Annales Univ. Sci. Budapest., Sect. Comp. 21 (2004) 81-118

VERTEX DISJOINT POLYP PACKING

G.Y. Katona (Budapest, Hungary)

Abstract. A graph is called a p-polyp if it consists of p simple paths of the same length and one endvertex of all these paths is a common vertex. The Polyp Packing problem is a generalization of the well known Bin Packing problem: How to pack a set of paths with different lengths to a set of polyps vertex disjointly? It is proved in [4] that the Polyp Packing problem is NP-complete. In the present paper we prove that a modification of the First Fit algorithm gives a reasonable approximation.

1. Introduction

We will use the standard terminology of graph theory through the paper, but a few terms have multiple meaning in general so they are defined next.

Let $G, H_1, H_2, \ldots, H_k$ be simple graphs on the disjoint vertex sets $V(G)$, $V(H_1), V(H_2), \ldots, V(H_k)$. An edge disjoint embedding of $H_1, \ldots, H_k$ into G is a mapping $f : V(G) \to \cup V_{i=1}^{k}(H_i)$ such that if u and v are adjacent vertices of H_i then $f(u)$ and $f(v)$ are adjacent in G. (Note that $u \neq v$ does not imply $f(u) \neq f(v)$.) A vertex disjoint embedding of $H_1, \ldots, H_k$ into G is an edge disjoint embedding for which $u \neq v$ implies $f(u) \neq f(v)$. The phrase path usually refers to a subgraph of a graph but in the present paper it will refer to the simple graph which is a path itself. So a simple path of length s refers to the graph with vertex set $\{v_0, v_1, \ldots, v_s\}$ and edge set $\{\{v_0, v_1, \}, \{v_1, v_2\}, \ldots, \{v_{s-1}, v_s\}\}$.

Both of the following problems may be considered as a generalization of the Bin Packing problem [2]: Given a simple path P, a set of simple paths

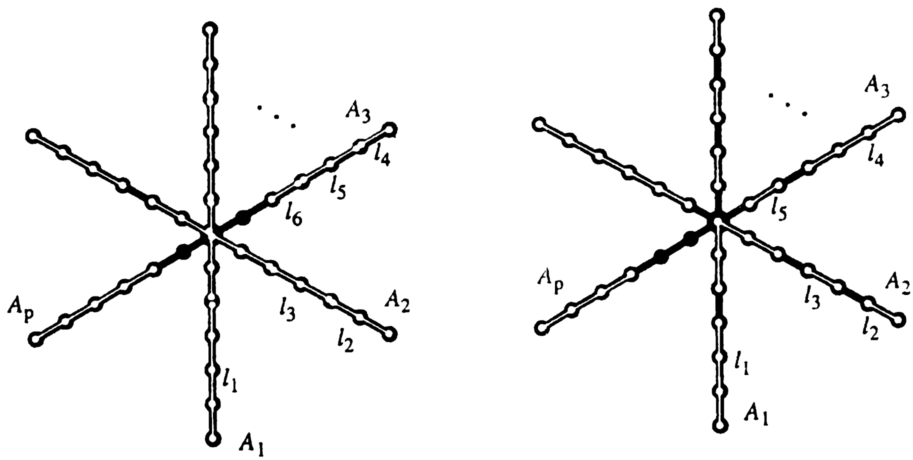
---

This work was partially supported by Hungarian National Foundation for Scientific Research under grants OTKA T4271 and T014302.

possibly with many different lengths and a positive integer K. Is it possible to embed all the paths edge/vertex disjointly into K copies of P?

G.O.H. Katona generalized this question [3]: Given a graph G, a set of simple paths possibly with many different lengths and a positive integer K. Is it possible to embed all the paths edge/vertex disjointly into K copies of G? (In the edge disjoint case the embedded path may contain a vertex more than once, but only once each edge.)

The vertex disjoint problem is equivalent to the Bin Packing problem if G has a Hamiltonian path. The edge disjoint problem is equivalent to the Bin Packing problem if G has an Eulerian path.



Edge disjoint problem Vertex disjoint problem

Figure 1.

In the present paper this question is investigated in a special case when the given graph is a polyp. A graph is called a p-polyp if it consists of p simple paths of the same length and one endvertex of all these paths is a common vertex - the center of the polyp. The paths attached to the center are called arms. These problems are called edge/vertex disjoint Polyp Packing problems. Obviously if the given graph is a p-polyp then the problem is called p-Polyp Packing. The main difference between the edge and vertex disjoint case is that several packed paths may contain the center of the polyp in the first case but only one in the second one. Note that there is no such difference if $p = 3$ since in both the edge and vertex disjoint cases only one path may contain edges of two different arms (see Fig. 1). Therefore in the present paper only the case $p \geq 4$ is considered. It was proved by the author in [4] that both Polyp Packing problems are NP-complete thus there is no hope for an exact polynomial algorithm. However, there is a simple near optimal algorithm.

In the case of the vertex disjoint packing there is at most one path in every polyp which has parts in two arms, thus the problem is very similar to the version of the Bin Packing problem where every $(p-2)$ bins of size 1 the next bin has size 2. (We combine the last two arms to be a bin of size 2.) This problem is not very far from the original problem so it does not look too difficult to handle. However, our problem is equivalent to this only in the sense that they have the same optimal solution (which is NP-hard to find) and some off-line algorithms give the same near optimal solution. On the other hand, for any on-line algorithm we usually obtain a better result for the vertex disjoint polyp packing problem, since "we can decide later" which arms we combine.

We consider a variant of the First Fit algorithm and determine how many more polyps are needed to pack the paths with the First Fit algorithm than with the optimal packing in the vertex disjoint case. The corresponding results for the edge disjoint problems were presented in [4]. The proofs are similar to the proofs in Johnson at al. [2] which proves similar theorems about the original Bin Packing problem but it seems to be more difficult to apply the same techniques than in the edge disjoint case.

2. Worst case of First Fit

Let us define the algorithm which will be investigated.

Let the p-polyps be indexed as $P_1, P_2, \ldots$, initially each empty. The arms of P_i are also indexed $A_{i,1}, \ldots, A_{i\,p}$. The paths $l_1, l_2, \ldots, l_t$ will be placed in that order. To place l_i, we try to put it into $A_{1,1}$ so that it does not contain the center. If it is not possible we try to put it into $A_{1,2}$ and so on. If we find an arm which is not "full" we put l_i to this arm as far as possible from the center. If all the arms of P_1 are "full" we try to put l_i to $A_{1,1}$ and $A_{1,2}$ so that l_i will contain the center of P_1. Then we try to put it into $A_{1,1}$ and $A_{1,3}$ and so on, we continue using the lexicographic order of the pair of arms. If we cannot put l_i to P_1, we move on to P_2, etc.

This is called the First Fit algorithm. (This algorithm is the same as in the edge disjoint case [4]. Notice that in the vertex disjoint case we can use the center only once.) The number of polyps one needs to pack a set L of paths is denoted by $FF_p(L)$, while the number of polyps in an optimal packing is $OPT_p(L)$. The set of paths may be viewed in the First Fit algorithm like a list since it gives the order of the paths in the First Fit algorithm. So L really denotes this list while $|L|$ denotes the number of paths in the list. One more notation:

$$R_p = \lim_{|L|\to\infty} \frac{FF_p(L)}{OPT_p(L)}.$$

To make computations easier from now on the arm length of the polyp will be the "unit". More precisely every length will be divided by s, where s denotes the number of edges in an arbitrary arm of the polyp (all arms have the same size). In addition in this vertex disjoint case the length of a path, despite the usual notation, always denotes the number of vertices of the path. Suppose for example that a path contains e edges where $e \le 2s$. The length of this path will be $\frac{e+1}{s}$ with the new notation. To make expressions shorter let $\varepsilon = \frac{1}{s}$. With this new notation a path of length $\frac{u}{v}$ contains $\frac{us}{v}$ vertices and $\frac{us}{v} - 1$ edges. Thus an arm of the polyp with the center as one endvertex will have size $\frac{s+1}{s} = 1 + \varepsilon$ and the sizes of the paths will be rational numbers between ε and $2 + \varepsilon$. This notation assures that if there are some paths such that the sum of their sizes is d, then ds consecutive vertices of an arm are needed to pack them to this arm. Since there is no difference between two paths of the same size b_i or l_i, they denote both the path and the size of the path.

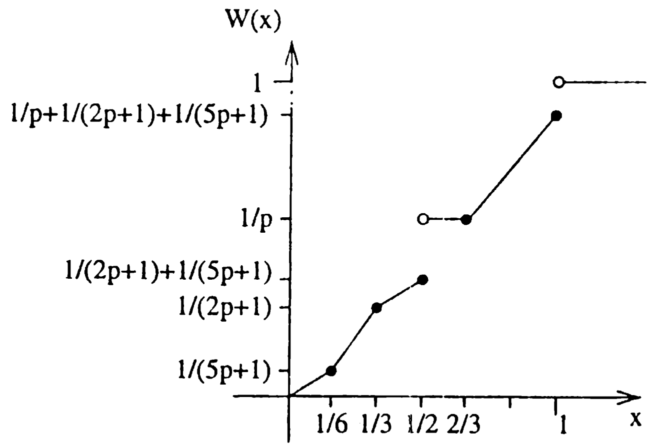


Figure 2.

