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A NON–INVERSE GENERAL MATRIX ANALYTICAL SOLUTION METHOD

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Abstract. A novel general solution method is derived in the paper for the general $GI/G/1$ type processes – infinite block-structured Markov chains with repetitive structure. The proposed method is shown to be accurate theory-wise. While matrix inversion is needed in each iterational step of other general (and of more special) numerical procedures, the method presented here uses matrix addition and matrix multiplication only. In exchange, the computational complexity and the memory requirement is increasing in each iterational step of the new method. This paper, however, lays priority on the theoretical aspect of the general solution. To help the better understanding of the proposed method, it is also being compared with another general solution method for a simple queueing model.

1. Introduction

With the growing requirement of the modelling of telecommunication and computer systems together with their more and more complicated and integrated world, the queueing systems are also becoming increasingly complex. The need of interconnecting compound systems (Internet, mobile phones, laptops, etc.) in the future communication networks is appearing lately. This phenomenon can be captured for example by the so-called batch-arrival-batch-departure queueing systems (varying number of customers enter queues with some number of servers). Like for most of the queues, finite or infinite Markov chains serve as the models of these queueing systems.

Often, the steady-state of the modelling Markov chains can be used for predicting the characteristics of the modelled system. From theoretical point of view, evaluating the π steady-state distribution (equilibrium distribution) of a homogeneous, positive recurrent Markov chain is an easy task, since it

is the solution of the so-called steady-state balance equation: $\pi^T P = \pi^T$ for discrete-time Markov chains with transition probability matrix P ($Pe = e$) and $\pi^T Q = 0^T$ for continuous-time Markov chain with generator matrix Q ($Qe = 0$); and the normalising equation $\pi^T e = 1$, where the row vector π^T is the transposed π column vector and e is a proper sized column vector of 1-s. For large finite Markov chains and for the infinite Markov chains the π is rarely available in explicit form, at least in numerically tractable manner. Thus numerical solution methods are frequently the feasible approach to deal with the large systems.

Guided by the new type of queueing systems, one can arrive at a very large and general class of Markov chains which can be coped with effective iterative algorithms that have probabilistic meaning. The probabilistic interpretation of computational procedures helps in understanding the stochastic systems and provides the possibility to find enhanced algorithms.

The rest of the paper is organised as follows. As an integral part of the introduction, the following two sections, 1.1 and 1.2, introduce the general Markov chain model and its appearance in the considerable literature – making it possible to describe and analyse the problem and set up the notation. The idea of censoring and state reduction is reviewed in Section 2, that leads to an algorithmically tractable general solution method. New theoretical aspects of the general model are discussed in Section 3 resulting in a new general solution algorithm. Section 4 then analyses the performance of the proposed solution method for a simple queueing model, that is introduced in Section 4.1. Numerical results for the queueing system, with the arrival process described in Section 4.2, are presented in Section 4.3. Finally, conclusions are drawn and some goals are set for further research in Section 5.

1.1. The general model

Let us define, just for unified notational purposes, for $1 \leq g, h \leq \infty$ the general $GI/G/1$ *type* (g,h)*-banded processes* as the infinite discrete-time Markov chains with the following P transition probability matrix:

$$P = \begin{bmatrix} B_0 & C_1 & C_2 & C_3 & C_4 & \dots \\ B_1 & Q_0 & Q_1 & Q_2 & Q_3 & \dots \\ B_2 & Q_{-1} & Q_0 & Q_1 & Q_2 & \dots \\ B_3 & Q_{-2} & Q_{-1} & Q_0 & Q_1 & \dots \\ B_4 & Q_{-3} & Q_{-2} & Q_{-1} & Q_0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \tag{1}$$

where the non-zero submatrices of the P matrix are the Q_i square submatrices of size $N \times N$ for all $-g \leq i \leq h$, the B_i submatrices of size $N \times N_0$ for all

$1 \leq i \leq g$, the C_j submatrices of size $N_0 \times N$ for all $1 \leq j \leq h$ and the B_0 square submatrix of size $N_0 \times N_0$. Following standard terminology, all states of the Markov chain associated with the same block row are said to be at the same *level* of the process. The terms *phase* and *sub-level* are used to distinguish the states at the same level, therefore each state of the Markov chain is uniquely determined by a pairing of a particular level and sub-level.

