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THE GROUP ANALYSIS OF STOCHASTIC DIFFERENTIAL EQUATIONS

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Abstract. In this work the criterion of invariance of Ito stochastic differential equation with respect to a one-parametric group of transformations is proved. This result allows us to find out possibility of constructing an ergodic process from the initial one by using the transformations of time and phase variables and demonstrates how it can be done.

On a complete probability space $(\Omega, \mathbf{F}, \mathbf{P})$, we consider the stochastic differential equation in R^1

$$du(t) = A\left(t, u(t)\right) dt + B\left(t, u(t)\right) dw(t), \tag{1}$$

$$t \in [0; T), \quad u(0) = u_0.$$

Here $w(t)$ is a standard Wiener process with respect to the filtration $\{\mathbf{F}_t\}_{t=0}^{+\infty}$, $\mathbf{F}_t \subseteq \mathbf{F}$ and u_0 is $\mathbf{F}_0$ - measurable random variable with $M\left|u_0\right|^2 < +\infty$. The functions $A(t, u)$ and $B(t, u)$ are non-random and have one derivative on the variables t and u. Suppose that the functions $A(t, u)$ and $B(t, u)$ satisfy conditions which guarantee the existence and uniqueness of the strong solution of equation (1). For example, suppose, that the conditions of Theorem 3.1 [1, p.109] are satisfied. Transform the variables t and $u:\ t = f(s, a),\ \ u = g(s, v, a)$, where $a \in R^1$. Let us demand that the following conditions are satisfied for the functions f and g:

A1) $f(s, 0) = s, \quad g(s, v, 0) = v$;

A2) $\exists f_s(s, a)$ and $f_s(s, a) \geq 0$ for every a;

A3) $\exists g^{(-1)}(s, u, a)$ such that $g^{(-1)}(s, g(s, v, a), a) = v$, $\forall s, v, a$;

A4) $f(f(s, a), b) = f(s, a + b), \quad g(f(s, a), g(s, v, a), b) = g(s, v, a + b)$, $\forall s, v, a$;

A5) $\exists f_{sa}, g_{sa}, g_{vva}$.

The alphabetic subindex means a partial derivative with respect to the appropriate variable.

Definition 1. The equation (1) is invariant with respect to the group of transformations $t = f(s,a), \quad u = g(s,v,a)$ if there exists the Wiener process $\tilde{w}(s)$ such that the process $v(s) = g^{-1}(s, u(f(s,a)), a)$ is a solution of equation

$$dv(s) = A\left(s, v(s)\right) ds + B\left(s, v(s)\right) d\tilde{w}(s), \tag{2}$$

$$s \in \left[s_0; s_T\right), \quad v\left(s_0\right) = g^{(-1)}\left(s_0, u_0, a\right).$$

Here $s_o = f^{(-1)}(0,a)$, $s_T = f^{(-1)}(T,a)$, $f^{(-1)}(t,a)$ is an inverse function to $f(s,a)$.

According to [2, p.25], we denote $(\xi(s), \eta(s,v))$ as a tangent vector of a considered group, i.e.

$$\xi(s) = f_a(s,a)|_{a=0}, \quad \eta(s,v) = g_a(s,v,a)|_{a=0}.$$

Theorem 1. *Equation (1) is invariant with respect to the group of transformations with the tangent vector $(\xi(s), \eta(s,v))$ if and only if*

$$\begin{cases} -\eta_s(s,v) + A_s(s,v)\xi(s) + A_v(s,v)\eta(s,v) - \frac{1}{2}\eta_{vv}(s,v)B^2(s,v) + \\ +A(s,v)\left(\xi_s(s) - \eta_v(s,v)\right) = 0, \\ \\ B_s(s,v)\xi(s) + B_v(s,v)\eta(s,v) + B(s,v)\left(\frac{1}{2}\xi_s(s) - \eta_v(s,v)\right) = 0, \end{cases} \tag{3}$$

$\forall s, v$.