Theorem 2. *For every list L and $p \geq 4$*

$$FF_p(L) \leq OPT_p(L)\left[\frac{p-1}{5p+1}+\frac{p-1}{2p+1}+\frac{p-1}{p}+1\right]+5.$$

For some values of p this gives

$$R_4 < 2.227,$$
$$R_5 < 2.318,$$
$$R_{15} < 2.570,$$
$$\lim_{i\to\infty} < 2.7.$$

Proof. The proof was inspired by the work of Johnson at al [2], they used a weight function on the items. Now a similar weight function is defined on the paths, but this function is much more complicated.

$$W(x)=\begin{cases}\dfrac{6}{5p+1}x, & \text{if } 0<x\leq\dfrac{1}{6},\\ \dfrac{18p}{(5p+1)(2p+1)}x-\dfrac{p-1}{(5p+1)(2p+1)}, & \text{if } \dfrac{1}{6}\leq x\leq\dfrac{1}{3},\\ \dfrac{6}{5p+1}x+\dfrac{p-1}{(5p+1)(2p+1)}, & \text{if } \dfrac{1}{3}\leq x\leq\dfrac{1}{2},\\ \dfrac{1}{2}, & \text{if } \dfrac{1}{2}<x\leq\dfrac{2}{3},\\ \dfrac{21p+6}{(2p+1)(5p+1)}x-\dfrac{4p^2-3p-1}{p(2p+1)(5p+1)}, & \text{if } \dfrac{2}{3}\leq x\leq 1,\\ 1, & \text{if } 1<x\leq 2+\varepsilon.\end{cases}$$

To make the picture more clear the key points of the function are (see Fig. 2)

$$W\left(\frac{1}{6}\right)=\frac{1}{5p+1},\quad W\left(\frac{1}{3}\right)=\frac{1}{2p+1},\quad W\left(\frac{1}{2}\right)=\frac{1}{5p+1}+\frac{1}{2p+1},$$
$$W(x)=\frac{1}{p}\quad\text{for } \frac{1}{2}<x<\frac{2}{3},\quad W(1)=\frac{1}{5p+1}+\frac{1}{2p+1}+\frac{1}{p}.$$

Lemma 1. *Let a polyp be filled with paths $l_1, l_2, \ldots, l_m$ in an arbitrary way. Then*

$$\sum_{i=1}^{m} W(l_i) \leq \frac{p-1}{5p+1}+\frac{p-1}{2p+1}+\frac{p-1}{p}+1.$$

Proof. There will be several cases. The first case is when there is no path is > 1. Choose a path which covers vertices in two different arms. It may be supposed that it is l_1. This path is ≤ 1 so it cannot cover $> \frac{1}{2}$ in both arms. Suppose it covers $\leq \frac{1}{2}$ in the arm A_1 and some portion of A_2. Replace l_1 and all the other paths in A_2 by a path which has the same size as the sum of the replaced paths or if this sum is ≤ 1 then by a path of size $1+\varepsilon$. Thus the length of the new path is $\leq \frac{3}{2}$ because ≤ 1 is contained by A_2 and $\leq \frac{1}{2}$ by A_1. If there is no path which covers the center, replace the content of any arm with a path of size $1+\varepsilon$.

One can check that if $0 < x \leq 1$ then $W(x)/x < \frac{2}{p}$ (see Fig. 3).

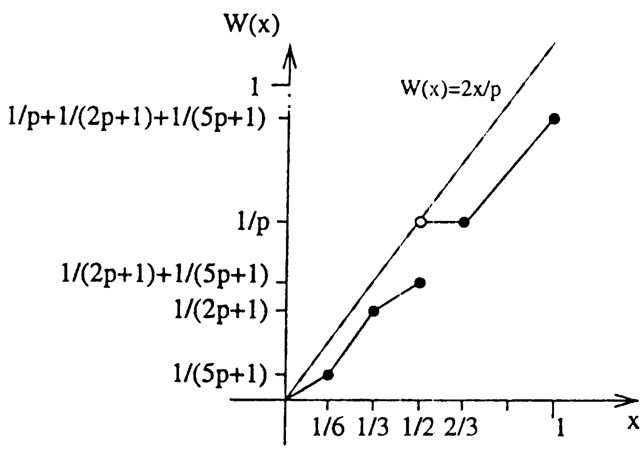


Figure 3.

The sum of the weights of the replaced paths was $< \frac{3}{2}\frac{2}{p}$. This is < 1 if $p \geq 4$. Since the weight of the new path is 1, the overall weight sum in the polyp is increased. Therefore the proof is reduced to the second case when there is a path of size > 1.

In the second case when there is a path of size > 1, this path must contain the center. The other paths are entirely contained in the arms. Now we prove that if there are paths with sizes $b_1 \geq b_2 \geq \ldots \geq b_t$ such that $\sum_{i=1}^{t} b_i \leq 1$ then

$$\sum_{i=1}^{t} W(b_i) \leq \frac{1}{5p+1} + \frac{1}{2p+1} + \frac{1}{p}.$$

The first subcase is when $1 \geq b_1 \geq \frac{2}{3}$. In this case $b_i \leq \frac{1}{2}$ holds for all other paths. The slope of $W(x)$ in the interval $\left[\frac{2}{3}, 1\right]$ is larger than anywhere in $\left[0, \frac{1}{2}\right]$. Replace all paths with a single path of size $\sum_{i=1}^{t} b_i$, then the weight sum increases. Since $W(x) \leq \frac{1}{5p+1} + \frac{1}{2p+1} + \frac{1}{p}$ if $x \leq 1$, this case is proved.

Suppose that there is no $b_i \geq \frac{2}{3}$, but $b_1 > \frac{1}{2}$. The slope of $W(x)$ in $\left(\frac{1}{2}, \frac{2}{3}\right]$ is less than anywhere in $\left[0, \frac{1}{2}\right]$, thus the weight sum increases if b_1 is replaced by two paths of length $\frac{1}{2} + \varepsilon$ and $b_1 - \frac{1}{2} - \varepsilon \leq \frac{1}{2}$. Now it is sufficient to prove that if $\sum_{i=1}^{t} b_i \leq \frac{1}{2}$ then

$$\sum_{i=1}^{t} W(b_i) \leq \frac{1}{5p+1} + \frac{1}{2p+1},$$

since $W\left(\frac{1}{2} + \varepsilon\right) = \frac{1}{p}$. W is almost the same in the interval $\left[0, \frac{1}{2}\right]$ as the weight function in [2] and this part was basically proved there, therefore the computations are omitted.

Only one case remains when $b_i \leq \frac{1}{2}$. However, $x \leq \frac{1}{2}$ implies

$$W(x)/x \leq \frac{3}{2p+1} < \frac{1}{5p+1} + \frac{1}{2p+1} + \frac{1}{p}.$$

To finish the proof note that if there is one path of length > 1 (with weight 1) and $p-1$ arms, each containing paths with weight sum less than $\frac{1}{5p+1} + \frac{1}{2p+1} + \frac{1}{p}$, then

$$\sum_{i=1}^{m} W(l_i) \leq \frac{p-1}{5p+1} + \frac{p-1}{2p+1} + \frac{p-1}{p} + 1.$$

Let us define the coarseness of a polyp. The coarseness of the polyp P_d is the largest α_d such that after completing the First Fit algorithm an additional path of size α_d would fit to some polyp with smaller index than d. The coarseness of P_1 is 0. Therefore if the First Fit algorithm puts a path of size l to polyp P_d then $l > \alpha_d$. Note that α_d could be different if a different set of paths were packed with the First Fit algorithm. It really corresponds to the subset of the paths that are packed to the first $d-1$ polyps.

Lemma 2. *Let a polyp P_d of coarseness $\alpha_d \leq 1$ be filled with paths $b_1 \geq b_2 \geq \ldots \geq b_t$ in the completed First Fit algorithm. If $\sum_{i=1}^{t} W(b_i) < 1$ then $\alpha_{d+1} > \alpha_d$.*

Proof. It may be supposed that there is no path > 1, otherwise the sum of the weights would be at least 1.

There will be a number of cases depending on the value of α_d.

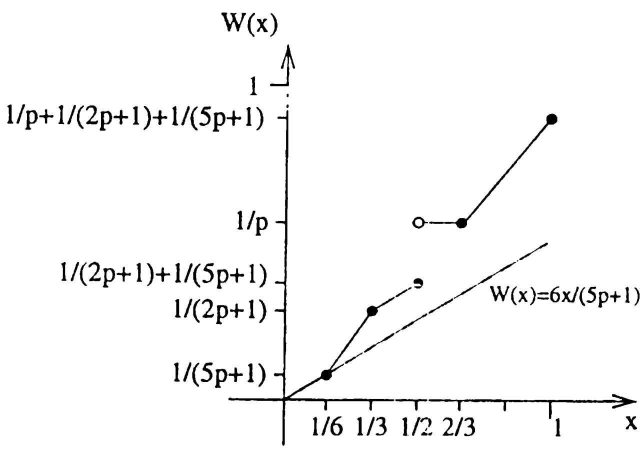


Figure 4.

Case 1. $\alpha_d \leq \dfrac{1}{6}$.

On Fig. 4 the reader may see that $W(x) \geq \frac{6}{5p+1}x$ holds in $[0,1]$, thus

$$1 > \sum_{i=1}^{t} W(b_i) \geq \frac{6}{5p+1} \sum_{i=1}^{t} b_i, \quad \text{therefore}$$

$$\frac{5p+1}{6} > \sum_{i=1}^{t} b_i$$

is obtained.

If there is a path containing the center then one of the arms must be covered completely and the empty part is "together" in the other arms since every path is placed to the arm as far from the center as possible. Hence there are at most $p-1$ empty parts. The sum of the sizes of the empty parts is

$$> p - \frac{5p+1}{6} = \frac{p-1}{6},$$

thus there is an empty part with size $> \frac{p-1}{6}\frac{1}{p-1} = \frac{1}{6} \geq \alpha_d$. By definition this implies $\alpha_{d+1} > \alpha_d$.

If there is no path containing the center then we can "combine" the empty parts of two arms together, the empty parts must be "adjacent" to the center, so there are $p-1$ empty parts again, so the conclusion is the same.

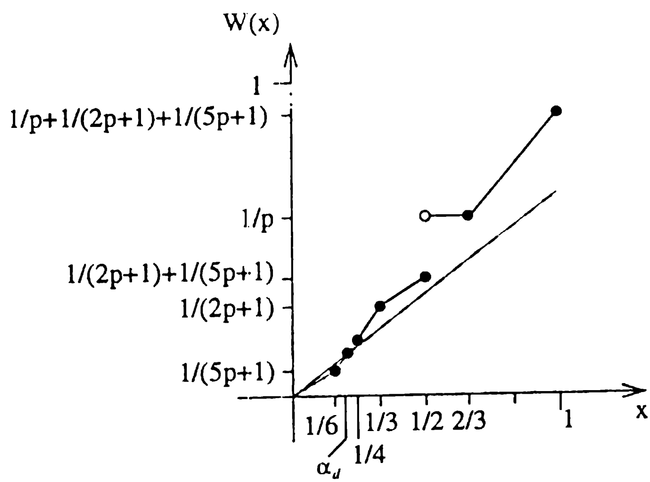


Figure 5.

Case 2. $\frac{1}{6} < \alpha_d \leq \frac{1}{4}$.

