Annales Univ. Sci. Budapest., Sect. Comp. 21 (2004) 3-18

ON THE CONVERGENCE OF STEFFENSEN–GALERKIN METHODS

I.K. Argyros (Lawton, OK, USA)

Abstract. In this study we prove the asymptotic mesh-independence principle for Steffensen-Galerkin methods. This principle asserts that when Steffensen's method is applied to a nonlinear equation between some Banach spaces, as well as to some finite-dimensional discretization of that equation, then the behavior of the discretized process is the same as that for the original iteration. Local and semilocal convergence results as well as an error analysis for Steffensen's method are also provided.

I. Introduction

In this study we are concerned with the problem of approximating a locally unique solution x^* of the equation

$$f(x) = 0, \tag{1}$$

where f is a nonlinear operator defined on an open convex subset D of a Banach space E with values in E.

Steffensen's method given by

$$x_{n+1} = x_n - [x_n, g(x_n)]^{-1} f(x_n) \quad (n \geq 0) \tag{2}$$

has been used [5], [6], [7] generates a sequence which converges quadratically to x^*. Here $g : D \to E$ is a continuous operator. $[x, y]$ denotes a divided difference of order one of f on D, satisfying

$$[x, y](y - x) = f(y) - f(x) \quad \text{for all } x, y \in D \text{ with } x \neq y \tag{3}$$

and

$$[x,x] = f'(x) \quad (x \in D) \tag{4}$$

if f is Fréchlet-differentiable on D.

Several authors have used various conditions to show convergence [5], [7], [8]. In the first part of this study we use a new affine invariant Mysovskii-type hypotheses [6] to provide semilocal-local convergence results as well as an error analysis for Steffensen's method given by (2). In the second part we consider discretized versions of (1) and (2) and try to relate the solution x^* with the solutions obtained through the discretized equations. This is important because in infinite dimensional spaces it is very difficult or even impossible to compute iterates given by (2). This leads to an asymptotic mesh-independence principle for Steffensen's method. Mesh independence of Steffensen's method means that Steffensen's method applied to a family of finite-dimensional discretizations of an operator equation behaves essentially the same for all sufficiently fine discretizations.

We also show that for special choices of the operator g our results reduce to the ones obtained in [6] for Newton's method. Another choice of g leads to the secant method. Many other choices are also possible. We denote by $U(x,r)$ the set $\{y \in E \mid \|x-y\| \leq r\}$, whereas $U^0(x_0,r)$ is the set $\{y \in E / \|x-y\| < r\}$.

II. Convergence analysis

We show the following semilocal result:

Theorem 1. *Let f,g be continuous operators defined on an open convex subset D of a Banach space E with values in E. Consider numbers $a,b > > 0$, $c \in [0,1]$ and a point $x_0 \in D$.*

Moreover, assume:

(a) *operators f,g satisfy:*

$$\|[y,g(y)]^{-1}([x,y]-[z,w])(y-x)\| \leq a(\|x-z\|+\|y-w\|)\|y-x\|, \tag{5}$$

$$\|x-g(x)\| \leq b\|[x,g(x)]^{-1}f(x)\|, \tag{6}$$

and

$$\|g(x)-g(y)\| \leq c\|x-y\| \tag{7}$$

for all $x, y, v, w \in D$.

(b) *The Newton-Kantorovich-type hypothesis*

$$h = d\eta \leq 1 \tag{8}$$

is true, where

$$d = 4a \max\{b, 2c\} \tag{9}$$

and

$$\|[x_0, g(x_0)]^{-1} f(x_0)\| \leq \eta. \tag{10}$$

(c) *The smallest solution* r^* *of the scalar equation*

$$f(r) = \frac{d}{4} r^2 - r + \eta = 0, \tag{11}$$

and number R *satisfy*

$$r^* \leq R, \quad U(x_0, R) \subseteq D, \tag{12}$$

$$r^* \geq \frac{\|x_0 - g(x_0)\|}{1 - c} \tag{13}$$

and

$$a[(2 + c)r^* + R + \|x_0 - g(x_0)\|] < 1. \tag{14}$$

Then

(i) *scalar iteration* $\{t_n\}$ $(n \geq 0)$ *generated by*

$$t_{n+2} - t_{n+1} = \frac{ac}{1 - ab(t_{n+1} - t_n)} (t_{n+1} - t_n)^2 \quad (n \geq 0) \tag{15}$$

with $t_0 = 0$ *and* $t_1 = \eta$ *is monotonically increasing, bounded above by* r^* *and* $\lim_{n \to \infty} t_n = r^*$.

