enough to yield some very strong theorems about complexity measures, see e.g. [1,3,8]. But on the other side they allow also a lot of unnatural and pathological measures violating our intuitive feeling about how natural complexity measures should behave. Therefore, attempts have been made to exclude pathological measures by further restrictions or axioms, see e.g. [2,6]. One deciding step into this direction was done by the help of Frege's principle, which may be formulated as:

The meaning of a complex expression is a function of the meanings of its parts and of the syntactic rules by which they are combined.

This principle was a guiding principle in the work of the famous mathematician and philosopher Gottlob Frege (1848 - 1925), and it was later exploited and referred to him by many researchers, see e.g. [6].

We take this principle and substitute the words *meaning, complex expression*, and *syntactic rules* by *complexity, composed function,* and *effective operation,* respectively, as it was done in [6]. Then we get the following which I want to call Frege's principle for complexity:

The complexity of a composed function is a function of the complexities of its parts and of the effective operation by which they are combined.

This principle seems to be quite natural. Its mathematical analysis in recursion theory yields the following definitions (see also [5,6]).

Let us denote by $\mathbb{R}$ the set of all total recursive functions over the natural numbers, and by $\mathbb{R}^n$ the set of all such functions with arity n.

h is called to be an *iteration function* iff $h \in \mathbb{R}^2$ and

$$\forall i \forall j \forall x (x \in D_{h(i,j)} \leftrightarrow x \in D_j \wedge \varphi_j(x) \in D_i) \wedge \forall i \forall j D_{h(i,j)} = D_i \cap D_j$$

and

$$\forall i \forall j \forall k \forall l \left(\varphi_i = \varphi_k \wedge \varphi_j = \varphi_l \rightarrow \varphi_{h(i,j)} = \varphi_{h(k,l)}\right).$$

(D_n abbreviates D_{φ_n} which is the domain of the function φ_n.)

Each iteration function h is associated with an effective binary operation P on the partial recursive functions such that $\varphi_{h(i,j)} = P(\varphi_i, \varphi_j)$ for all i and j.

σ is called to be a $\mathcal{K}$-function or a *size function* iff $\sigma \in \mathbb{R} \wedge \forall n(n \in \mathbb{N} \rightarrow \rightarrow \sigma^{-1}(n)$ is finite) $\wedge\ \exists g(g \in \mathbb{R}^1 \wedge \forall n(n \in \mathbb{N} \rightarrow g(n) = card(\sigma^{-1}(n))))$ (it is a total recursive function giving the size of the full inverse image for each number).

The following equations are called conservation theorems (where σ is a $\mathcal{K}$-function and h is an iteration function):

(1) $$\Phi_{h(i,j)}(x) = \sigma(\Phi_i(\varphi_j(x)), \Phi_j(x)) \text{ a.e.}$$

(2) $$\Phi_{h(i,j)}(x) = \sigma(\Phi_i(\varphi_j(x)), \Phi_j(x), \varphi_j(x)) \text{ a.e.}$$

(3) $$\Phi_{h(i,j)}(x) = \sigma(\Phi_i(x), \Phi_j(x)) \text{ a.e.}$$

(4) $$\Phi_{h(i,j)}(x) = \sigma(\Phi_i(x), \Phi_j(x), \varphi_j(x)) \text{ a.e.}$$

(a.e.=almost everywhere; this means the equation holds for all but finitely many x).

A Blum measure Φ is called to be a *conservation measure* or a *Blum measure with the property of compositionality* iff, for suitable σ and h, and for all i and j, one of the conservation theorems is true.

Examples of binary operations with associated iteration functions are the substitution of two functions realized by the simple composition of Turing machines, the sum of two functions, and the product of two functions. Examples of $\mathcal{K}$-functions are the sum and the maximum. Examples of conservation measures fulfilling equation (1), where σ is the sum or a variant of the maximum, are the time and space for Turing machines, respectively, where h corresponds to the simple composition of the machines.

It is obvious that the conservation theorems reflect exactly Frege's principle for complexity, and the conservation measures have been taken as abstract representatives of natural complexity measures. Nevertheless, the "naturalness" of the conservation measures remains moot and raises some philosophical question which are discussed in [6].

It should be emphasized that in the original literature [5] there is still a further type of conservation theorems, namely

(5) $$\Phi_{h(i,j)}(x) = \sigma(\Phi_i(x), \Phi_j(x), \varphi_i(x), \varphi_j(x)) \quad \text{a.e.,}$$

and it could be shown in [6] that no other type of equation can be fulfilled by a Blum measure. But by some reason which will be clear later, we restrict here to the types (1)-(4).