The definition of the coarseness implies $b_1, b_2, \ldots \geq \alpha_d > \frac{1}{6}$. It is clear from Fig. 5 that the line connecting $(0,0)$ and $(\alpha_d, W(\alpha_d))$ on the diagram of the weight function lies under the weight function in the interval $[\alpha_d, 1]$. The "worst case" is when $\alpha_d = \frac{1}{4}$, then the line contains $\left(\frac{1}{2}, W\left(\frac{1}{2}\right)\right)$. Using this observation

$$1 > \sum_{i=1}^{t} W(b_i) \geq \frac{W(\alpha_d)}{\alpha_d} \sum_{i=1}^{t} b_i,$$

thus

$$\frac{\alpha_d}{W(\alpha_d)} > \sum_{i=1}^{t} b_i$$

holds. There are $p-1$ empty parts again, like in the previous case, so it remains to show that

$$\frac{p - \frac{\alpha_d}{W(\alpha_d)}}{p-1} > \alpha_d,$$

or equivalently

$$\frac{p(-18\alpha_d^2 + 9\alpha_d - 1)}{18\alpha_d p - p + 1} > 0,$$

which is clearly true if $\frac{1}{6} < \alpha_d \leq \frac{1}{4}$ and $p \geq 4$.

Case 3. $\frac{1}{4} < \alpha_d \leq \frac{1}{3}$, *the center is not covered.*

Suppose that $\alpha_{d+1} \not> \alpha_d$ which means that $\alpha_{d+1} = \alpha_d$. It will be shown that this implies $\sum_{i=1}^{t} W(b_i) \geq 1$, a contradiction. Two major cases are distinguished. The central vertex may be covered by a path or it may remain uncovered. When it is covered there is a path which has some parts in two arms. These two arms will be called paired. If the central vertex is the endvertex of the covering path then any other arm may be paired with the arm containing the covering path.

If the central vertex is not covered then let s_1 and s_2 be the sum of the sizes of the covered parts in the two least filled arms. Since a path of size over

α_d should not fit in, we obtain $s_1 + s_2 \geq 2 + \varepsilon - \alpha_d > 2 - \alpha_d$. On the other hand

$$s_1 + s_2 \leq \frac{2}{p} \sum_{i=1}^{t} b_i,$$

assuring that

$$p - p\frac{\alpha_d}{2} \leq \sum_{i=1}^{t} b_i.$$

It is clear from Fig. 6 that

$$W(x) \geq \left(\frac{2}{5p+1} + \frac{2}{2p+1}\right) x$$

holds in $\left(\frac{1}{4}, 1\right]$.

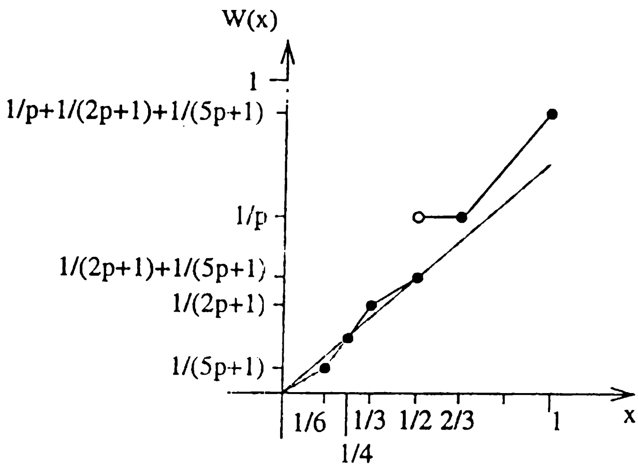


Figure 6.

Thus

$$\sum_{i=1}^{t} W(b_i) \geq \sum_{i=1}^{t} b_i \left(\frac{2}{5p+1} + \frac{2}{2p+1}\right) \geq \left(p - p\frac{\alpha_d}{2}\right)\left(\frac{2}{5p+1} + \frac{2}{2p+1}\right) > 1$$

holds, which is a contradiction.

The case when the center is covered is somewhat more complicated. Two subcases, $\frac{1}{4} < \alpha_d \le \frac{2}{7}$ and $\frac{2}{7} < \alpha_d \le \frac{1}{3}$ are distinguished.

Case 4. $\frac{1}{4} < \alpha_d \le \frac{2}{7}$, *the center is covered.*

Suppose that $\alpha_{d+1} \not> \alpha_d$. It will be shown that this implies $\sum_{i=1}^{t} W(b_i) \ge 1$, a contradiction. In this case it will be shown that the sum of the weights in the paired arms together is at least $\frac{7}{3p+1}$ and the sum of the weights in the other arms is at least $\frac{3}{3p+1}$. The latter claim is proved first. If there is only one path in the arm then it must be larger than $\frac{2}{3}$, otherwise $\alpha_{d+1} > \alpha_d$. In this way its weight is at least $\frac{1}{p} > \frac{3}{3p+1}$.

Now a few subcases will be considered when there are two paths in the arm.

If one of the paths is $> \frac{1}{2}$ then the sum of the weights is again at least $\frac{1}{p} > \frac{3}{3p+1}$.

When both paths are $\le \frac{1}{2}$ and $> \frac{1}{3}$ then $b_1 + b_2 \ge 1 - \alpha_d \ge 1 - \frac{2}{7}$ must hold, thus

$$W(b_1) + W(b_2) \ge \left(1 - \frac{2}{7}\right)\frac{6}{5p+1} + 2\frac{p-1}{(2p+1)(5p+1)} > \frac{3}{3p+1}.$$

If $\frac{1}{2} \ge b_1 > \frac{1}{3} \ge b_2 > \frac{2}{7} \ge \alpha_d > \frac{1}{4}$ then $b_1 + b_2 \ge 1 - \alpha_d$ must be true, therefore

$$\begin{aligned} W(b_1) + W(b_2) &= \\ &= \frac{6}{5p+1}b_1 + \frac{p-1}{(2p+1)(5p+1)} + \frac{18p}{(2p+1)(5p+1)}b_2 - \frac{p-1}{(2p+1)(5p+1)} = \\ &= (b_1+b_2)\frac{6}{5p+1} + b_2\frac{18p-12p-6}{(2p+1)(5p+1)} \ge \\ &\ge (1-\alpha_d)\frac{6}{5p+1} + b_2\frac{6p-6}{(2p+1)(5p+1)} > \frac{3}{3p+1}, \end{aligned}$$

since $b_2 > \frac{2}{7}, \alpha_d \le \frac{2}{7}$ and $p \ge 4$.

If $\frac{1}{2} \geq b_1 > \frac{1}{3} > \frac{2}{7} \geq b_2 > \alpha_d > \frac{1}{4}$ then $b_1 + b_2 \geq 1 - \alpha_d > 1 - b_2$ must be true, therefore

$$\begin{aligned}
&W(b_1) + W(b_2) = \\
&= \frac{6}{5p+1} b_1 + \frac{p-1}{(2p+1)(5p+1)} + \frac{18p}{(2p+1)(5p+1)} b_2 - \frac{p-1}{(2p+1)(5p+1)} = \\
&= (b_1 + b_2) \frac{6}{5p+1} + b_2 \frac{18p - 12p - 6}{(2p+1)(5p+1)} > \\
&> (1 - b_2) \frac{6}{5p+1} + b_2 \frac{6p - 6}{(2p+1)(5p+1)} = \\
&= \frac{6}{5p+1} - b_2 \frac{6p+12}{(2p+1)(5p+1)} \geq \frac{3}{3p+1}
\end{aligned}$$

since $b_2 \leq \frac{2}{7}$.

If both paths are $\geq \frac{1}{3}$ then $\alpha_{d+1} > \alpha_d$.

If there are at least three paths, all $> \frac{1}{4}$, then

$$W(b_1) + W(b_2) + W(b_3) \geq \frac{3}{4} \frac{18p}{(2p+1)(5p+1)} - \frac{3p-3}{(2p+1)(5p+1)} > \frac{3}{3p+1}.$$

The other claim will be proved now, that the sum of the weights is at least $\frac{7}{3p+1}$ in the paired arms.

The sum of the sizes must be at least $2 + \varepsilon - \alpha_d \geq \frac{12}{7}$, so there must be at least two paths in the paired arms since there is no path of size ≤ 1 now.

Suppose there are exactly two paths. If both paths are $> \frac{1}{2}$ then

$$W(b_1) + W(b_2) \geq (b_1 + b_2) \frac{3}{2p} \geq \frac{12}{7} \frac{3}{2p} \geq \frac{7}{3p+1} \tag{1}$$

since the weight function satisfies $W(x)/x \geq \frac{3}{2p}$ in $\left(\frac{1}{2}, 1\right]$ as the reader may see on Fig. 7.

If one path is $\leq \frac{1}{2}$ then the sum of the sizes is at most $\frac{3}{2} < \frac{12}{7}$ since the other path is ≤ 1 in this case.

If there are 3 paths and each one is $> \frac{1}{2}$ then (1) also holds for 3 paths.

The next case is when $b_1 \geq b_2 > \frac{1}{2} \geq b_3 \geq \frac{1}{3}$, which implies $b_1 + b_2 \geq$ $\geq \left(2 + \varepsilon - \frac{2}{7} - b_3\right)$. Hence we obtain

$$\sum_{i=1}^{3} W(b_i) \geq \frac{3}{2p}(b_1 + b_2) + b_3 \frac{6}{5p+1} + \frac{p-1}{(2p+1)(5p+1)} >$$
$$> \frac{3}{2p}\left(2 - \frac{2}{7} - b_3\right) + b_3 \frac{6}{5p+1} + \frac{p-1}{(2p+1)(5p+1)} > \frac{7}{3p+1}.$$

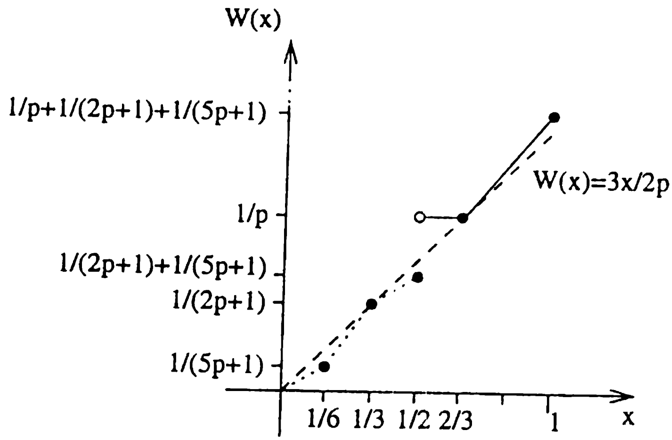


Figure 7.

since $b_3 \leq \frac{1}{2}$.

The same argument works with different coefficients for the case $b_1 \geq b_2 > \frac{1}{2} \geq \frac{1}{3} \geq b_3 > \frac{1}{4}$ and the case $b_1 > \frac{1}{2} \geq b_2 \geq b_3 \geq \frac{1}{3}$.