(ii) *Steffensen's iteration* $\{x_n\}$ $(n \geq 0)$ *generated by (2) is well defined, remains in* $U(x_0, r^*)$ *for all* $n \geq 0$ *and converges to a solution* x^* *of equation* $f(x) = 0$, *which is unique in* $U(x_0, R)$.

Moreover the following estimates are true for all $n \geq 0$

$$\|x_{n+2} - x_{n-1}\| \leq \frac{ac}{1 - ab\|x_{n+1} - x_n\|}\|x_{n+1} - x_n\|^2 \leq t_{n+2} - t_{n+1} \tag{16}$$

and

$$\|x_n - x^*\| \leq r^* - t_n. \tag{17}$$

Proof. (i) Using the initial conditions we get $t_1 \geq t_0 \geq 0$, and $t_{n+2} \geq t_{n+1}$ if $t_{n+1} \geq t_n \geq 0$ $(n \geq 0)$ (by (15)). Hence $\{t_n\}$ $(n \geq 0)$ is monotonically increasing and nonnegative. Moreover from the initial conditions and the definition of r^* we have $t_0 \leq t_1 \leq r^*$. Let us assume that $t_k \leq r^*,\ k = 0, 1, 2, \ldots, n$. Then from (15) we obtain in turn

$$\begin{aligned} t_{k+2} &= t_{k+1} + \frac{ac}{1 - ab(t_{k+1} - t_k)}(t_{k+1} - t_k)^2 \leq \\ &\leq t_{k+1} + \frac{acr^*}{1 - abr^*}(t_{k+1} - t_k) \leq \\ &\leq \ldots \leq t_1 + \frac{acr^*}{1 - abr^*}(t_{k+1} - t_0) \leq \\ &\leq \eta + 2ac(r^*)^2 = r^* \end{aligned}$$

by the choice of r^* and (8). That is $\{t_n\}$ $(n \geq 0)$ is bounded above by r^*. Since r^* is the minimum number satisfying (11) it follows that $\lim_{n \to \infty} t_n = r^*$.

(ii) By hypotheses (7) and (13) it follows $x_1, g(x_0), g(x_1) \in U(x_0, r^*)$. Let us assume $x_k, g(x_k) \in U(x_0, r^*),\ k = 0, 1, 2, \ldots, n+1$. We first show that $g(x_{k+1}) \in U(x_0, r^*)$. Indeed from (7) and (13) we get

$$\begin{aligned} \|g(x_{k+1}) - x_0\| &\leq \|g(x_{k+1}) - g(x_0)\| + \|g(x_0) - x_0\| \leq \\ &\leq c\|x_{k+1} - x_0\| + \|g(x_0) - x_0\| \leq cr^* + \|x_0 - g(x_0)\| \leq r^*. \end{aligned}$$

That is $g(x_{k+1}) \in U(x_0, r^*)$.

Using induction on $n \geq 0$ we will show

$$\|x_{n+1} - x_n\| \leq t_{n+1} - t_n \quad (n \geq 0). \tag{18}$$

Estimate (18) is true for $n = 0$ by the initial conditions. Suppose (18) is true for $k = 0, 1, 2, \ldots, n+1$. Starting from the approximation

$$\begin{aligned} f(x_{k+1}) &= f(x_{k+1}) - f(x_k) - [x_k, g(x_k)](x_{k+1} - x_k) = \\ &= ([x_k, x_{k+1}] - [x_k, g(x_k)])(x_{k+1} - x_k), \end{aligned}$$

hypotheses (5), (6) and (7) give in turn

$$\|[x_{k+1}, g(x_{k+1})]^{-1} f(x_{k+1})\| \le a\|x_{k+1} - g(x_k)\| \; \|x_{k+1} - x_k\|, \tag{19}$$

but

$$\begin{aligned} \|x_{k+1} - g(x_k)\| &\le \|x_{k+1} - g(x_{k+1})\| + \|g(x_{k+1}) - g(x_k)\| \le \\ &\le b\|[x_{k+1}, g(x_{k+1})]^{-1} f(x_{k+1})\| + c\|x_{k+1} - x_k\|. \end{aligned} \tag{20}$$