Proof. Let equation (1) be invariant with respect to the given group. We shall prove, that the coefficients A and B satisfy the system (3). Transform the time variable $t = f(s,a)$ and denote $\hat{u}(s) = u(f(s,a))$. Then

$$d\hat{u}(s) = A\left(f(s,a), \hat{u}(s)\right) f_s(s,a) ds + B\left(f(s,a), \hat{u}(s)\right) \sqrt{f_s(s,a)} d\tilde{w}(s),$$

$$\hat{u}\left(s_0\right) = u_0.$$

Here $\tilde{w}(s)$ is a Wiener process such that $w(f(s,a)) = \int\limits_{s_0}^{s} \sqrt{f_h(h,a)} d\tilde{w}(h)$.

Transform the phase variable

$$v(s) = g^{(-1)}\left(s, \hat{u}(s), a\right) = g^{(-1)}\left(s, u(f(s,a)), a\right).$$

By the Ito formula we obtain

(4)
$$dv(s) = \Big[g_s^{(-1)}(s, g(s, v(s), a), a) +$$
$$+ g_u^{(-1)}(s, g(s, v(s), a), a) A(f(s, a), g(s, v(s), a)) f_s(s, a) +$$
$$+ \frac{1}{2} g_{uu}^{(-1)}(s, g(s, v(s), a), a) B^2(f(s, a), g(s, v(s), a)) f_s(s, a)\Big] ds +$$
$$g_u^{(-1)}(s, g(s, v(s), a), a) B(f(s, a), g(s, v(s), a)) \sqrt{f_s(s, a)} d\tilde{w}(s).$$

Substituting

$$g_u^{(-1)}(s, g, a) = \frac{1}{g_v(s, v, a)}, \qquad g_s^{(-1)}(s, g, a) = -\frac{g_s(s, v, a)}{g_v(s, v, a)},$$
$$g_{uu}^{(-1)}(s, g, a) = -\frac{g_{vv}(s, v, a)}{(g_v(s, v, a))^3}$$

into (4) we obtain

$$dv(s) = \Big[-\frac{g_s(s, v, a)}{g_v(s, v, a)} + \frac{1}{g_v(s, v, a)} A(f(s, a), g(s, v(s), a)) f_s(s, a) -$$
$$-\frac{1}{2} \frac{g_{vv}(s, v, a)}{(g_v(s, v, a))^3} B^2(f(s, a), g(s, v(s), a)) f_s(s, a)\Big] ds +$$
$$+ \frac{1}{g_v(s, v, a)} B(f(s, a), g(s, v(s), a)) \sqrt{f_s(s, a)} d\tilde{w}(s),$$
$$v(s_0) = u_0.$$

The invariance of equation (1) with respect to the considered group of transformations is equivalent to the equalities

(5)
$$\begin{cases} -\dfrac{g_s}{g_v} + \left[A(f, g) - \dfrac{g_{vv}}{2(g_v)^2} B^2(f, g)\right] \dfrac{f_s}{g_v} = A(s, v), \\ \dfrac{\sqrt{f_s}}{g_v} B(f, g) = B(s, v), \end{cases}$$

$$\forall s, v, a.$$

The condition (5) is equivalent to following equalities

$$(6) \qquad \begin{cases} \dfrac{d}{da} H\left(s, v, a\right) = 0, \\ \frac{d}{da} G\left(s, v, a\right) = 0, \end{cases}$$

where

$$H = -\frac{g_s}{g_v} + \left[A(f,g) - \frac{g_{vv}}{2\left(g_v\right)^2} B^2(f,g)\right] \frac{f_s}{g_v},$$

$$G = \frac{\sqrt{f_s}}{g_v} B(f,g).$$

Consider the second equality of system (6). According to the Lie equations [2, p.29]

$$\frac{df}{da} = \xi\left(f\right), \quad \frac{dg}{da} = \eta(f,g).$$

Then

$$\frac{df_s}{da} = \xi_f f_s, \quad \frac{dg_s}{da} = \eta_f f_s + \eta_g g_s,$$

$$(7) \qquad \frac{dg_v}{da} = \eta_g g_v, \quad \frac{dg_{vv}}{da} = \eta_{gg}\left(g_v\right)^2 + \eta_g g_{vv}.$$

Substituting (7) in the second equality of system (6) we obtain

$$\frac{dG}{da} = \frac{\sqrt{f_s}}{g_v}\left[\xi(f)B_f + \eta(f,g)B_g + \left(\frac{1}{2}\xi_f(f) - \eta_g(f,g)\right)B\right] = 0.$$

Since this equality should be satisfied for all a, hence we obtain the second equality of system (3).