The next case is explained more detailed than it is necessary, because there will be some other cases when this general method will be used again. If $b_1 >$

$> \frac{1}{2} \geq b_2 > \frac{1}{3} \geq b_3 > \frac{1}{4}$ then $b_1 \geq 2 + \varepsilon - \frac{2}{7} - b_2 - b_3$ is obtained. Therefore

$$
\begin{aligned}
\sum_{i=1}^{3} W(b_i) \geq & \frac{3}{2p} b_1 + \frac{6}{5p+1} b_2 + \frac{p-1}{(2p+1)(5p+1)} + \\
& + \frac{18p}{(2p+1)(5p+1)} b_3 - \frac{p-1}{(2p+1)(5p+1)} > \\
& > \frac{3}{2p}\left(2 - \frac{2}{7} - b_2 - b_3\right) + \frac{6}{5p+1} b_2 + \frac{18p}{(2p+1)(5p+1)} b_3 = \\
& = \frac{18}{7p} - b_2 \frac{3p+3}{2p(5p+1)} + b_3 \frac{6p^2 - 21p - 3}{2p(2p+1)(5p+1)}
\end{aligned} \tag{2}
$$

holds. In general the coefficients of b_2 and b_3 may be independently positive or negative for some fixed p, but the expression can reach its minimum only if b_2 and b_3 have minimum or maximum value. If we are interested in the minimum of the expression the following four cases must be checked:

1) $b_2 = \frac{1}{2}$ and $b_3 = \frac{1}{3}$,
2) $b_2 = \frac{1}{2}$ and $b_3 = \frac{1}{4} + \varepsilon$,
3) $b_2 = \frac{1}{3} + \varepsilon$ and $b_3 = \frac{1}{3}$,
4) $b_2 = \frac{1}{3} + \varepsilon$ and $b_3 = \frac{1}{4} + \varepsilon$.

In each case an inequality is obtained with one variable and, though it can be quite complicated, it may be proved that the inequality is true. In fact the author used the computer software MAPLE to find the proofs faster.

In this particular case it is easy to see that the coefficient of b_2 is always negative and the coefficient of b_3 is positive if $p \geq 4$, so we need to check only Case 2) for $p \geq 4$:

$$
\frac{18}{7p} - b_2 \frac{3p+3}{2p(5p+1)} + b_3 \frac{6p^2 - 21p - 3}{2p(2p+1)(5p+1)} - \frac{7}{3p+1} =
$$

$$
= \frac{274p^3 + 859p^2 + 586p + 81}{56p(5p+1)(2p+1)(3p+1)} > 0.
$$

If $1 \geq b_1 > \frac{1}{2} \geq \frac{1}{3} > b_2 \geq b_3$ then $b_1 + b_2 + b_3 < \frac{5}{3} < \frac{12}{7}$, so the proof may be continued with the case of four paths.

When two of the four paths are $> \frac{1}{2}$ then we obtain

$$\sum_{i=1}^{4} W(b_i) \geq 2\frac{1}{p} + 2W\left(\frac{1}{4}\right) > \frac{7}{3p+1}$$

if $p \geq 4$, since the other two paths are at least $\frac{1}{4}$.

The next case is $b_1 > \frac{1}{2} \geq b_2 \geq b_3 \geq b_4 > \frac{1}{3}$, when

$$\sum_{i=1}^{4} W(b_i) > \frac{1}{p} + 3\frac{1}{2p+1} \geq \frac{7}{3p+1}$$

holds.

If $b_1 > \frac{1}{2} \geq b_2 \geq b_3 > \frac{1}{3} \geq b_4 > \frac{1}{4}$ then one can apply the detailed method explained above and obtain

$$\sum_{i=1}^{4} W(b_i) \geq$$

$$\geq \left(2 - \frac{2}{7} - b_2 - b_3 + b_4\right)\frac{3}{2p} + (b_2+b_3)\frac{6}{5p+1} + 2\frac{p-1}{(2p+1)(5p+1)} +$$
$$+ b_4\frac{18p}{(2p+1)(5p+1)} - \frac{p-1}{(2p+1)(5p+1)} > \frac{7}{3p+1},$$

because $b_2 + b_3$ may be considered as one variable.

The same method works with the appropriate coefficients if $b_1 > \frac{1}{2} \geq b_2 > > \frac{1}{3} \geq b_3 \geq b_4 > \frac{1}{4}$.

If $b_1 > \frac{1}{2} > \frac{1}{3} \geq b_2 \geq b_3 \geq b_4 > \frac{1}{4}$ then

$$\sum_{i=1}^{4} W(b_i) \geq \left(2 - \frac{2}{7} - b_2 - b_3 - b_4\right)\frac{3}{2p} +$$
$$+ (b_2+b_3+b_4)\frac{18p}{(2p+1)(5p+1)} - 3\frac{p-1}{(2p+1)(5p+1)} > \frac{7}{3p+1},$$

because the coefficient of $(b_2+b_3+b_4)$ will be positive if $p \geq 4$ and $b_2+b_3+b_4 > > \frac{3}{4}$.

In the rest of this case all paths have length $\leq \frac{1}{2}$. If all paths are also $> \frac{1}{3}$ then

$$\sum_{i=1}^{4} W(b_i) \geq \left(2 - \frac{2}{7}\right)\frac{6}{5p+1} + 4\frac{p-1}{(2p+1)(5p+1)} > \frac{7}{3p+1}$$

is obtained.

If $\frac{1}{2} \geq b_1 \geq b_2 \geq b_3 > \frac{1}{3} \geq b_4 > \frac{1}{4}$ then similarly to the previous cases

$$\sum_{i=1}^{4} W(b_i) \geq \left(2 - \frac{2}{7} - b_4\right)\frac{6}{5p+1} + 3\frac{p-1}{(2p+1)(5p+1)} + \\ + b_4\frac{18p}{(2p+1)(5p+1)} - \frac{p-1}{(2p+1)(5p+1)} > \frac{7}{3p+1}.$$

If $\frac{1}{2} \geq b_1 \geq b_2 > \frac{1}{3} \geq b_3 \geq b_4 > \frac{1}{4}$ then $2 - \frac{2}{7} < 2 + \varepsilon - \alpha_d \leq \sum_{i=1}^{4} b_i \leq \frac{5}{3}$ is a contradiction.

This was the last case of four paths so we can continue with five paths.

If one of the paths is $> \frac{1}{2}$ then we obtain

$$\sum_{i=1}^{5} W(b_i) \geq \frac{1}{p} + 4W\left(\frac{1}{4}\right) = \frac{1}{p} + \frac{2}{2p+1} + \frac{2}{5p+1} > \frac{7}{3p+1}$$

is obtained because all the other paths are $> \frac{1}{4}$.

If $\frac{1}{2} \geq b_1 \geq b_2 \geq b_3 \geq b_4 \geq b_5 > \frac{1}{3}$ then

$$\sum_{i=1}^{5} W(b_i) > 5W\left(\frac{1}{3}\right) > \frac{7}{3p+1}$$

is obtained.

If $\frac{1}{2} \geq b_1 \geq b_2 \geq b_3 \geq b_4 > \frac{1}{3} \geq b_5 > \frac{1}{4}$ then

$$\sum_{i=1}^{5} W(b_i) \geq$$

$$\geq \left(2 - \frac{2}{7} - b_5\right) \frac{6}{5p+1} + b_5 \frac{18p}{(2p+1)(5p+1)} + 3\frac{p-1}{(2p+1)(5p+1)} > \frac{7}{3p+1}$$

holds because the coefficient of b_5 is positive if $p \geq 4$ and $b_5 > \frac{1}{4}$.

If $\frac{1}{2} \geq b_1 \geq b_2 \geq b_3 > \frac{1}{3} \geq b_4 \geq b_5 > \frac{1}{4}$ then the proof is the same as in the previous case.

If $\frac{1}{2} \geq b_1 \geq b_2 > \frac{1}{3} \geq b_3 \geq b_4 \geq b_5 > \frac{1}{4}$ then

$$\sum_{i=1}^{5} W(b_i) \geq (b_1 + b_2)\frac{6}{5p+1} +$$
$$+ \left(2 - \frac{2}{7} - b_1 - b_2\right) \frac{18p}{(2p+1)(5p+1)} + \frac{p-1}{(2p+1)(5p+1)} > \frac{7}{3p+1}$$

holds because the coefficient of $b_1 + b_2$ is negative if $p \geq 4$ and $b_1 + b_2 \leq 1$.

If $\frac{1}{2} \geq b_1 > \frac{1}{3} \geq b_2 \geq b_3 \geq b_4 \geq b_5 > \frac{1}{4}$ then

$$\sum_{i=1}^{5} W(b_i) \geq$$

$$\geq b_1 \frac{6}{5p+1} + \left(2 - \frac{2}{7} - b_1\right) \frac{18p}{(2p+1)(5p+1)} + 3\frac{p-1}{(2p+1)(5p+1)} > \frac{7}{3p+1}$$

is obtained since the coefficient of b_1 is negative if $p \geq 4$ and $b_1 \leq \frac{1}{2}$.

If all 5 paths are $< \frac{1}{3}$ then the sum of their sizes is $< \frac{5}{3} < \frac{12}{7}$ which is a contradiction therefore we continue with the case of six paths.

If one path is $> \frac{1}{2}$ then

$$\sum_{i=1}^{6} W(b_i) > \frac{1}{p} + 5W(\left(\frac{1}{4}\right) > \frac{7}{3p+1}$$

holds since the other paths are $> \frac{1}{4}$.

The proof is similar if $\frac{1}{2} \geq b_1 \geq b_2 > \frac{1}{3}$ since $p \geq 4$.

If $\frac{1}{2} \geq b_1 > \frac{1}{3} \geq b_2 \geq \ldots \geq b_6 > \frac{1}{4}$ then

$$\sum_{i=1}^{6} W(b_i) > b_1 \frac{6}{5p+1} + \frac{p-1}{(2p+1)(5p+1)} + \\ + \left(2 - \frac{2}{7} - b_1\right) \frac{18p}{(2p+1)(5p+1)} - 5\frac{p-1}{(2p+1)(5p+1)} > \frac{7}{3p+1}$$

since $b_1 \leq \frac{1}{2}$.

The last case is when there are seven or more paths. All of them must be over $\frac{1}{4}$, thus

$$\sum_{i=1}^{t} W(b_i) > 7W\left(\frac{1}{4}\right) > \frac{7}{3p+1}.$$

To complete the proof of this part a contradiction is shown. It is supposed that $\alpha_{d+1} = \alpha_d$ which implies that the sum of the weights is $\geq \frac{7}{3p+1}$ in the paired arms and the weight sum is $\geq \frac{3}{3p+1}$ in each of the $p-2$ single arms. This is a contradiction since

$$\frac{7}{3p+1} + (p-2)\frac{3}{3p+1} = 1.$$

Case 5. $\frac{2}{7} < \alpha_d \leq \frac{1}{3}$, *the center is covered.*

Suppose that $\alpha_{d+1} \not> \alpha_d$. It will be shown that this implies $\sum_{i=1}^{t} W(b_i) \geq 1$, a contradiction. Unfortunately we have to go through a number of cases again but as α_d increases, the number of the cases decreases.