This notation unifies some well-known paradigms from the literature introduced by Neuts [9], [10]:

- the *quasi birth and death (QBD) processes* represented as the $GI/G/1$ type $(1,1)$-banded processes by the new notation,
- the *Markov chains of* $GI/M/1$ *type* as the $GI/G/1$ type $(\infty,1)$-banded processes,
- the $M/G/1$ *type process* represented as the $GI/G/1$ type $(1,\infty)$-banded processes by the new notation,

and some from [4] or Ch. 6 of [5] as well:

- the $GI/G/1$ *type process* represented as the $GI/G/1$ type (∞,∞)-banded processes and
- the *banded* $GI/G/1$ *type processes*, *left-banded* by finite g and/or *right-banded* by finite h, represented as the $GI/G/1$ type (g,h)-banded processes. Note that for finite g and h the $GI/G/1$-type (g,h)-banded processes are also QBD processes, the $GI/G/1$-type (g,∞)-banded processes are also $M/G/1$ type processes and the $GI/G/1$-type (∞,h)-banded processes are also $GI/M/1$ type processes – by block size enlargement of submatrices.

1.2. Overview of the literature

There are two different methodologies for analysing either the general infinite model or the more special ones.

The first class of methods uses eigenvalues and eigenvectors. One way is to transform the (infinite) steady-state balance equations to a linear equation involving vector *generating function* and some unknown probabilities (that can be determined through finding some characteristic roots) as in Ch. 7 of [5] (for $M/G/1$ and $GI/M/1$ type processes). Another approach, closely related to the generating function method, known as the *spectral expansion*, is based on searching eigenvalues and left eigenvectors of a certain matrix polynomial as proposed in [2] and can be used for QBD processes and $GI/G/1$-type (g,h)-banded processes with finite g and h.

The second class of methods, which this paper is also concerned with, is the class of the well-known and widely used *matrix analytic methods* that

are based on some non-linear matrix equation and its minimal nonnegative solution. Matrix analytic methods propose iterative algorithms for finding the solution to the matrix equation and are popular since the successive steps in the iterative algorithms have a probabilistic interpretation (the eigenvalue-eigenvector methods are one-step (non-iterative) methods, but do not have this probabilistic interpretation advantage). There were even three conferences dedicated to matrix analytic methods, see [3], [1] and [7].

A possible approach to analyse the general infinite Markov chains is to truncate them into a finite state one using it as an approximation of the original processes. Then the repeating structure can be exploited to arrive at very efficient iterative solution algorithms as for example in [8] for (bounded) QBD processes.

The other approach is to deal with the infinite Markov chains and their repetitive structure without truncation. With no intention of giving an exhaustive survey of the huge literature these methods have, the author would like to mention some of the pioneering works and some of the latest ones: QBD processes, $M/G/1$ type and $GI/M/1$ type processes are analysed in [9], [10] or [12] and $GI/G/1$ type processes and some other general processes are analysed in [6], [4] or Ch. 6 of [5] for example.

As mentioned before, any $GI/G/1$ type (g,h)-banded process, with at least one of the parameters g or h being finite, can be represented as a QBD, an $M/G/1$ or a $GI/M/1$ process (by re-sizing the block matrices). The price of the block size enlargement, though, is the significant increase of computational complexity and memory requirement. Therefore, the solution methods for the general $GI/G/1$ type processes have the advantage of keeping the block sizes as low as possible and providing general computational procedures (since all the processes can be treated as special cases of the general model).

2. Censoring as a state elimination method

Consider a Markov chain with the transition matrix P of an arbitrary structure in general. One starts with a state space that is partitioned into two sets: E and the complement set E^c. We denote the transition matrix of the Markov chain imbedded in E by P^E. To formulate this mathematically, let X_t be the state of the original process at time t and suppose that the successive visits to E take place at time $t_1, t_2, \ldots$. The process $\{Y_n,\ n = 1, 2, \ldots\}$ with $Y_n = X_{t_n}$ is then called the *process imbedded in* E and the states in E^c are

said to be *censored*. Whenever the process is in E^c, we say that we are on a *sojourn in* E^c.