Let us do for the first equation of system (6) the calculations similar to what were done for the second equation. Using (7) we obtain

$$\frac{dH}{da} = \frac{f_s}{g_v}\left\{A_f\xi + A_g\eta + A\left(\xi_f - \eta_g\right) - \eta_f - \frac{1}{2}B^2\eta_{gg} - \right.$$

$$\left. - \frac{g_{vv}}{\left(g_v\right)^2} B\left[B_f\xi + B_g\eta + \left(\frac{1}{2}\xi_f - \eta_g\right)B\right]\right\} = 0.$$

The expression in square brackets is equal to zero, as it was proved earlier (this is the second equality of system (3)). The remaining part in braces coincides with the left part of the first equality of system (3).

Now we shall prove the inverse assertion. Let us prove, that the second equality of system (3) implies the equality $G(s,v,a) = B(s,v) \quad \forall s,v,a$. Compute

$$\frac{dG(s,v,a)}{da} = \frac{\sqrt{f_s(s,a)}}{g_v(s,v,a)} \times$$

$$\times \left[\xi(t)B_t + \eta(t,u)B_u + \left(\frac{1}{2}\xi_t(t) - \eta_u(t,u) \right) B \right].$$

By the system (3) and condition A1 the function G is a solution of the problem

$$\frac{dG(s,v,a)}{da} = 0, \qquad G(s,v,0) = B(s,v).$$

This means that $G(s,v,a) = B(s,v) \quad \forall s,v,a$. Let us do for the first equation of system (3) the calculations similar to what were done for the second one. Then $H(s,v,a) = A(s,v) \quad \forall s,v,a$. Thus, Theorem 1 is proved.

Theorem 1 allows us to solve the following tasks:

Task 1. *The coefficients of equation (1) are given. It is required to find all one-parametric groups of transformations satisfying to the conditions A1 - A5, with respect to which equation (1) is invariant.*

Task 2. *The group of transformations is given by the tangent vector. It is required to find a general form of equation (1), which is invariant with respect to the given group.*

Let us demonstrate the obtained results by some examples.

Example 1. *We shall construct all one-parametric groups of transformations with respect to which the equation*

$$du(t) = Au^{2\gamma-1}(t)dt + Bu^{\gamma}(t)dw(t), \tag{8}$$

$$u(0) = u_0 > 0$$

is invariant.

If $\gamma > 1, \quad A \leq -\frac{\gamma^2 B^2}{2(2\gamma-1)}$, then the conditions of Theorem 3.1 [1, p.109] are satisfied and there exists a unique solution of equation (8).

Lemma. *If* $\gamma > 1$, $A \leq -\frac{\gamma^2 B^2}{2(2\gamma-1)}$, $u_0 > 0$, *then the solution of equation (8) is non-negative for all* t *with probability 1.*

Proof. Use Pardoux's method [3, p.152]. Consider the equation

$$du(t) = A|u(t)|^{2(\gamma-1)}u(t)dt + B|u(t)|^{\gamma-1}u_+(t)dw(t), \tag{9}$$

$$u(0) = u_0 > 0.$$

Let $\gamma > 1, A \leq -\frac{\gamma^2 B^2}{2(2\gamma-1)}$, then the conditions of Theorem 3.1 [1, p.109] are satisfied and there exists the unique solution of equation (9). Denote $u_+ = \sup\{0, u\}, u_- = \sup\{0, -u\}, \quad \phi(r) = |r_-|^2$,

$$\phi_\epsilon(r) = \begin{cases} r^2 - \dfrac{\epsilon^2}{6}, & \text{if } r \leq -\epsilon, \\ -\dfrac{r^4}{2\epsilon^2} - \dfrac{4r^3}{3\epsilon}, & \text{if } -\epsilon < r \leq 0, \\ 0, & \text{if } r > 0. \end{cases}$$