There is an exceptional case which cannot be covered by the ususal method, so it has to be proved in a different way. This is the case when there are 5 paths in the paired arms (the center is covered) and $\frac{1}{2} \geq b_1 \geq b_2 > \frac{1}{3} \geq b_3 \geq b_4 \geq b_5 > \alpha_d > \frac{2}{7}$.

Let us prove this case first. It will be proved that the weight sum is at least

$$\frac{23p+13}{(2p+1)(5p+1)} + \alpha_d \frac{6p-24}{(2p+1)(5p+1)},$$

in the paired arms and the weight sums are at least

$$\frac{10p^2-16p-12-6\alpha_d p+24\alpha_d}{(2p+1)(5p+1)(p-2)}$$

in the single arms. The total weight sum is the sum of the weights in the paired arms and in the $(p-2)$ single arms

$$\frac{23p+13}{(2p+1)(5p+1)}+\alpha_d\frac{6p-24}{(2p+1)(5p+1)}+\frac{10p^2-16p-12-6\alpha_d p+24\alpha_d}{(2p+1)(5p+1)}\geq 1,$$

which is a contradiction.

It is easy to prove the first claim. Now $b_1+b_2\geq 2+\varepsilon-b_3-b_4-b_5-\alpha_d$ holds which implies

$$\begin{aligned}\sum_{i=1}^{5}W(b_i)&>(2-b_3-b_4-b_5-\alpha_d)\frac{6}{5p+1}+\\ &+(b_3+b_4+b_5)\frac{18p}{(2p+1)(5p+1)}-\frac{p-1}{(2p+1)(5p+1)}=\\ &=(b_3+b_4+b_5)\frac{6p-6}{(2p+1)(5p+1)}+\frac{12-6\alpha_d}{5p+1}-\frac{p-1}{(2p+1)(5p+1)}>\\ &>\alpha_d\frac{18p-18}{(2p+1)(5p+1)}+\frac{12-6\alpha_d}{5p+1}-\frac{p-1}{(2p+1)(5p+1)}=\\ &=\frac{23p+13}{(2p+1)(5p+1)}+\alpha_d\frac{6p-24}{(2p+1)(5p+1)}.\end{aligned}$$

The second claim about the single arms is proved next.

If there is only one path in the arm it must be over $\frac{2}{3}$ otherwise $\alpha_{d+1}>\alpha_d$. The weight of a path over $\frac{2}{3}$ is at least $\frac{1}{p}$, and the inequality

$$\frac{1}{p}>\frac{10p^2-16p-12-6\alpha_d p+24\alpha_d}{(2p+1)(5p+1)(p-2)}$$

holds for $\alpha_d\leq\frac{1}{3}$, proving our claim.

If there are two paths, $b_1\geq b_2>\frac{1}{3}$ then

$$\begin{aligned}&W(b_1)+W(b_2)\geq\\ &\quad>(1-\alpha_d)\frac{6}{5p+1}+2\frac{p-1}{(2p+1)(5p+1)}\geq\frac{10p^2-16p-12-6\alpha_d p+24\alpha_d}{(2p+1)(5p+1)(p-2)}\end{aligned}$$

holds, since $\alpha_d \le \frac{1}{3}$.

If $b_1 > \frac{1}{3} \ge b_2 > \frac{2}{7}$ then

$$\begin{aligned}
W(b_1) + W(b_2) \ge b_1 \frac{6}{5p+1} + \frac{p-1}{(2p+1)(5p+1)} + \\
+ b_2 \frac{18p}{(2p+1)(5p+1)} - \frac{p-1}{(2p+1)(5p+1)} > \\
> (1 - b_2 - \alpha_d)\frac{6}{5p+1} + b_2 \frac{18p}{(2p+1)(5p+1)} = \\
= b_2 \frac{6p-6}{(2p+1)(5p+1)} + \frac{6-6\alpha_d}{5p+1} \ge \\
\ge \frac{10p^2 - 16p - 12 - 6\alpha_d p + 24\alpha_d}{(2p+1)(5p+1)(p-2)}
\end{aligned}$$

is obtained.

$\frac{1}{3} \ge b_1 \ge b_2$ is not possible so the only case left is when there are at least 3 paths. The weight of each path is at least $W(\alpha_d)$ and

$$3W(\alpha_d) \ge \frac{10p^2 - 16p - 12 - 6\alpha_d p + 24\alpha_d}{(2p+1)(5p+1)(p-2)}$$

can be proved easily.

The proof of the exceptional case is completed. Notice that

$$\frac{10p^2 - 16p - 12 - 6\alpha_d p + 24\alpha_d}{(2p+1)(5p+1)(p-2)} \ge \frac{2}{2p+1}$$

holds for the previous lower bound if $p \ge 4$ and $\alpha_d \le \frac{1}{3}$. Thus it can be proved that the sum of the weights is at least $\frac{2}{2p+1}$ in the single arms since these inequalities do not depend on the conditions of the exceptional case.

In order to finish this case it will be proved that the weight sum is at least $\frac{5}{2p+1}$ in the paired arms (except the exceptional case).

It is easy to see that there are least 2 paths, otherwise the weight of the only path is > 1. If both paths are $> \frac{1}{2}$ then the lower bound $\frac{3}{2p}x$ on the weight of a path of length x can be used as earlier. The other observation is that $b_1 + b_2 \geq 2 + \varepsilon - \alpha_d > \frac{5}{3}$. Thus

$$W(b_1) + W(b_2) \geq \frac{3}{2p}(b_1 + b_2) > \frac{3}{2p}\frac{5}{3} > \frac{5}{2p+1}$$

holds. The same proof can be applied if there are more than 2 paths and all are $> \frac{1}{2}$. In the rest of the proof there will be at least one path which is $\leq \frac{1}{2}$. Therefore if there are only two paths then $\alpha_{d+1} > \alpha_d$, since the other path is ≤ 1.

The cases when there are 3 paths in the paired arms are proved next. If $b_1 \geq b_2 > \frac{1}{2} \geq b_3 \geq \frac{1}{3}$ then

$$\sum_{i=1}^{3} W(b_i) \geq 2\frac{1}{p} + \frac{1}{2p+1} > \frac{5}{2p+1}$$

holds.

If $b_1 \geq b_2 > \frac{1}{2} > \frac{1}{3} > b_3 > \alpha_d > \frac{2}{7}$ then a better lower bound must be used for the paths $> \frac{1}{2}$. It is obvious that

$$W(x) \geq x\frac{21p+6}{(2p+1)(5p+1)} - \frac{4p^2-3p-1}{p(2p+1)(5p+1)}$$

in $\left(\frac{1}{2}, 1\right]$. Using the bound $b_1 + b_2 > 2 + \varepsilon - \alpha_d - b_3$

$$\begin{aligned}
\sum_{i=1}^{3} W(b_i) \geq (b_1+b_2)&\frac{21p+6}{(2p+1)(5p+1)} - 2\frac{4p^2-3p-1}{p(2p+1)(5p+1)} + \\
&+ b_3\frac{18p}{(2p+1)(5p+1)} - \frac{p-1}{(2p+1)(5p+1)} > \\
&> (2-\alpha_d-b_3)\frac{21p+6}{(2p+1)(5p+1)} + \\
&+ b_3\frac{18p}{(2p+1)(5p+1)} - \frac{9p^2-7p-2}{p(2p+1)(5p+1)} = \\
&= -b_3\frac{3p+6}{(2p+1)(5p+1)} + \frac{13p+2}{p(5p+1)} \geq \frac{5}{2p+1}
\end{aligned}$$

is obtained since now $b_3 < \dfrac{1}{3}$ and $\alpha_d \leq \dfrac{1}{3}$.

If $b_1 > \dfrac{1}{2} \geq b_2 \geq b_3 \geq \dfrac{1}{3}$ then none of our usual methods works. However, if we take into consideration that these paths got their place in the polyp with the First Fit algorithm then we can settle it easily. The following more general claim is proved because this method is used in other case, too.

Claim 1. *If the paired arms contain the paths $b_1, \ldots, b_t$, for every $1 \leq i \leq t$ and every partition of $\{b_1, \ldots, b_t\} - \{b_i\}$ into two sets (one of them may be empty) the sum of the sizes in one of the parts is $\leq 1 - b_i$, then the center cannot be covered.*

Proof of Claim 1. The only reason to cover the center with b_i is that it does not fit to neither arm of the paired arms. This happens only if the sum of the sizes in both of these arms is $> 1 - b_i$.

If $b_1 > \dfrac{1}{2} \geq b_2 \geq b_3 \geq \dfrac{1}{3}$ then the conditions of Claim 1 clearly hold, the center cannot be covered.

If $b_1 > \dfrac{1}{2} \geq b_2 \geq \dfrac{1}{3} > b_3 > \dfrac{2}{7}$ then

$$\begin{aligned}
\sum_{i=1}^{3} W(b_i) &\geq b_1 \frac{21p+6}{(2p+1)(5p+1)} - \frac{4p^2-3p-1}{p(2p+1)(5p+1)} + \\
&\quad + b_2 \frac{6}{5p+1} + \frac{p-1}{(2p+1)(5p+1)} + \\
&\quad + b_3 \frac{18p}{(2p+1)(5p+1)} - \frac{p-1}{(2p+1)(5p+1)} \geq \\
&\geq \left(2 - \frac{1}{3} - b_2 - b_3\right) \frac{21p+6}{(2p+1)(5p+1)} - \frac{4p^2-3p-1}{p(2p+1)(5p+1)} + \\
&\quad + b_2 \frac{6}{5p+1} + b_3 \frac{18p}{(2p+1)(5p+1)} > \frac{5}{2p+1}
\end{aligned}$$

using the method explained in detail earlier.

If $b_1 > \dfrac{1}{2} > \dfrac{1}{3} > b_2 \geq b_3 > \dfrac{2}{7}$ then

$$\begin{aligned}\sum_{i=1}^{3} W(b_i) &\geq b_1 \frac{21p+6}{(2p+1)(5p+1)} - \frac{4p^2-3p-1}{p(2p+1)(5p+1)} + \\ &+ (b_2+b_3)\frac{18p}{(2p+1)(5p+1)} - 2\frac{p-1}{(2p+1)(5p+1)} \geq \\ &\geq \left(2-\frac{1}{3}-b_2-b_3\right)\frac{21p+6}{(2p+1)(5p+1)} - \frac{4p^2-3p-1}{p(2p+1)(5p+1)} + \\ &+ (b_2+b_3)\frac{18p}{(2p+1)(5p+1)} - 2\frac{p-1}{(2p+1)(5p+1)} > \frac{5}{2p+1}\end{aligned}$$

since the coefficient of b_2+b_3 is negative and $b_2+b_3 < \frac{2}{3}$.