It is not hard to construct Blum measures which do not fulfill any of the equations (1)-(5). This is done in [5,6] by fixing some small run-times in a given Blum measure. But what about large run-times? It is a known phenomenon in recursion theory and in complexity theory that some properties are always true for sufficiently large functions, that means, for all functions greater than some recursive function f. A typical example is the following

Theorem (Hartmanis [2,3]). *For an arbitrary Blum measure Φ the complexity classes C_t^Φ are recursively enumerable for sufficiently large bounding functions $t \in \mathbb{R}^1$, where $C_t^\Phi =_{Df} \{\varphi_i : \Phi_i(x) \leq t(x) \text{ a.e.}\}$.*

Thus in Blum's sky (that means, for sufficiently large functions) the recursive enumerability of all complexity classes is always true. We show that Frege's principle is not true in Blum's sky, that means, there are Blum measures Φ having arbitrarily large run-times for which no conservation theorem of any type (but (5)) with any suitable operation P is true.

2. The result

We consider Blum measures fulfilling (or not) any of the conservation theorems (1)-(4) with arbitrary $\mathcal{K}$-function σ and iteration function h to an operation P. The effective operation P should be a reasonable one in the following sense. If we combine very different functions $f, f' \in \mathbb{R}$ with the same function $g \in \mathbb{R}$ then it should be possible to get different results: $P(f,g) \neq$ $\neq P(f',g)$.

More precisely, we make the following definition.

Definition. An effective binar operation P on the partial recursive functions is useful iff there exists a total recursive function g such that for each infinite set F of total recursive functions which are pairwise different at infinitely many places, the set $P(F,g)$ is infinite.

Formally: $\exists g \forall F (g \in \mathbb{R}^1 \wedge F \subseteq \mathbb{R}^1 \wedge F$ infinite $\wedge \forall f \forall f' (f, f' \in F \wedge f \neq f' \rightarrow$ $\rightarrow f(x) \neq f'(x)$ i.o.)$\rightarrow P(F,g)$ infinite).

Each reasonable operation - e.g. sum, product or substitution of two functions - has this property.

Now we formulate the main result.

Theorem 1. *There exist Blum measures (φ, Φ) having arbitrarily large run-times for which none of the conservation theorems (1)-(4) is true with any $\mathcal{K}$-function σ and any iteration function h to any useful operation P.*

Proof. We construct such a measure (φ, Φ) starting from an arbitrary Blum measure (φ', Φ') by definining for each i and x:

$\varphi_{2i}(x) =_{Df} \varphi'_k(x),\ \varphi_{2i+1}(x) =_{Df} \varphi'_i(x) + i,$

$\Phi_{2i}(x) = \Phi_{2i+1}(x) =_{Df} c(\Phi'_i(x), i)$, where c is a recursive 1-1 map from $\mathbb{N} \times \mathbb{N}$ onto $\mathbb{N}$ with the property $c(m,n) \geq m$ for all $m, n \in \mathbb{N}$ (for instance, a Cantor numbering).

It is easy to see that (φ, Φ) is a Blum measure again. Assume, it fulfills one of the equations (1)-(4) above f for some sufficiently large function $f \in \mathbb{R}^1$, where h is the corresponding iteration function with the useful operation P belonging to it. Let j_0 be a fixed index of such a function g existing by the definition of usefulness of P and such that $\Phi_{j_0} \geq f$. Let ψ be an arbitrary total recursive function. There exist infinitely many indices i of ψ in the Gödel numbering φ' such that $\Phi'_i \geq f$ (the existence of all these i and of j_0 is guaranteed by general properties of Blum measures and Gödel numberings, see e.g. [1,8], and Theorem 2 below). Therefore, we have infinitely many even numbers $2i$ such that $\varphi_{2i} = \psi$ and $\Phi_{2i} \geq f$. Because of the conservation theorem we get for each of these i, $\Phi_{h(2i,j_0)}(x) = \Phi_{h(2i+1,j_0)}(x)$ a.e., where $\varphi_{h(2i,j_0)} = = P(\varphi_{2i}, \varphi_{j_0}) = P(\psi, g)$ is a fixed function η, and because of the 1-1-dependence of Φ_n from the number $[n/2]$ we must have

$$\varphi_{h(2i+1,j_0)} = P(\varphi'_i + i, g) = P(\psi + i, g) \in \{\eta', \eta' + l\}$$

for some constant l and $\eta' \in \{\eta, \eta - m\}$ for some constant m (depending of whether $h(2i, j_0)$ is even or not). Thus we have an infinite set F of total recursive functions - namely the functions $\psi + i$ - which are pairwise different at all places and $P(F, g)$ is finite contradicting to the presumption of P.

Unfortunately, we have not been able to prove the following.

Claim. *There exist Blum measures (φ, Φ) having arbitrarily large runtimes for which none of the conservation theorems (1)-(5) is true with any $\mathcal{K}$-function σ and any iteration function h to any useful operation P.*

But comparing Frege's principle for complexity with the equations (1)-(5) we see that in a very strong sense only (1) and (3) meet the principle. Hence we can say with a clear conscience that Frege's principle is not true in Blum's sky.

Nevertheless, the following problems remain:

1. Prove the claim above.

2. Can one prove Theorem 1 (and the claim) with a weaker definition of the usefulness of the operation P?

3. Index sets and iteration functions

The following result is related to our problems and may be helpful to prove the claim. First, we agree on some notations.

For a set F of recursive functions, ΩF denotes the set of all Gödel numbers of functions belonging to F, that is $\Omega F =_{Df} \{i : \varphi_i \in F\}$.