By Ito's formula

$$\phi_\epsilon(u(t)) = A\int_0^t |u(s)|^{2(\gamma-1)}u(s)\phi'_\epsilon(u(s))ds + \tag{10}$$

$$\frac{1}{2}B^2\int_0^t |u(s)|^{2(\gamma-1)}u_+^2(s)\phi''_\epsilon(u(s))ds + B\int_0^t |u(s)|^{\gamma-1}u_+(s)\phi'_\epsilon(u(s))dw(s).$$

For each s we have either $u_+(s) = 0$ or $\phi'_\epsilon(u(s)) = 0$, also either $u_+(s) = 0$ or $\phi''_\epsilon(u(s)) = 0$. Then last two terms in (10) are equal to zero. As $\forall u \in R^1$

$$\lim_{\epsilon \to 0} u\phi'_\epsilon(u) = 2\phi(u),$$

then from (10) under $\epsilon \to 0$ we obtain

$$\phi(u(t)) = 2A\int_0^t |u(s)|^{2(\gamma-1)}\phi(u(s))ds.$$

From here it follows, that $\phi(u(t)) = 0$, i.e. $u(t) \geq 0$ for all t with probability 1. If $u(t) \geq 0$, then equations (9) and (8) coincide and because of the uniqueness of their solutions we obtain the assertion of lemma.

From Lemma it follows, that R_+ is a support of coefficients of equation (8). For equation (8), system (3) is equivalent to

$$\begin{cases} -\eta_t + A(2\gamma-1)u^{2(\gamma-1)}\eta - \frac{1}{2}B^2\eta_{uu}u^{2\gamma} + A\left(\xi_t - \eta_u\right)u^{2\gamma-1} = 0, \\ B\gamma\eta u^{\gamma-1} + B\left(\frac{1}{2}\xi_t - \eta_u\right)u^{\gamma} = 0. \end{cases} \tag{11}$$

From the second equation we obtain $\eta_u - \frac{\gamma}{u}\eta = \frac{1}{2}\xi_t$. As $\xi(t)$ does not depend on u, $\eta(t,u) = \frac{\xi_t(t)}{2(1-\gamma)}u + \eta^{(1)}(t)u^{\gamma}$. Substituting this equality in the first equation of system (11) we obtain

$$-\frac{\xi_{tt}(t)}{2(1-\gamma)}u - \eta_t^{(1)}(t)u^{\gamma} + (\gamma-1)\left(A - \frac{\gamma B^2}{2}\right)\eta^{(1)}(t)u^{3\gamma-2} = 0.$$

As $\gamma > 1, A \leq -\frac{\gamma^2 B^2}{2(2\gamma-1)}$, the last equality is possible only if $\xi(t) = C_1 t + C_2$ and $\eta^{(1)}(t) = 0$. Finally, $\xi(t) = C_1 t + C_2$ and $\eta(t,u) = \frac{C_1}{2(1-\gamma)}u$. Under the conditions $C_1 = 1, C_2 = 0$ we obtain the subgroup of non-uniform strains with the tangent vector $\left(t, \frac{u}{2(1-\gamma)}\right)$. Under the conditions $C_1 = 0, C_2 = 1$ we obtain the subgroup of transfers along the axis Ot with the tangent vector $(1;0)$.

The invariance of equation (8) with respect to the group of transfers along the axis Ot signifies, that equation (8) has an invariant measure [4, p.176]. However, this measure is not a probability measure as

$$\int\limits_0^{+\infty} x^{-2\gamma}\exp\left\{\frac{2A}{B^2}\int\limits_{u_0}^{x}\frac{du}{u}\right\}dx = +\infty$$

and the necessary and sufficient condition of existence of stationary distribution is not satisfied [5, p.337-338]. Thus, the solution of equation (8) is not an ergodic process.