Hence from (19) and (20) we get

$$\begin{aligned} \|[x_{k+1}, g(x_{k+1})]^{-1} f(x_{k+1})\| &\le \frac{ac}{1 - ab\|x_{k+1} - x_k\|}\|x_{k+1} - x_k\|^2 \le \\ &\le \frac{ac}{1 - ab(t_{k+1} - t_k)}(t_{k+1} - t_k)^2 = t_{k+2} - t_{k+1} \end{aligned} \tag{21}$$

which shows (16) for all $n \ge 0$. Estimate (16) shows that iteration $\{x_n\}$ $(n \ge 0)$ is Cauchy in a Banach space E and as such it converges to some $x^* \in U(x_0, r^*)$ (since $U(x_0, r^*)$ is a closed set). By letting $k \to \infty$ in (21) we deduce $f(x^*) = 0$. That is the point x^* is a solution of equation (1). Estimate (17) follows immediately from (16) by using standard majorization techniques [1], [5].

To show uniqueness of the solution $x^* \in U(x_0, R)$, let us assume that there exists a solution $y^* \in U(x_0, R)$. From the approximation

$$x_{n+1} - y^* = -[x_n, g(x_n)]^{-1}([y^*, x_n] - [x_n, g(x_n)])(x_n - y^*),$$

we get

$$\|x_{n+1} - y^*\| \le a(\|x_n - y^*\| + \|x_n - g(x_n)\|)\|x_n - y^*\|. \tag{22}$$

But we also have

$$\|x_n - y^*\| \le \|x_n - x_0\| + \|x_0 - y^*\| \le r^* + R$$

and

$$\begin{aligned} \|x_n - g(x_n)\| &\le \|x_n - x_0\| + \|x_0 - g(x_0)\| + \|g(x_0) - g(x_n)\| \le \\ &\le r^* + \|x_0 - g(x_0)\| + cr^* = (1 + c)r^* + \|x_0 - g(x_0)\|. \end{aligned}$$

Estimate (22) now gives

$$\|x_{n+1} - y^*\| \le \varepsilon_0\|x_n - y^*\| \le \ldots \le \varepsilon_0^{n+1}\|x_0 - y^*\| \le R\varepsilon_0^{n+1}, \tag{23}$$

where

$$\varepsilon_0 = a[(2+c)r^* + R + \|x_0 - g(x_0)\|].$$

But from (14) we have $0 \le \varepsilon_0 < 1$. Letting $n \to \infty$ in (23) we get $\lim_{n\to\infty} x_n = y^*$. We have already showed $\lim_{n\to\infty} x_n = x^*$. Hence, we deduce $x^* = y^*$. That completes the proof of the theorem.

Remark 1. (a) Condition (8) and the choice of r^* given by (11) can be replaced by the following weaker hypothesis: there exists a minimum nonnegative number r_1^* satisfying

$$T(r_1^*) \le r_1^*,$$

and the number r_1^* must also satisfy

$$abr_1^* < 1.$$

This follows immediately from part (i).

(b) Moreover the condition on uniqueness (14) can be replaced by the hypothesis

$$0 \le \varepsilon_1 < 1, \quad r_1^* \le R,$$

where

$$\varepsilon_1 = a[(1+2b)r_1^* + R].$$

Indeed this follows from (22) and the estimate

$$\begin{aligned}\|x_n - g(x_n)\| &\le b\|[x_n, g(x_n)]^{-1} f(x_n)\| \le b\|x_{n+1} - x_n\| \le \\ &\le b(\|x_{n+1} - x_0\| + \|x_0 - x_n\|) \le b(r_1^* + r_1^*) = 2br_1^*.\end{aligned}$$

(c) If f is Fréchet-differentiable on D and we choose $g(x) = x$ $(x \in D)$, then iteration (2) reduces to Newton's method. In this case (6), (7) are satisfied for $b = 0$ and $c = 1$, whereas by (9) $d = 8a$, and by (5) $4a = \ell$, where ℓ denotes the usual Lipschitz constant in (5). Hence (8) becomes

$$2\ell\eta \le 1,$$

which is the Newton-Kantorovich hypothesis for Newton's method [5]. If operator g is chosen so that $g(x_n) = x_{n-1}$ $(n \ge 0)$, then iteration (2) reduces to the secant method. Another common choice is given by $g(x) = x - f(x)$ $(x \in D)$. Many other choices for g are also possible.