If there is no path $> \frac{1}{2}$ then the sum of the sizes may reach only $\frac{3}{2}$, but then $\alpha_{d+1} > \alpha_d$, so we can continue with the case of four paths in the paired arms.

If there are at least two paths $> \frac{1}{2}$ then, since the other two must be $> \frac{2}{7}$, we obtain

$$\sum_{i=1}^{3} W(b_i) \geq \frac{2}{p} + 2W\left(\frac{2}{7}\right) > \frac{5}{2p+1}.$$

If $b_1 > \frac{1}{2} \geq b_2 \geq b_3 \geq b_4 \geq \frac{1}{3}$ then

$$\sum_{i=1}^{4} W(b_i) \geq \frac{1}{p} + 3\frac{1}{2p+1} > \frac{5}{2p+1}$$

holds.

If $b_1 > \frac{2}{3} > \frac{1}{2} \geq b_2 \geq b_3 \geq \frac{1}{3} > b_4 > \frac{2}{7}$ then the complicated method must be used again:

$$\begin{aligned}\sum_{i=1}^{4} W(b_i) &\geq b_1 \frac{21p+6}{(2p+1)(5p+1)} - \frac{4p^2-3p-1}{p(2p+1)(5p+1)} + (b_2+b_3)\frac{6}{5p+1} + \\ &+ 2\frac{p-1}{(2p+1)(5p+1)} + b_4\frac{18p}{(2p+1)(5p+1)} - \frac{p-1}{(2p+1)(5p+1)} > \\ &> b_1 \frac{21p+6}{(2p+1)(5p+1)} + (2-\alpha_d-b_1-b_4)\frac{6}{5p+1} + \\ &+ b_4\frac{18p}{(2p+1)(5p+1)} - \frac{3p^2-2p-1}{p(2p+1)(5p+1)} > \frac{5}{2p+1}\end{aligned}$$

holds since $\alpha_d < b_4$, the coefficient of b_1 is positive, the coefficient of b_4 is negative, $b_1 > \frac{2}{3}$ and $b_4 < \frac{1}{3}$.

If $\frac{2}{3} \geq b_1 > \frac{1}{2} \geq b_2 \geq b_3 \geq \frac{1}{3} > \frac{2}{7}$ then

$$\sum_{i=1}^{4} W(b_i) > \frac{1}{p} + \left(2 - \frac{2}{3} - b_4\right)\frac{6}{5p+1} + 2\frac{p-1}{(2p+1)(5p+1)} + \\ + b_4\frac{18p}{(2p+1)(5p+1)} - \frac{p-1}{(2p+1)(5p+1)} > \frac{5}{2p+1}$$

since $b_1 \leq \frac{2}{3}$, the coefficient of b_4 is positive and $b_4 > \frac{2}{7}$.

If $\frac{2}{3} \geq b_1 > \frac{1}{2} \geq b_2 \geq \frac{1}{3} > b_3 \geq b_4 > \frac{2}{7}$ then

$$\sum_{i=1}^{4} W(b_i) \geq \frac{1}{p} + \left(2 - \frac{2}{3} - b_3 - b_4\right)\frac{6}{5p+1} + \frac{p-1}{(2p+1)(5p+1)} + \\ + (b_3 + b_4)\frac{18p}{(2p+1)(5p+1)} - 2\frac{p-1}{(2p+1)(5p+1)} > \frac{5}{2p+1}$$

because $b_1 \leq \frac{2}{3}$, the coefficient of $b_3 + b_4$ is positive and $b_3 + b_4 > \frac{4}{7}$.

If $\frac{2}{3} \geq b_1 > \frac{1}{2} > \frac{1}{3} > b_2 \geq b_3 \geq b_4 > \frac{2}{7}$ then it is easy to see that $\alpha_{d+1} > \alpha_d$.

If $\frac{1}{2} \geq b_1 \geq b_2 \geq b_3 \geq b_4$ then the center cannot be covered by Claim 1. This completes the case of 4 paths.

In the next cases there are five paths in the paired arms. If at least one of them is $> \frac{1}{2}$ then

$$\sum_{i=1}^{5} W(b_i) > \frac{1}{p} + 4W\left(\frac{2}{7}\right) > \frac{5}{2p+1}$$

is a good lower bound, since the other paths are over $\frac{2}{7}$.

If all paths are $\geq \frac{1}{3}$ then the weight sum is clearly $\geq \frac{5}{2p+1}$.

If $\frac{1}{2} \geq b_1 \geq b_2 \geq b_3 \geq b_4 \geq \frac{1}{3} > b_5 > \alpha_d > \frac{2}{7}$ then the usual method works, thus we obtain

$$\sum_{i=1}^{5} W(b_i) \geq (b_1 + b_2 + b_3 + b_4)\frac{6}{5p+1} + 4\frac{p-1}{(2p+1)(5p+1)} + \\ + b_5\frac{18p}{(2p+1)(5p+1)} - \frac{p-1}{(2p+1)(5p+1)} > \\ > (2 - \alpha_d - b_5)\frac{6}{5p+1} + b_5\frac{18p}{(2p+1)(5p+1)} + 3\frac{p-1}{(2p+1)(5p+1)} > \\ > \frac{5}{2p+1},$$

since $\alpha_d < b_5$. The same idea works if $\frac{1}{2} \geq b_1 \geq b_2 \geq b_3 \geq \frac{1}{3} > b_4 \geq b_5 > \alpha_d > \frac{2}{7}$.

The next case, $\frac{1}{2} \geq b_1 \geq b_2 \geq \frac{1}{3} > b_3 \geq b_4 \geq b_5 > \alpha_d > \frac{2}{7}$, is the exceptional case which is already proved.

If $\frac{1}{2} \geq b_1 \geq \frac{1}{3} > b_2 \geq b_3 \geq b_4 \geq b_5 > \alpha_d > \frac{2}{7}$ then the center remains uncovered. By Claim 1 the only way the center could be covered if two of the paths are $< \frac{1}{3}$ are in one of the paired arms, the other two are in the other side and the path of size $\geq \frac{1}{3}$ covers the center. However this cannot occur since any 3 of the 4 paths of size $< \frac{1}{3}$ fits completely to one arm, thus we do not place a path of size $< \frac{1}{3}$ into a new arm until there are three of them in the first arm.

The last case is when there are at least 6 paths in the paired arms. However, it is easy to prove this, because

$$\sum_{i=1}^{t} W(b_i) \geq 6W\left(\frac{2}{7}\right) > \frac{5}{2p+1}.$$

Case 6. $\frac{1}{3} < \alpha_d \leq \frac{1}{2}$, *the center is not covered.*

As usual, the case when the center is not covered must be distinguished. If there are two arms where the empty space is more than $\frac{\alpha_d}{2}$ then a path of size α_d fits to this place. Hence it may be supposed that there is only one arm, A_1, where the sum of the sizes is $1 - \frac{\alpha_d}{2} - z$ such that $0 < z \leq \frac{\alpha_d}{2}$. If $z > \frac{\alpha_d}{2}$ then a path of size $> \alpha_d$ already fits to this arm. In this way the sum of the sizes in all other arms is $\geq 1 - \frac{\alpha_d}{2} + z$. It will be proved that if the sum of the sizes in an arm is $S > \frac{1}{2}$ then the sum of the weights is

$$\geq \frac{6}{5p+1}S + \frac{2p-2}{(2p+1)(5p+1)}.$$

If there is only one path and $S > \frac{2}{3}$ then

$$W(S) = \frac{(21p+6)}{(2p+1)(5p+1)}S - \frac{4p^2-3p-1}{p(2p+1)(5p+1)} > \frac{6}{5p+1}S + \frac{2p-2}{(2p+1)(5p+1)}$$

holds.

If there is only one path and $\frac{1}{2} < S < \frac{2}{3}$ then

$$W(S) = \frac{1}{p} > \frac{6}{5p+1}\frac{2}{3} + \frac{2p-2}{(2p+1)(5p+1)} \geq \frac{6}{5p+1}S + \frac{2p-2}{(2p+1)(5p+1)}.$$

If there are two paths then the claim is true, because

$$W(x) \geq \frac{6}{5p+1}x + \frac{p-1}{(2p+1)(5p+1)}$$

holds in the interval $\left[\frac{1}{3}, 1\right]$.

Therefore the proof is complete if $\alpha_d \leq \frac{2}{5}$, because

$$\sum_{i=1}^{t} W(b_i) \geq$$

$$\geq \left[\left(1 - \frac{\alpha_d}{2} + z\right)(p-1) + \left(1 - \frac{\alpha_d}{2} - z\right)\right]\frac{6}{5p+1} + p\frac{2p-2}{(2p+1)(5p+1)} >$$

$$> p\left(1 - \frac{\alpha_d}{2}\right)\frac{6}{5p+1} + p\frac{2p-2}{(2p+1)(5p+1)} > 1$$

holds in this case.

If $\frac{2}{5} < \alpha_d \le \frac{1}{2}$ then it can be shown that the sum of the sizes in each arm is at least $\frac{1}{p}$. If there is a path $> \frac{1}{2}$ in the arm then this is clearly true. If there is only one path but it is $\le \frac{1}{2}$ then the empty space with the center is larger than α_d which is a contradiction. If there are two paths then both must be larger than $\alpha_d > \frac{2}{5}$, thus

$$W(b_1) + W(b_2) > \frac{4}{5}\frac{6}{5p+1} + 2\frac{p-1}{(2p+1)(5p+1)} > \frac{1}{p}.$$

Therefore the weight sum is at least $\frac{1}{p}$ in each arm, thus the weight sum in the polyp is at least 1, proving our claim.

Case 7. $\frac{1}{3} < \alpha_d \le \frac{5}{12}$, *the center is covered.*

Like in the previous cases a lower bound will be proved for the paired arms and an other lower bound for the other arms. It will be proved that the weight sum is $> \frac{5}{2p+1}$ in the paired arms and $> \frac{2}{2p+1}$ in the other arms.

There are only two subcases in the latter case, so we begin with the single arm. If there is only one path in the arm, b_1, then $b_1 \ge 1 - \alpha_d \ge 1 - \frac{5}{12}$, otherwise $\alpha_{d+1} > \alpha_d$, so clearly $W(b_1) \ge \frac{1}{p} > \frac{2}{2p+1}$.