However, the invariance of equation (8) with respect to the group of strains allows us to calculate a multiplier, after multiplication by which the solution $u(t)$ is transformed to an ergodic process. Find the invariant J of group from the equation

$$\frac{dt}{C_1 t + C_2} = 2(1-\gamma)\frac{du}{C_1 u}.$$

Then $J = \left(t + \frac{C_2}{C_1}\right)^{\frac{1}{2(\gamma-1)}} u$. Consider the process $v(t) = \left(t + \frac{C_2}{C_1}\right)^{\frac{1}{2(\gamma-1)}} u(t)$, satisfying the initial condition $v(0) = u_0$. Then $v(t) = (t+1)^{\frac{1}{2(\gamma-1)}} u(t)$. The process $v(t)$ satisfies to the equation

$$v(t) = u_0 + \int\limits_0^t \left[\frac{v(s)}{2(\gamma-1)} + Av^{2\gamma-1}(s)\right] \frac{ds}{1+s} + B \int\limits_0^t \frac{v^\gamma(s)}{\sqrt{1+s}} dw(s).$$

By replacing $r = \ln(1+t)$ and denoting $V(r) = v\left(e^r - 1\right)$, we obtain

$$V(r) = u_0 + \int\limits_0^r \left[\frac{V(h)}{2(\gamma-1)} + AV^{2\gamma-1}(h)\right] dh + B \int\limits_0^r V^\gamma(h) d\tilde{w}(h), \tag{12}$$

where $\tilde{w}(h)$ is a Wiener process such that $w\left(e^r - 1\right) = \int\limits_0^r e^{\frac{h}{2}} d\tilde{w}(h)$.

If $2A < B^2(2\gamma - 1)$, then for equation (12) the conditions of criterion of existence of stationary distribution are satisfied [5, p.337-338] and $V(r)$ is an ergodic process. The density of stationary distribution looks like

$$g(x) = Cx^{2\left(\frac{A}{B^2} - \gamma\right)} \exp\left\{-\frac{x^{2(1-\gamma)}}{2B^2(1-\gamma)^2}\right\}, \quad x > 0,$$

where C is a normalizing constant.

As $V(r) = v\left(e^r - 1\right)$, so $V(\ln(1+t)) = v(t) = (1+t)^{\frac{1}{2(\gamma-1)}} u(t)$. Thus, multiplying the solution of equation (8) by $(1+t)^{\frac{1}{2(\gamma-1)}}$ we obtain an ergodic process with distribution $g(x)$.

Example 2. *Let us find the general form of equation (1), which is invariant with respect to the group of non-uniform strains with the tangent vector $\xi(t) = t$, $\eta(t,u) = ku$, where $k \in R^1$.*

Theorem 2. *Equation (1) is invariant with respect to the group of non-uniform strains with the tangent vector (t, ku) if and only if its coefficients look like*

$$A(t,u) = t^{k-1} A\left(t^{-k} u\right), \quad B(t,u) = t^{k-\frac{1}{2}} B\left(t^{-k} u\right).$$

Proof. According to Theorem 1 the coefficients $A(t,u)$ and $B(t,u)$ should satisfy to system (3), which for the group of strains looks like

$$\begin{cases} tA_t(t,u) + kuA_u(t,u) + (1-k)A(t,u) = 0, \\ \\ tB_t(t,u) + kuB_u(t,u) + \left(\frac{1}{2} - k\right) B(t,u) = 0. \end{cases} \tag{13}$$

The first integrals of the first equation are $J_1 = t^{-k}u$, $J_2 = t^{1-k}A$. According to [2, p.40], they are the invariants of group of non-uniform strains with the tangent vector $(t, ku, (k-1)A)$. The infinitesimal operator of this group looks so

$$X = t\frac{\partial}{\partial t} + ku\frac{\partial}{\partial u} + (k-1)A\frac{\partial}{\partial A},$$

and its first continuation

$$X_1 = X + (k-2)A_t\frac{\partial}{\partial A_t} - A_u\frac{\partial}{\partial A_u}.$$

Compute the value of operator X_1 on the manifold $D = tA_t + kuA_u + (1-k)A = 0$.

$$X_1(D)|_{D=0} =$$

$$= \left(tA_t + k^2uA_u - (k-1)^2A + (k-2)tA_t - kuA_u\right)|_{(k-1)A=tA_t+kuA_u} \equiv 0.$$

According to [2, p.65], the first equation of system (13) is invariant with respect to the group with the tangent vector $(t, ku, (k-1)A)$. Then, according to [2, p.38], there exists the function $A(x)$ such that $A(t,u) = t^{k-1}A\left(t^{-k}u\right)$ and this equality sets a general form of coefficient $A(t,u)$ in equation (1).