(d) Condition (13) can be replaced by the stronger $\|x_0 - g(x_0)\| \le (1-c)\eta$, since $r^* \ge \eta$.

(e) It can easily be seen by (6) that condition (13) is satisfied if $b + c \le 1$ for $r^* \ne 0$.

We will also need the following

Theorem 2. *Let f, g be continuous operators defined on an open convex subset D of a Banach space E with values in E, and numbers $a, b > 0,\ c \in [0, 1]$, and a point $x^* \in D$ such that conditions (5)-(7) are satisfied and $f(x^*) = 0$. Let $x_0 \in D$ be such that*

$$U(x^*, \|x_0 - x^*\|) \subseteq D, \tag{24}$$

and

$$x_0 \in U^0(x^*, r_2^*),\ \ \|x_0 - x^*\| \le r_2^*, \tag{25}$$

where

$$0 \le r_2^* < \frac{1}{a(2+c)}. \tag{26}$$

Then Steffensen's iteration $\{x_n\}$ $(n \ge 0)$ generated by (2) is well defined, remains in $U(x^*, \|x_0 - x^*\|)$, and converges to x^* which is the unique solution of equation (1) in $U^0(x^*, r_2^*)$ with

$$\|x_{n+1} - x^*\| \le a(2+c)\|x_n - x^*\|^2 \quad (n \ge 0). \tag{27}$$

Proof. We first show that all iterates $x_n \in U(x^*, \|x_0 - x^*\|)$. From the approximation

$$\begin{aligned} x_{n+1} - y^* &= x_n - x^* - [x_n, g(x_n)]^{-1}(f(x_n) - f(x^*)) = \\ &= -[x_n, g(x_n)]^{-1}([x^*, x_n] - [x_n, g(x_n)])(x_n - x^*), \end{aligned}$$

hypotheses (5), (6), (7) and (26) we obtain, if $x_n \in U(x^*, \|x_0 - x^*\|)$

$$\begin{aligned} \|x_{n+1} - x^*\| &\le a(\|x^* - x_n\| + \|x_n - g(x_n)\|)\|x_n - x^*\| \le \\ &\le a(\|x^* - x_n\| + \|x_n - x^*\| + \|g(x^*) - g(x_n)\|)\|x_n - x^*\| \le \\ &\le a(2+c)\|x_n - x^*\|^2 \le a(2+c)\|x_0 - x^*\|^2 < \|x_0 - x^*\|, \end{aligned}$$

which shows $x_{n+1} \in U^0(x^*, \|x_0 - x^*\|)$.

To show uniqueness let $y^* \ne x^*,\ f(y^*) = 0$ and $y^* \in U^0(x^*, r_2^*)$. Set $x_0 = x^*$ in (2), then $x_1 = x^*$ also. Hence, the above inequality gives

$$\|y^* - x^*\| \le a(2+c)\|y^* - x^*\| < \|y^* - x^*\|,$$

which contradicts the hypothesis $x^* \neq y^*$. That completes the proof of the Theorem.

We show uniqueness of the solution x^* inside a special set by the following

Theorem 3. *Assume that hypotheses of Theorem 1 are satisfied: for d^0 replacing d, given by*

$$d^0 = 2a \max\{2b, 4c, a(2+c)\}, \tag{28}$$

r^* denoted by δ^* in this case, and (8) being true as a strict inequality. Define the set

$$D^* = \bigcup_{n=0}^{\infty} U^0(x_n, \delta^*) \cap D. \tag{29}$$

Then the following are true:

(i) $x_{n+1} \in U^0(x_n, \delta^*)$ *and* $x^* \in U^0(x_0, \delta^*)$*;*
(ii) D^* *is a connected set;*
and
(iii) x^* *is the unique solution of equation (1) in* D^**.*

Proof. (i) Since the iteration $\{x_n\}$ $(n \geq 0)$ converges, we have

$$\|x_{n+1} - x_n\| \leq \|x_1 - x_0\| \leq \eta < \delta^* \quad (n \geq 0).$$

Hence we deduce $x_{n+1} \in U^0(x_n, \delta^*)$ $(n \geq 0)$. Moreover we have

$$\|x^* - x_0\| < \delta^*.$$

That is $x^* \in U^0(x_0, \delta^*)$.