Then for $\psi \in \mathbb{R}^1 : \Omega\psi =_{Df} \Omega\{\psi\} = \{i : \varphi_i = \psi\}$.

Using such index sets there is another property used in the literature as a candidate property for naturalness of complexity measures. It concerns the index sets of complexity classes defined above in the Theorem of Hartmanis.

Definition [4]. The Blum measure Φ has the *conformity property* or *conforms* iff the index sets for any two non-trivial (i.e. non-empty) complexity classes are recursively isomorphic.

This property is also true in Blum's sky.

Theorem (Lewis [4]). *Each Blum measure conforms for sufficiently large bounding functions.*

Now let h be an iteration function with the corresponding effective operation P. We are interested in sets $h(\Omega\psi, j_0) =_{Df} \{h(i, j_0) : i \in \Omega\psi\}$ for a fixed number j_0 of some recursive function g, and even more for such sets where we restrict to numbers i of ψ, where $\Phi_i \geq f$ for some $f \in \mathbb{R}^1$. Let us denote $\Omega^f\psi =_{Df} \{i : \varphi_i = \psi \wedge \Phi_i \geq f\}$.

Of course, we have $h(\Omega^f\psi, j_0) = \{h(i, j_0) : \varphi_i = \psi \wedge \Phi_i \geq f\} \subseteq \Omega P(\psi, g)$. It is a simple consequence from the Theorem of Rice (see it in [7]) that the set $\Omega P(\psi, g)$ is not decidable. Now we show that also the subset $h(\Omega^f\psi, j_0)$ is not decidable (if g has the whole set $\mathbb{N}$ as its range).

Theorem 2. *Let h be an iteration function Φ a Blum measure, $f, g, \psi \in$ $\in \mathbb{R}^1$, where g has the range $\mathbb{N}$ and $j_0 \in \Omega g$. Then $h(\Omega^f\psi, j_0)$ is not decidable.*

Proof. Assume, the presumptions of the theorem are given. We use two central theorems for Gödel numberings, the s-m-n Theorem and the Recursion Theorem which are to find for instance in [7,8]. First, by the s-m-n-Theorem there exists a function $\tau \in \mathbb{R}^2$ such that

$$\varphi_{\tau(i,j)}(x) =$$

$$= \begin{cases} \psi(x) & \text{if } \varphi_j(x') = \psi(x') \text{ and } \Phi_i(x') \geq f(x') \text{ for all } x' \leq x, \\ \text{not defined} & \text{if } \varphi_j(x') \neq \psi(x') \text{ for some } x' \leq x, \\ \varphi_i(x) + 1 & \text{otherwise.} \end{cases}$$

By the Recursion Theorem there exists $\sigma \in \mathbb{R}^1$ such that $\varphi_{\tau(\sigma(i),i)} = \varphi_{\sigma(i)}$ for all $i \in \mathbb{N}$.

Claim (a). $i \in \Omega\psi \leftrightarrow D_{\sigma(i)} = \mathbb{N} \leftrightarrow \sigma(i) \in \Omega^f \psi$.

Claim (b). Let R_j *denote the range of* φ_j. *If* $D_j = R_j = \mathbb{N}$ *then* $D_i = \mathbb{N} \leftrightarrow$ $\leftrightarrow D_{h(i,j)} = \mathbb{N}$.

(b) is an immediate consequence from the definition of an iteration function.

For the proof of (a) see that

$$\varphi_{\sigma(i)}(x) = \begin{cases} \psi(x) & \text{if } \varphi_i(x') = \psi(x') \wedge \Phi_{\sigma(i)}(x') \geq f(x') \text{ for all } x' \leq x, \\ \text{not defined} & \text{if } \varphi_i(x') \neq \psi(x') \text{ for some } x' \leq x, \\ \varphi_{\sigma(i)}(x) + 1 & \text{otherwise.} \end{cases}$$

If $\varphi_i = \psi$ then either $\varphi_{\sigma(i)} = \psi$ and $\Phi_{\sigma(i)} \geq f$ (which means, $\sigma(i) \in \Omega^f$ and $D_{\sigma(i)} = \mathbb{N}$) or $\varphi_{\sigma(i)}(x) = \varphi_{\sigma(i)}(x) + 1$ and $\Phi_{\sigma(i)}(x) < f(x)$ for some x. But the latter is not possible. If $\varphi_i \neq \psi$ then $\varphi_{\sigma(i)}(x)$ is not defined for some x, and therefore $D_{\sigma(i)} \neq \mathbb{N}$ and $\sigma(i) \notin \Omega^f \psi$.

From (a) and (b) we get

$$i \in \Omega\psi \leftrightarrow h(\sigma(i), j_0) \in h(\Omega^f \psi, j_0)$$

because of $D_{h(i,j_0)} = \mathbb{N}$ for each $j \in \Omega\psi$.

If $h(\Omega^f \psi, j_0)$ were decidable then also $\Omega\psi$ should be decidable contradicting to the Theorem of Rice.

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(Received September 27, 2001)

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