The reasonings, similar above mentioned, give a general form of coefficient $B(t,u) = t^{k-\frac{1}{2}}B\left(t^{-k}u\right)$. Theorem 2 is proved.

Theorem 2 can be used for the investigation of limiting behaviour of solution of equation (1). Consider the equation

$$du(t) = \left(1+\frac{t}{T}\right)^{k-1} A\left(\left(1+\frac{t}{T}\right)^{-k} u\right)dt+$$

$$+\left(1+\frac{t}{T}\right)^{k-\frac{1}{2}} B\left(\left(1+\frac{t}{T}\right)^{-k} u\right)dw(t), \tag{14}$$

$$t \in [0;+\infty), \quad u(0) = u_0, T > 0, \quad k \in R^1.$$

The functions $A(x)$ and $B(x)$ should satisfy to the following conditions for all $x, y \in R^1$

$$2xA(x) + B^2(x) \leq 0, \tag{15}$$

$$2(x-y)\left(A(x) - A(y)\right) + \left(B(x) - B(y)\right)^2 \leq 0. \tag{16}$$

Then the conditions of Theorem 3.1 [1, p.109] are satisfied and there exists the unique solution of equation (14). Theorem 2 allows us to calculate a multiplier $h(t)$ such that the process $h(t)u(t)$ is ergodic and its stationary distribution is easily calculated. After the replacement $s = 1 + \frac{t}{T}$ in equation (14) we get the equation, which is invariant with respect to the group of non-uniform strains. As it is mentioned above, $J = t^{-k}u$ is the invariant of this group. Define the process $v(t)$ by the equality $v(t) = \left(1 + \frac{t}{T}\right)^{-k} u(t)$. Then

$$dv(t) = \left[-\frac{k}{T} v(t) + A(v(t)) \right] \frac{dt}{\left(1 + \frac{t}{T}\right)} + \frac{B(v(t))}{\sqrt{1 + \frac{t}{T}}} dw(t),$$

$$v(0) = u_0.$$

Transform the variable $r = \ln\left(1 + \frac{t}{T}\right)$ and denote $V(r) = v\left(T\left(e^r - 1\right)\right)$. Then we obtain

$$dV(r) = \left[-kV(r) + TA(V(r))\right] dr + \sqrt{T} B(V(r)) d\tilde{w}(r),$$

$$r \in [0, +\infty)\,, \quad V(0) = u_0.$$

Note, that with the help of last replacement of variables the initial equation is transformed to a form which is invariant with respect to the transfers along the axis Or. According to [5, p.338], $V(r)$ is an ergodic process if and only if

$$\int\limits_{x_1}^{x_2} \frac{1}{B^2(x)} \exp\left\{ \frac{2}{T} \int\limits_{u_0}^{x} \frac{-ky + TA(y)}{B^2(y)} dy \right\} dx < +\infty, \tag{17}$$

where (x_1, x_2) is a support of function $B(x)$. Thus, we proved the following theorem.

Theorem 3. *If the coefficients of equation (14) satisfy the conditions (15) - (17), then the process* $V\left(\ln\left(1 + \frac{t}{T}\right)\right) = \left(1 + \frac{t}{T}\right)^{-k} u(t)$ *is an ergodic process and its stationary distribution has the density*

$$g(x) = \frac{C}{B^2(x)} \exp\left\{ \frac{2}{T} \int\limits_{u_0}^{x} \frac{-ky + TA(y)}{B^2(y)} dy \right\}, \quad x \in (x_1, x_2)\,,$$

where C is a normalizing constant.

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