(ii) It follows immediately from part (i).

(iii) Let $y^* \in D$ with $f(y^*) = 0$, and choose $x_0 \in U^0(y^*, \delta^*)$, which implies $y^* \in U^0(x_0, \delta^*)$. By Theorem 2 y^* is unique in $U^0(y^*, \delta^*)$. Assume there exists another solution $z^* \in U^0(x_0, \delta^*) \backslash U^0(y^*, \delta^*)$. It follows that z^* is the unique solution in $U^0(z^*, \delta^*)$. Moreover the Steffensen's iteration initiating at x_0 must converge to z^*, which contradicts the uniqueness of the iteration. That is y^* is unique in $U^0(x_0, \delta^*)$. The same argument applies to all iterates $x_n (n \geq 0)$, which leads to the definition of D^* given by (29). That completes the proof of the Theorem.

In the following section we follow the formulation first introduced in the elegant study [6].

III. The asymptotic mesh independence for Steffensen-Galerkin methods

Iteration $\{x_n\}$ $(n \geq 0)$ can rarely be computed for infinite dimensional space E. That is why in practice equation (1) is replaced by a family of discretized equations of the form

$$f_i(x_i) = 0 \quad (i \geq 0), \tag{31}$$

where $f_i : D_i \subseteq E_i \to E_i$ is a nonlinear operator defined on a convex domain D_i of a finite dimensional subspace $E_i \subseteq E$ with values in itself. Let x_i^* be a solution of (31). Then f_i must be chosen so that

$$\lim_{i\to\infty} x_i^* = x^*. \tag{32}$$

To achieve this we introduce a disretized Steffensen's method of the form

$$x_i^{n+1} = x_i^n - [x_i^n, g_i(x_i^n)]_i^{-1} f_i(x_i^n) \quad (n \geq 0), \tag{33}$$

where $[,]_i$ denotes divided difference of order one on the space E_i and $g_i : D_i \subseteq$ $\subseteq E_i \to E_i$ is a given family of continuous nonlinear operators.

Assume:

(I_1): there exists a family $\{p_i\}$ $(i \geq 0)$ with $p_i : E \to E_i$ of linear projection operators and a scalar sequence $\{d_i\}$ $(i \geq 0)$ such that

$$\|p_i(x)\| \leq d_i\|x\|, \quad x \in E, \quad d_i \leq e < \infty \quad (i \geq 0); \tag{34}$$

(I_2): there exists a scalar sequence $\{e_i\}$ $(i \geq 0)$ such that

$$\|x - p_i(x) \leq e_i\|x\|, \quad x \in E(i \geq 0), \tag{35}$$

$$e_{i+1} \leq e_i \quad (i \geq 0) \tag{36}$$

and

$$\lim_{i\to\infty} e_i = 0. \tag{37}$$

The discretization method is described by a family

$$\{f_i, p_i, g_i, e_i\} \quad (i \geq 0) \tag{38}$$

(39) $$(I_3): \ U_i^0(p_i(x^*), \delta) \subseteq D_i \subseteq D \quad (i \geq 0).$$

(I_4) : If $u_i \in E_i$ and $u \in E$ are solutions of the linear equations

$$[p_i(x), g_i(p_i(x))]_i u_i = f_i(p_i(x)),$$
$$[x, g(x)]u = f(x),$$

where $x \in E, \ p_i(x) \in D_i$, then there exists a positive constant c_0 such that

(40) $$\|u_i - p_i(u)\| \leq c_0 e_i \quad (i \geq 0).$$

(I_5): If $w_i \in E_i$ and $w \in E$ are solutions of the linear equations

$$[z_i, g_i(z_i)]_i w_i = ([x_i, x_i + v_i]_i - [\overline{z}_i, \overline{w}_i]_i)v_i,$$
$$[z_i, g(z_i)]w = ([x_i, x_i + v_i] - [\overline{z}_i, \overline{w}_i])v_i,$$

where $x_i, z_i, \overline{z}_i, \overline{w}_i \in D_i, v_i \in \mathrm{span}(z_i - x_i)$, then there exists a positive constant c_1 such that