If we partition the state space into E and E^c, it induces the partition of the transition matrix P into

$$P = \begin{bmatrix} P_{(E\to E)} & P_{(E\to E^c)} \\ P_{(E^c\to E)} & P_{(E^c\to E^c)} \end{bmatrix} \tag{2}$$

and then the transition matrix P^E of the Markov chain imbedded in E is given by

$$P^E = P_{(E\to E)} + P_{(E\to E^c)} \underbrace{\sum_{n=0}^{\infty} P^n_{(E^c\to E^c)}}_{(I-P_{(E^c\to E^c)})^{-1}} P_{(E^c\to E)}, \tag{3}$$

where I is the identity matrix of proper size (see for example [4] or Ch. 6 of [5] for the details).

From now on we consider Markov chains with transition probability matrix P of form (1).

Denote the corresponding submatrices of the P^{E_n} transition matrix of the Markov chain imbedded in E_n by $Q_i^{E_n}$ $(-n < i < n)$, $B_i^{E_n}$ $(0 \le i \le n)$ and $C_i^{E_n}$ $(0 < i \le n)$ for all the states in the levels $E_n = \{0, 1, \ldots, n\}$, $n > 0$.

The element of the k-th row and l-th column of the matrix $Q_i^{E_n}$, for $i \ge 0$, is by definition the probability of making a transition from state $(n-i, k)$, that is from phase k in level $n-i$, to state (n, l), that is to phase l in level n, through a possible sojourn in states above level n. Similarly, the element of the k-th row and l-th column of the matrix $Q_i^{E_n}$, for $i \le 0$, is also by definition the probability of having a transition from state (n, k), that is from phase k in level n, to state $(n+i, l)$, that is to phase l in level $n+i$, through a possible sojourn in states above level n.

Using the presence of repeating rows, it is easy to see from (3) that $Q_i^{E_n} = Q_i^*$ is independent of $n > 0$ $(-n < i < n)$.

It is then shown in [4] and Ch. 6 of [5] that the matrix sequence $\ldots, Q_{-1}^*, Q_0^*, Q_1^*, \ldots$ is the minimal non-negative solution of the matrix equation

$$Q_i^* = Q_i + \sum_{j=1}^{\infty} Q_j^* (I - Q_0^*)^{-1} Q_{i-j}^*, \quad i \le 0, \tag{4}$$

$$Q_i^* = Q_i + \sum_{j=1}^{\infty} Q_{i+j}^* (I - Q_0^*)^{-1} Q_{-j}^*, \quad i \ge 0. \tag{5}$$

The following straightforward recursion then turns out to produce a monotone non-decreasing sequence of matrices

$$Q_i[0] = 0, \qquad -g \le i \le h, \tag{6}$$

$$Q_i[k+1] = Q_i + \sum_{j=1}^{\min\{h,g+i\}} Q_j[k](I - Q_0[k])^{-1} Q_{i-j}[k], \tag{7}$$

$$-g \le i \le 0 \quad (k \ge 0),$$

$$Q_i[k+1] = Q_i + \sum_{j=1}^{\min\{h-i,g\}} Q_{i+j}[k](I - Q_0[k])^{-1} Q_{-j}[k], \tag{8}$$

$$0 \le i \le h \quad (k \ge 0),$$

which converge to the minimal solution, that is $\lim_{k\to\infty} Q_i[k] = Q_i^*$ for $-g \le i \le h$. The numerical method suggested by the recursion leads to a well-defined numerical iteration method for generating the Q_i^* matrices, especially in the case of finite g and h.

On the other hand, it is also shown in [4] and Ch. 6 of [5] that this recursion is in accordance with the step-by-step state reduction of levels by censoring. Theoretically, cut the infinite transition matrix P at level ω and censor out levels $\omega, \omega-1, \ldots$ successively exactly according to equations (7) and (8). If ω is large enough, $Q_i[k]$ approaches the limiting value Q_i^* and equations (7) and (8) transform to equations (4) and (5) in the succeeding reduction steps ($Q_i[k]$ does not change any more). Hence the reduction steps for the corresponding boundary matrices of B_i and C_i yield equations