If there are two paths then both must be $> \frac{1}{3}$ assuring that the weight sum is $> \frac{2}{2p+1}$. Finally there is not enough place for three paths each $> \frac{1}{3}$.

Now the other claim is proved for the paired arms. If there is only one path in the paired arms then $\alpha_{d+1} > \alpha_d$ or the weight sum is ≥ 1. If there are two paths then it may be supposed that both are ≤ 1. However, it is easy to see that the center cannot be covered in this case.

We may suppose now that there are three paths. If all three are $> \frac{1}{2}$ then we obtain

$$\sum W(b_i) > \frac{3}{p} > \frac{5}{2p+1}.$$

If $b_1 \geq b_2 > \frac{1}{2} \geq b_3 > \frac{1}{3}$ then

$$\sum W(b_i) \geq 2\frac{1}{p} + \frac{1}{2p+1} > \frac{5}{2p+1}$$

holds.

The next case is $b_1 > \frac{1}{2} \geq b_2 \geq b_3 > \alpha_d > \frac{1}{3}$. In this case the center cannot be covered again by Claim 1.

There are no more cases with three paths in the paired arms, since if there are three paths each $\leq \frac{1}{2}$ then $\alpha_{d+1} > \alpha_d$.

There are two more cases when there are four paths but at most one is $> \frac{1}{2}$, $b_1 > \frac{1}{2} \geq b_2 \geq b_3 \geq b_4 > \alpha_d > \frac{1}{3}$. In this case

$$\sum_{i=1}^{4} W(b_i) \geq \frac{1}{p} + 3\frac{1}{2p+1} > \frac{5}{2p+1}$$

holds. If $\frac{1}{2} \geq b_1 \geq b_2 \geq b_3 \geq b_4 > \alpha_d > \frac{1}{3}$ then the center cannot be covered by Claim 1.

Therefore there is only one case left, when there are at least five paths in the paired arms. Now we obtain

$$\sum_{i=1}^{t} W(b_i) \geq 5\frac{1}{2p+1}$$

immediately completing the proof for this set of cases.

Case 8. $\frac{5}{12} < \alpha_d \leq \frac{1}{2}$, *the center is covered.*

It will be proved in this case that the lower bound for the weight sum is $\frac{2}{p}$ in the paired arms and $\frac{1}{p}$ in the single arms. If there is only one path in a single arm then its weight is at least $\frac{1}{p}$ since it is $\geq 1 - \alpha_d$ and $> \alpha_d$. If there are at least two paths then

$$W(b_1) + W(b_2) \geq 2W\left(\frac{5}{12}\right) = \frac{12p+3}{(2p+1)(5p+1)} > \frac{1}{p}$$

holds.

It is clear that two paths in the paired arms, each ≤ 1, cannot cover the center with the First Fit algorithm. If there are three paths but one of them is $> \frac{1}{2}$ then, similarly to the previous case,

$$W(b_1) + W(b_2) + W(b_3) \geq \frac{1}{p} + 2W\left(\frac{5}{12}\right) > \frac{2}{p}$$

is obtained. If all three paths are $\leq \frac{1}{2}$ then the center cannot be covered by Claim 1.

If there are at least four paths in the paired arms then

$$\sum_{i=1}^{t} W(b_i) \geq 4W\left(\frac{5}{12}\right) > \frac{2}{p}$$

holds.

Case 9. $\frac{1}{2} < \alpha_d \leq 1$.

Now $W(b_1) \geq \ldots \geq W(b_t) \geq \frac{1}{p}$ so if $t \geq p$ then $\sum_{i=1}^{t} W(b_i) \geq 1$ holds. Suppose that there are less than p paths. From the definition of the algorithm it is clear that in this way these paths must be in the ends of the arms, thus the center is not covered. But if there are less than p paths then one of the arms is empty, so a path of size > 1 fits in, thus $\alpha_{d+1} > \alpha_d$. This completes the proof of Lemma 2.

Lemma 3. *Let a polyp P_d of coarseness $\alpha_d \leq 1$ be filled with paths $b_1 \geq \geq b_2 \geq \ldots \geq b_t$ in the completed First Fit algorithm. If $\sum_{i=1}^{t} W(b_i) = 1 - \beta_d$ $(0 < \beta_d < 1)$ then either $\alpha_{d+1} \geq \min\left(\alpha_d + \frac{1}{3}\beta_d, 1\right)$ holds or there is an arm containing at most one path of size $\leq \frac{1}{2}$.*

Proof. The second alternative is satisfied unless there are at least two paths or one with size $> \frac{1}{2}$ in every arm. Suppose that $\alpha_{d+1} < \alpha_d + \frac{1}{3}\beta_d \leq 1$ when the list of paths L is packed to the polyps with the First Fit algoithm. A modified list L' will be created in the following way. Some of the paths in P_d will be enlarged. The proof is similar if $\alpha_d + \frac{1}{3}\beta_d > 1$.

Case 1. *The center is covered.*

Let A_1 and A_2 denote the paired arms and b_c the path which covers the center. The empty part is "together" in this pair of arms, because the First Fit algorithm was used to pack the paths. Let e denote the length of the empty part. This empty part does not contain the center, so it is contained by an arm completely, suppose that it is in A_1. A_1 contains a part of b_c and it must contain some paths which are as far from the center as possible, otherwise the First Fit algorithm could not cover the center. Let b_e be the closest path to the center among these paths. If $b_e > \frac{1}{2}$ then this path will be enlarged, let $b'_e = b_e + \min\left(\frac{1}{3}\beta_d, e\right)$. In this way the size of this empty part decreases by $\frac{1}{3}\beta_d$ or is becomes 0. If $b_e \leq \frac{1}{2}$ then other paths will be enlarged in the following way. Suppose that there is another path, b_f, in A_1 besides b_e. If $b_f > \frac{1}{2}$ then let $b'_f = b_f + \min\left(\frac{1}{3}\beta_d, e\right)$. If $b_f \leq \frac{1}{2}$ then let $b'_e = b_e + x$ and $b'_f = b_f + y$ such that $x + y = \min\left(\frac{1}{3}\beta_d, e\right)$, $x \leq \frac{1}{2} - b_e$ and $y \leq \frac{1}{2} - b_f$. It is easy to see that such x, y exist since $\min\left(\frac{1}{3}\beta_d, e\right) \leq 1 - b_e - b_f$.

There is one more possibility that $b_c \leq \frac{1}{2}$, A_1 contains a part of b_c but there is no other path in it along b_e. If $b_e > \frac{1}{2}$ then let $b'_e = b_e + \min\left(\frac{1}{3}\beta_d, e\right)$. If $b_c > \frac{1}{2}$ then $b'_c = b_c + \min\left(\frac{1}{3}\beta_d, e\right)$. If $b_c \leq \frac{1}{2}$ and $b_e \leq \frac{1}{2}$ then let b_f a path from A_2. If $b_f > \frac{1}{2}$ then let $b'_e = b_e + x,\ b'_c = b_c + y$ and $b'_f = b_f + z$ such that $x + y + z = \min\left(\frac{1}{3}\beta_d, e - \varepsilon\right),\ x \leq \frac{1}{2} - b_e,\ y \leq \frac{1}{2} - b_c$ and $z \leq 1 - b_f$. It is easy to see that such x, y, z exist since $\min\left(\frac{1}{3}\beta_d, e - \varepsilon\right) < 2 - b_e - b_c - b_f$. If $b_f \leq \frac{1}{2}$ then choose a further path, b_g, from A_2. It may be supposed that it is $\leq \frac{1}{2}$. Now enlarge b_e, b_c, b_f, b_g similarly as above.

In each case the empty part decrease by $\frac{1}{3}\beta_d$, e or $e - \varepsilon$ such that if a path $b_x \leq \frac{1}{2}$ then $b'_x \leq \frac{1}{2}$, if $b_x \leq 1$ then $b'_x \leq 1$ and the sum of the enlarged paths in A_2 is ≤ 1. The last condition ensures that if L' is packed with the First Fit algorithm then b'_c covers the center.

It is easy to see that such an enlargement can be done in the unpaired arms in the same way.

Case 2. *The center is not covered.*

The empty part in these arms are attached to the center and all paths in the arms are as far from the center as possible. (There must be at least one path in each arm.) Let $e_1 \geq e_2 \geq \ldots \geq e_k$ be the sizes of the empty parts in the unpaired arms, b_{e_i} the closest path to the center in arm A_i. By our assumption $a_d + \frac{1}{3}\beta_d > e_1 + e_2$. If $e_1 < \alpha_d + \frac{1}{6}\beta_d$ then let $b'_{e_i} = b_{e_i} + \min\left(\frac{1}{6}\beta_d, e_i\right)$ for all i. If $e_1 \geq \alpha_d + \frac{1}{6}\beta_d$ then $\frac{1}{6}\beta_d > e_2 \geq e_3 \geq \ldots$ holds since $e_1 + e_2 < \alpha_d + \frac{1}{3}\beta_d$. Let $b'_{e_1} = b_{e_1} + \frac{1}{3}\beta_d - e_2 + \varepsilon$ and $b'_{e_i} = b_{e_i} + e_i - \varepsilon$ for $i = 2, 3, \ldots, k$, where ε is as small as possible. If $b_{e_i} \leq \frac{1}{2}$ for some i then there must be at another path $b_{e'_i} \leq \frac{1}{2}$ in the same arm where b_i is. Then it is easy to see that it is possible to enlarge b_i and $b_{i'}$ together with the same size, such that $b'_i \leq \frac{1}{2}$ and $b'_{i'} \leq \frac{1}{2}$. This completes Case 2.

The modified list $L' = b'_1, b'_2, \ldots$ is obtained by replacing the above paths by the enlarged paths. If L' is packed with the First Fit algorithm then every paths will go the same arm of the same polyp in the same order as the corresponding path in L, because no path fits to an earlier place, but every path fits to its "old" place. On the other hand it is easy to see that after packing L' there will not be enough empty space to insert an additional path of size $> \alpha_d$.

The slope of W in the interval $\left[0, \frac{1}{2}\right]$ and $\left(\frac{1}{2}, 1\right]$ (we did not "jump" over $\frac{1}{2}$ during the enlargements) is less than $\frac{3}{p}$, the sum of the enlargements is $\leq (p-1)\frac{1}{3}\beta_d$, therefore

$$\sum W(b'_i) \leq \sum W(b_i) + (p-1)\frac{3}{p}\frac{1}{3}\beta_d = 1 - \frac{\beta_d}{p} < 1$$

is obtained. Thus L' would be a counterexample to Lemma 2 which is a contradiction.