(41) $$\|w_i - p_i(w)\| \leq c_1 e_i(\|x_i - \overline{z}_i\| + \|x_i + v_i - \overline{w}_i\|)\|v_i\| \quad (i \geq 0).$$

(I_6): If $w_i \in E_i$ and $w \in E$ are solutions of the linear equations

$$[x_i, g_i(x_i)]_i(w_i - g_i(w_i)) = f_i(x_i),$$
$$[x_i, g(x_i)](w - g(w)) = f(x_i),$$

where $x_i \in D_i$, then there exists a scalar sequence $\{b_i\}$ $(i \geq 0)$ satisfying

(42) $$\lim_{i \to \infty} b_i = b$$

and

(43) $$\|(w_i - g_i(w_i)) - p_i(w - g(w))\| \leq b_i \|[x_i, g_i(x_i)]_i^{-1} f_i(x_i)\| - \|p_i(w - g(w))\|.$$

Note that from the triangle inequality it follows that

(44) $$\begin{aligned} \|w_i - g_i(w_i)\| &\leq \|(w_i - g_i(w_i)) - p_i(w - g(w))\| + \|p_i(w - g(w))\| \leq \\ &\leq b_i \|[x_i, g_i(x_i)]_i^{-1} f_i(x_i)\| \quad (i \geq 0). \end{aligned}$$

(I_7): Assume that there exists a scalar sequence $\{h_i\}$ $(i \geq 0)$ such that

(45) $$\lim_{i \to \infty} h_i = c$$

and

$$\|g_i(v_i) - g_i(w_i)\| \leq h_i \|v_i - w_i\|$$

for all $v_i, w_i \in E_i$ $(i \geq 0)$.

In this next result we examine the relationship between the constant a appearing in (5) for the operator equation (1) and the associated a_i for the finite-dimensional equation (31).

Lemma. *Let $f, g : D \subseteq E \to E$ be nonlinear operators. Assume:*

(a) *condition (5) is satisfied;*

(b) *the discretization method (38) satisfies (34), (35), (36), (39) and (I_5).*

Then for all $i \geq 0$

$$\|[z_i, g_i(z_i)]_i^{-1}([x_i, x_i+v_i]_i - [\overline{z}_i, \overline{w}_i]_i)v_i\| \leq a_i(\|x_i - \overline{z}_i\| + \|x_i + v_i - \overline{w}_i\|)\|v_i\|, \tag{46}$$

where

$$a_i = c_1 e_i + a d_i \quad (i \geq 0). \tag{47}$$

Moreover if conditions (37) and $d_i \leq 1 + e_i$ $(i \geq 0)$ are satisfied, then

$$\lim_{i \to \infty} a_i = a.$$

Proof. Using (I_5), (5) we can write

$$\|w_i\| \leq a_i(\|x_i - \overline{z}_i\| + \|x_i + v_i - \overline{w}_i\|)\|v_i\|,$$
$$\|w\| \leq a(\|x_i - \overline{z}_i\| + \|x_i + v_i - \overline{w}_i\|)\|v_i\|,$$

and by (34) and (41) we get

$$\|w_i\| \leq \|w_i - p_i(w)\| + \|p_i(w)\| \leq (c_1 e_i + a d_i)(\|x_i - \overline{z}_i\| + \|x_i + v_i - \overline{w}_i\|)\|v_i\|$$

which shows (47).

The projection property $p_i^2 = p_i$ $(i \geq 0)$ and (34) imply $d_i \geq 1$, and by the hypothesis we deduce $\lim_{i \to \infty} d_i = 1$. The first result follows by letting $i \to \infty$ in (47). That completes the proof of the Lemma.

We can now prove the first part of the asymptotic mesh-independence principle for Steffensen-Galerkin methods.