$$B_i^* = B_i + \sum_{j=1}^{\min\{h,g-i\}} Q_j^*(I - Q_0^*)^{-1} B_{i+j}^*, \quad 0 < i \le g, \tag{9}$$

$$C_i^* = C_i + \sum_{j=1}^{\min\{h-i,g\}} C_{i+j}^*(I - Q_0^*)^{-1} Q_{-j}^*, \quad 0 < i \le h \tag{10}$$

and in the final reduction step, for B_0 (reducing to level 0) it gives

$$B_0^* = B_0 + \sum_{j=1}^{\min\{h,g\}} C_j^*(I - Q_0^*)^{-1} B_j^*. \tag{11}$$

Then one solves

$$\alpha_0^T B_0^* = \alpha_0^T \tag{12}$$

and generates recursively (by back-substitution in the state reduction procedure)

$$\alpha_k^T = \sum_{i=1}^{\min\{k-1,h\}} \alpha_{k-i}^T Q_i^* (I - Q_0^*)^{-1} + \alpha_0^T C_k^* (I - Q_0^*)^{-1} \sharp_{\{k \le h\}}, \quad k > 0, \tag{13}$$

where α_0 is a column vector of size N_0, α_k for $k > 0$ are vectors of size N and where $\sharp_{\{expression\}}$ equals 1 if the *expression* is true and it equals 0 if the *expression* is false.

Finally, the infinite equilibrium vector $\pi^T = (\pi_0^T, \pi_1^T, \pi_2^T, \ldots)$, namely the steady-state distribution of the Markov chain with transition matrix of form (1) can be attained as

$$\pi_i = \frac{1}{s} \alpha_i, \quad i \ge 0, \tag{14}$$

where s is the normalising constant, $s = \alpha_0^T e_0 + \sum_{k=1}^{\infty} \alpha_k^T \bar{e}$ (e_0 and $\bar{e}$ are column vectors of sizes N_0 and N respectively with all their elements equalling to one).

CSR-method ("censoring, state reduction"): Summarizing the general solution method, the steady-state distribution of the Markov chain with transition matrix of form (1) can be found through generating the Q_i^* for $-g \le i \le h$ (therefore exact for finite g and h) by the numerical iteration method of equations (6), (7) and (8), then these can be used to find B_0^* through equations (9), (10) and (11) and, finally, the equilibrium vector can be calculated by equations (12), (13) and (14).

3. A novel general solution

Remember the probabilistic interpretation of the matrices $Q_i^{E_n} = Q_i^*$ from the previous section:

$Q_i^{E_n}$ is the matrix of the probabilities of going from the corresponding sublevels of level $n - i$ for $i \ge 0$ (from level n for $i \le 0$) to the corresponding sublevels of level n for $i \ge 0$ (to level $n + i$ for $i \le 0$) making an arbitrary sojourn in states above level n.

Now, an alternative recursion is to be introduced resulting in the same minimal non-negative solution, Q_i^*, to the equations (4) and (5) as the one

defined by the previous section in equations (6), (7) and (8). The latter recursion can also be interpreted in probabilistic terms ("the state reduction") as seen in the previous section, but the following alternative one has a more direct probabilistic interpretation (building on limited step sojourns).

Let us define $Q'_i[k]$ as the matrix of the probabilities of going from the corresponding sublevels of level $n-i$ for $i \geq 0$ (from level n for $i \leq 0$) to the corresponding sublevels of level n for $i \geq 0$ (to level $n+i$ for $i \leq 0$) making *at most* $k \geq 0$ steps in the sojourn in states above level n. This definition was introduced by Grassmann and Heyman in [4], yet it was not used for attaining the minimal non-negative solution sequence Q_i^*.

It is easy to see that $\lim_{k \longrightarrow \infty} Q'_i[k] = Q_i^*$. To obtain a recursion though for $Q'_i[k]$ resulting in an iterative approximation method for Q_i^*, an auxiliary definition is needed.

Definition. Define $Q_i^{(j)}[k]$ as the matrix of the probabilities of going from the corresponding sublevels of level n to the corresponding sublevels of level $n+i$ for $i \leq 0$ making **exactly** $k \geq 0$ steps in the sojourn in states above the level $n+i+j$ for $j \geq 0$.