Now we are prepared to complete the proof of Theorem 2.

Proof of Theorem 2. Let $Q_1, \ldots, Q_f$ denote the polyps for which $\sum_{i=1}^{t} W(b_j) = 1 - \beta_i$ with $\beta_i > 0$ in the First Fit algorithm in the same order as they appear in the list of the polyps. The other polyps have weight sum at least 1. Furthermore let γ_i be the coarseness of Q_i. $\gamma_i \leq 1$ holds for $i = 1, \ldots, f$, since Q_i cannot contain a path of size > 1.

By Lemma 3 and the definition of coarseness,

$$\gamma_{i+1} > \gamma_i + \frac{1}{3}\beta_i \quad \text{for} \quad i = 1, \ldots, f-1,$$

except when Q_i has an arm containing at most one single path of size $\leq \frac{1}{2}$. Suppose Q_j has an arm containing one path of size $\leq \frac{1}{2}$, then a path of size $\geq \frac{1}{2}$ would fit to Q_j, thus all the paths in the larger indexed polyp must be $> \frac{1}{2}$. Thus for $i > j$ Q_i cannot have an arm which contains one path of size $\leq \frac{1}{2}$. However, it is possible that Q_k for some $k > j$ has an empty arm. Then all the larger indexed polyps must have paths only of size > 1 which implies that their weight sum is not less than 1. Therefore if such k exists then $k = f$. So

$$\beta_i < 3(\gamma_{i+1} - \gamma_i)$$

holds for $i = 1, \ldots, j-1,\ j+1, \ldots, f-1$. Obviously $\beta_k, \beta_f \leq 1$ and it was also shown that $\gamma_1 < \ldots < \gamma_f \leq 1$, thus

$$\sum_{i=1}^{f} \beta_i < 3\sum_{i=1}^{j-1}(\gamma_{i+1} - \gamma_i) + 3\sum_{i=j+1}^{f-1}(\gamma_{i+1} - \gamma_i) + \beta_j + \beta_f < 3\sum_{i=1}^{f-1}(\gamma_{i+1} - \gamma_i) + 2 =$$

$$= 3(\gamma_f - \gamma_1) + 2 \leq 5.$$

Summing the weights for all paths in L

$$FF_p(L) - 5 \leq \sum W(l_i)$$

is obtained. Finally, applying Lemma 1 we conclude

$$FF_p(L) \leq \sum_{i=1}^{m} W(l_i) + 5 \leq OPT_p(L)\left[\frac{p-1}{5p+1} + \frac{p-1}{2p+1} + \frac{p-1}{p} + 1\right] + 5$$

completing the proof.

Conjecture. *If the arm size of the polyp is suitably large then there exists a list L for any enough $OPT_p(L)$ such that for*

$$FF_p(L) \geq OPT_p(L)\left[\frac{p-1}{5p+1}+\frac{p-1}{2p+1}+\frac{p-1}{p+1}+1\right]-const.$$

Note that this ratio is very close to the bound in Theorem 2. The difference is only $\dfrac{1}{p(p+1)}$. For some values of p this would give the following lower bounds for the ratio (the upper bounds are given by Theorem 2):

$$\begin{aligned} 2.076 &< R_4(< 2.227),\\ 2.184 &< R_5(< 2.318),\\ 2.260 &< R_6(< 2.380),\\ 2.522 &< R_{15}(< 2.570),\\ 2.7 &= \lim_{i\to\infty} R_i. \end{aligned}$$

The following method may give a proof for this conjecture. However, the proof may be very long and complicated.

As it was mentioned before the arm size of the polyp will be 1 and the sizes of the rational numbers. All sizes are multiplied by a suitable integer to get integer sizes after having the complete list.

The elements of L will belong to four regions. In the first region all the elements have size near $\frac{1}{6}$, in the second one all the sizes are near $\frac{1}{3}$, in the third one each length is $\frac{1}{2}+\varepsilon$ and finally in the fourth region all the paths have size $1+\varepsilon$. In the optimal packing every polyp should contain one element with size 1 and every other arm should contain one element from each other region. In the list they will be ordered so that all the paths from the first region in a given order are first, then paths from region two and so on. There may also be a constant number of exceptional paths packed to a constant number of polyps.

The exact sizes should be determined so that in the First Fit algorithm $5p+1$ paths with size about $\frac{1}{6}$ fit into the first polyp, five to each arm and one through the center. The rest of the paths with size about $\frac{1}{6}$ will fit into the next few polyps the same way. The paths with sizes near $\frac{1}{3}$ and $\frac{1}{2}$ will fit into the next number of polyps in a similar way, $2p+1$ and $p+1$ to each one, respectively. Finally only one path with size $1+\varepsilon$ fits to each of the remaining polyps.

Perhaps one may prove that it is possible to determine the exact sizes in such a way. Moreover there may be an algorithm which gives these sizes. I suppose Johnson et al. [2] probably used such an algorithm, but since there were only 12 different sizes, it was enough to give the sizes created by the algorithm in their paper.

A somewhat weaker bound will be proved now.

Theorem 3. *If the arm size of the polyp is suitably large then there exists a list L for any large enough M such that $OPT_p(L) \geq M$* and

$$FF_p(L) \geq OPT_p(L)\left[\frac{p-1}{42p} + \frac{p-1}{6p+1} + \frac{p-1}{2p+1} + \frac{p-1}{p+1} + 1\right].$$

Note that this ratio is still very close to the bound in Theorem 2. For some values of p this gives the following lower bounds for the ratio (the upper bounds are given by Theorem 2):

$$\begin{aligned} 2.071 &< R_4(< 2.227), \\ 2.178 &< R_5(< 2.318), \\ 2.253 &< R_6(< 2.380), \\ 2.502 &< R_{15}(< 2.570), \\ 2.69047 &< \lim_{i\to\infty}(\leq 2.7). \end{aligned}$$

Unfortunately the asymptotic result still has a 0.01 gap.

Proof. Let N be a positive integer divisible by $42p$, $(6p+1)$, $(2p+1)$ and $(p+1)$ and let ε denote $\dfrac{1}{s}$ where s, the length of the arms of the polyps, is chosen suitably large. The list of paths denoted by L consists of the following regions of paths in the order of appearance

a) $N(p-1)$ paths of size $\dfrac{1}{42} - 3\varepsilon$,

b) $N(p-1)$ paths of size $\dfrac{1}{7} + \varepsilon$,

c) $N(p-1)$ paths of size $\dfrac{1}{3} + \varepsilon$,

d) $N(p-1)$ paths of size $\dfrac{1}{2} + \varepsilon$,

e) N paths of size $1+\varepsilon$.

If these paths are packed with the First Fit algorithm then each arm of the first $\dfrac{N(p-1)}{42p}$ polyp will contain 42 paths of region a). The coarseness of the next polyp will be $(2\cdot 42\cdot 3+1)\cdot\varepsilon$. We choose s large enough, and thus ε small enough, to exclude that one of the later coming paths could fit in. (Note that sum of the sizes of two arms is $2+\varepsilon$). Namely $253\varepsilon < \dfrac{1}{7}$ must be true. In this way all the paths in region b) will go to later polyps. Exactly 6 will fit to the first arm, then 6 to the second one etc., and finally 1 through the center. This makes $6p+1$ in one polyp. Thus region b) will fill up $\dfrac{N(p-1)}{6p+1}$ polyps. The coarseness of the next polyp will be $\dfrac{1}{7}-5\varepsilon$, so no later paths will fit into these polyps. In the same way region c) will fill up $\dfrac{N(p-1)}{2p+1}$ polyps and region d) $\dfrac{N(p-1)}{p+1}$ polyps. Finally only one path of size $1+\varepsilon$ may fit into a polyp. Thus we have additional N polyps filled. In this way

$$N\left[\frac{p-1}{42p}+\frac{p-1}{6p+1}+\frac{p-1}{2p+1}+\frac{p-1}{p+1}+1\right]$$

polyps are needed to pack these paths.

If an optimal packing of list L is considered, it is possible to pack the list into N polyps. Put one path from region e) into the first arm such that it completely covers the first arm and the center. Then put one path from each other region to each arm. In this way one polyp is completely filled. Follow the same procedure until there are some paths left. It is easy to see that N polyps are needed. This packing is optimal since the sum of the sizes of the paths is equal to the sum of the sizes of the polyps.

3. Conclusions

We have seen that although it looks very hard to find an optimal packing, the very easily applicable First Fit algorithm works quite well. So if we can model some practical problem by the Polyp-packing one might use the First Fit algorithm.

The vertex disjoint p-polyp packing may be considered as a variant of the Bin Packing problem. Now we have groups of bins each group containing p bins. It is possible to break at most one item into two pieces and place the two parts into two different bins of the same group, but it is allowed to do this at most once in each group. In [2] the reader may find some problems which are modeled by the Bin Packing problem. With this modification we may model some modified problems. For example file allocation. It is desired to place files of varying sizes on as few disks as possible, where files may not be broken between tracks except one file per disk.

There are some open problems left. First of all our results are not sharp so they may be improved.

There are on-line algorithms for the Bin Packing problem which give packing closer to the optimal. Probably such an algorithm would give a better result here also.

Further investigation of this problem may include other graphs not just polyps. However, there should be some restrictions. If we want to pack arbitrary paths into arbitrary graphs then, with any algorithm, when we want to pack a particular path, it must be decided that this path fits into the graph or not. But this problem is NP-complete in general, since it contains the hamiltonian path problem as a special case. One possible restriction is that the sizes of the paths have a constant upper bound. For example if all paths are at most 100 long, this problem does not arise.

Most of the known variants of the Bin Packing problem can be formulated as a special case of the general graph packing problem by choosing a suitable graph class for "bins" and an other class for "items". For example if the "bins" and the "items" are graphs of rectangular grids we obtain the 2-dimensional parallel rectangle packing.

References

[1] **Fernandez de la Vega W. and Lueker G.S.,** Bin packing can be solved within $1+\varepsilon$ in linear time, *Combinatorica,* **1** (1981), 349-355.

[2] **Johnson D.S., Demers A., Ullman J.D., Garey M.R. and Graham R.L.,** Worst case bounds for simple one-dimensional packing algorithms, *SIAM J. Comptg.,* **3** (1974), 299-325.

[3] **Katona G.O.H.,** personal communication

[4] **Katona G.Y.,** Edge-disjoint polyp packing, *Discrete Applied Mathematics,* **78** (1997), 133-152.

(Received September 27, 2001)

G.Y. Katona
Mathematical Institute
Hungarian Academy of Sciences
H-1364 Budapest, P.O.B. 127
Hungary
kiskat@cs.elte.hu