Theorem 4. *Assume:*

(a) hypotheses of Theorems 1,2 and 3 are satisfied;

(b) conditions $(I_1)-(I_7)$ *are satisfied.*

Then

(i) *there exist an integer* $i^* \in N$ *such that for any* $i \geq i^*$ *equation (31) has a solution* x_i^* *satisfying*

$$\|x_i^* - p_i(x^*)\| \leq 2c_0 e_i \quad (i \geq 0). \tag{48}$$

(ii) *Moreover* x_i^* *is the unique solution of equation (31) in*

$$U_i^0(p_i(x^*), \delta_i^*) \cap D_i. \tag{49}$$

Proof. We will make use of Theorem 1. Define $x_i^0 = p_i(x^*)$ $(i \geq 0)$. Denote

$$u_i^* = [p_i(x^*), g_h(p_i(x^*))]^{-1} f_i(p_i(x^*)), \quad \alpha_i^* = \|u_i^*\|.$$

Since x^* is a solution of equation (1), we get

$$u^* := [x^*, g(x^*)]^{-1} f(x^*) = 0,$$

and by condition (I_4) we get

$$\alpha_i^* = \|u_i^*\| = \|u_i^* - p_i(u^*)\| \leq c_0 e_i \quad (i \geq 0). \tag{50}$$

By (8) we can set

$$h_i^* = d_i^0 \alpha_i^* \quad (i \geq 0), \tag{51}$$

where

$$d_i^0 = 2a_i \max\{2b_i, 4c_i, a_i(2 + c_i)\}. \tag{52}$$

By (42), (45), (47) and (51) we see that the Newton-Kantorovich sequence $\{h_i^*\}$ is null. Hence there exists $i^* > 0$ such that

$$h_i^* \leq 1 \quad \text{for} \quad i \geq i^*. \tag{53}$$

By the definition of δ_i^* we obtain

$$\delta_i^* \leq 2\alpha_i^*. \tag{54}$$

By Theorem 1 Steffensen's method (33) starting at x_i^0 converges to a solution $x_i^* \in U_i^0(p_i(x^*), \delta_i^*)$ which is unique in

$$D_i^* = \bigcup_{n=0}^{\infty} U_i^0(x_i^n, \alpha_i^*) \cap D$$

and in the set given by (49). That completes the proof of the Theorem.

As in [6] choose $x_i^0 \in D_i$ and define

$$\alpha_i(x_i^0) = \|[x_i^0, g_i(x_i^0)]^{-1} f_i(x_i^0)]. \tag{55}$$

If

$$h_i(x_i^0) = d_i^0 \alpha_i(x_i^0) \leq 1, \tag{56}$$

then choose

$$\delta_i(x_i^0) \in [\alpha_i(x_i^0), \delta_i^*). \tag{57}$$

Moreover if we also have

$$U_i^0(x_i^0, \delta_i(x_i^0)) \subseteq D_i, \tag{58}$$

it follows from Theorem 1, 4 and the Lemma that iteration $\{x_i^n\}_{n=0}^{\infty}$ $(i \geq 0)$ generated by (33) converges to a solution x_j^* provided that

$$\|x_i^0 - x_i^*\| < \lambda_i^0 \quad \text{with} \quad \lambda_i^0 = \frac{1}{d_i^0} \quad (i \geq 0). \tag{59}$$

Furthermore, whenever the Steffensen iterates (33) remain in D_i and converge to x_i^* the following are true

$$\|x_i^n - x_i^{n+1}\| \leq \frac{a_i c_i}{1 - a_i b_i \|x_i^{n+1} - x_i^n\|} \|x_i^{n+1} - x_i^n\|^2 \quad (n \geq 0,\ i \geq 0) \tag{60}$$

and

$$\|x_i^n - x_i^*\| \leq a_i(2 + c_i)\|x_i^{n-1} - x_i^*\|^2 \quad (n \geq 1,\ i \geq 0). \tag{61}$$

We want to find integers $\beta_i = \beta_i(x_i^0, \varepsilon),\ \beta = \beta(x_0, \varepsilon)$ for $\varepsilon > 0$ such that

$$\|x_i^n - x_i^*\| \leq \varepsilon, \quad n \geq \beta_i \tag{62}$$

and

$$\|x^n - x^*\| \le \varepsilon, \quad n \ge \beta, \tag{63}$$

provided that

$$\|x_0 - x^*\| < \lambda_0 \quad \text{with} \quad \lambda_0 = \frac{1}{d^0}. \tag{64}$$

It can easily be seen from (27), (61), (62) and (63) that β_i, β can be chosen to be

$$\beta_i = \left]\log_2\left(\frac{\ln(a_i(2+c_i)\varepsilon)}{\ln(\|x_i^0 - x_i^*\|a_i(2+c_i))}\right)\right[\tag{65}$$

and

$$\beta = \left]\log_2\left(\frac{\ln(a(2+c)\varepsilon)}{\ln(\|x_0 - x^*\|a(2+c))}\right)\right[. \tag{66}$$

It follows that since β_i and β are integer valued, they will differ by at most one whenever $\|x_i^0 - x_i^*\|d_i$ is close enough to $\|x_0 - x^*\|d$. This is the case when

$$\lim_{i\to\infty} d_i^0 = d^0 \quad (i \ge 0), \tag{67}$$

and

$$x_i^0 = p_i(x^0) \quad (i \ge 0) \tag{68}$$

are true. Hence we arrived at the second part of the asymptotic mesh-independence principle for Steffensen-Galerkin methods.

Theorem 5. *Assume:*

(a) *the hypotheses of Theorem 4 are true;*
(b) *condition (67) is satisfied;*
(c) *moreover the following is true*

$$h^0 = d^0\eta < 1. \tag{69}$$

Then

(i) *there exists $i_1 \ge i^*$ such that Steffensen's iteration generated by (33) with starting point x_i^0 given by (68) converges to x_i^0.*

(ii) *Estimates (59) and (64) are satisfied and*

$$|\beta - \beta_i| \leq 1 \quad \text{for all} \quad i \geq i_1. \tag{70}$$

Proof. (i) By Theorems 1,2 and (25) we obtain

$$\|x_0 - x^*\| \leq \delta^* < \lambda_0,$$

which shows (64). Using (67), (68), Theorem 4 and the estimate

$$|\|p_i(x_0 - x^*)\| - \|p_i(x^*) - x_i^*\|| \leq \|x_i^0 - x_i^*\| \leq \|p_i(x_0 - x^*)\| + \|p_i(x^*) - x_i^*\|,$$

we deduce

$$\lim_{i \to \infty} \|x_i^0 - x_i^*\| = \|x_0 - x^*\|. \tag{71}$$

Moreover from (67) and (64) we get

$$\lim_{i \to \infty} d_i^0 \|x_i^0 - x_i^*\| = d^0 \|x_0 - x^*\| < 1. \tag{72}$$

It follows from (71) and (72) that (59) and (70) are satisfied. That completes the proof of the Theorem.

Remark 2. For $g(x) = x$ $(x \in D)$, $b = 0$, $c = 1$, $b_i = 0$ and $h_i = 1$ $(i \geq 0)$ our results reduce to the ones obtained in [6] for Newton's method.

References

[1] **Allgower E.L., Böhmer K., Potra F.A. and Rheinboldt W.C.,** A mesh independence principle for operator equations and their discretizations, *SIAM J. Numer. Anal.,* **23** (1) 160-169.

[2] **Argyros I.K.,** A mesh independence principle for operator equations and their discretizations under mild differentiability conditions, *Computing*, **45** (1990), 265-268.

[3] **Argyros I.K.,** On a mesh independence principle for operator equations and the secant method, *Acta Math. Hungar.*, **60** (1-2) (1992), 7-19.

[4] **Argyros I.K.,** On the discretization of Newton-like methods, *Intern. J. Computer Math.,* **52** (1994), 161-170.

[5] **Argyros I.K. and Szidarovszky F.,** *The theory and application of iteration methods*, C.R.C. Press Inc., Boca Raton, Florida, 1993.

[6] **Deufehard P. and Potra F.,** Asymptotic mesh independence of Newton-Galerkin methods via a refined Mysovskii theorem, *SIAM J. Numer. Anal.,* **29** (5) (1992), 1395-1412.

[7] **Catinas E.,** On some iterative methods for solving nonlinear equations, *Revue d'analyse numérique et de théorie de l'approximation*, **23** (1) (1994), 47-53.

[8] **Pavaloiu I.,** Sur une généralisation de la méthode de Steffensen, *Revue d'analyse numérique et de théorie de l'approximation*, **21** (1) (1992), 59-65.

(Received October 30, 1996)

I.K. Argyros
Department of Mathematics
Cameron University
Lawton, OK 73505, USA