Therefore, using the probability equality

$$Q'_i[k+1] = Q'_i[k] +$$

$$+Pr\{\text{exactly k} + 1 \text{ steps in the sojourn of the corresponding transition}\}$$

and then conditioning on the first transition, the following recursion is reached:

$$Q'_i[0] = Q_i, \qquad -g \leq i \leq h, \tag{15}$$

$$Q'_i[k+1] = Q'_i[k] + \sum_{m=1}^{\min\{h,(k+1)g+i\}} Q_m Q_{i-m}^{(-i)}[k],$$

$$-g \leq i \leq 0 \quad (k \geq 0), \tag{16}$$

$$Q'_i[k+1] = Q'_i[k] + \sum_{m=1}^{\min\{h-i,(k+1)g\}} Q_{i+m} Q_{-m}^{(0)}[k],$$

$$0 \leq i \leq h \quad (k \geq 0), \tag{17}$$

for the "main recursion" for $Q'_i[k]$ and

$$Q_i^{(j)}[0] = Q_i, \qquad -g \leq i \leq 0, \quad 0 \leq j < g, \tag{18}$$

$$Q_i^{(j)}[k] = \sum_{m=\max\{i+j+1,-g\}}^{\min\{h,kg+i\}} Q_m Q_{i-m}^{(j)}[k-1], \tag{19}$$

$$-(k+1)g \leq i \leq 0, \quad 0 \leq j < g \quad (k > 0),$$

for the "auxiliary recursion" for $Q_i^{(j)}[k]$.

It is clear from the recursion (15), (16) and (17) that the convergence $Q'_i[k] \overset{k\to\infty}{\to} Q_i^*$ is monotone from below similarly to the $Q_i[k] \overset{k\to\infty}{\to} Q_i^*$ convergence.

Especially for finite g and h, the numerical method suggested by the novel recursion also provides a well-defined numerical iteration method for generating the Q_i^* matrices resulting in a new general solution method.

FS-method ("finite sojourn"). The steady state distribution of the Markov chain with transition matrix of form (1) can be found through generating the Q_i^* for $-g \leq i \leq h$ (therefore exact for finite g and h) by the numerical iteration method of equations (15), (16), (17), (18) and (19) and hence, in accordance with the previous method, these can be used to find B_0^* through equations (9), (10) and (11) and finally the equilibrium vector can be calculated by equations (12), (13) and (14) as in the previous case.

4. Performance analysis of the new method

Some of the advantages and the disadvantages of the novel numerical iteration method can already be stated prior to the actual performance analysis.
Advantages:

- The equations the method is based on are theory-wise exact;
- The method provides a general solution method for the general $GI/G/1$ type (g, h)-banded processes;
- The numerical iteration algorithm does not rely on matrix inverses, only matrix addition and matrix multiplication is needed.

Disadvantages:

- The performance of general solution methods is usually less suitable for special models than the methods utilising the special properties of the models;
- Expanding computational and storage capacity (in the number of iterations) is required in the iteration method for the "auxiliary recursion", see equation (19);
- At the end, the matrix inverse $(I - Q_0^*)^{-1}$ still needs to be determined for the final steps, see equations (9), (10), (11) and (13). Thus supposing matrix inversion is not available, some approximation is necessary for this only matrix inverse.

Since these general processes are best captured in the context of queueing or other applications from which they stem, a simple queueing system is now evaluated to investigate the performance of the proposed general method (and compared to the other general method from [4] and Ch. 6 of [5]).

4.1. The queueing model

Consider the following discrete-time queueing model:

- Let $\{X_n,\ n = 0, 1, \ldots\}$ be a homogeneous, aperiodic, irreducible and finite-state discrete-time Markov chain (thus positive recurrent with unique steady-state distribution) denoting the number of customers arriving at time $t_n \stackrel{def}{=} t_0 + n\Delta$ (X_n is the size of the n-th batch arrival, $X_n \in \in \{0, 1, \ldots, N\}$, where $N < \infty$).
- Customers arrive in an infinite storage capacity queue with $S \geq 1$ identical parallel servers. Each server has the service capacity of one customer per the Δ time unit.
- If we denote the queue length immediately before the n-th batch arrival by L_n, we will have the evolutional equation $L_{n+1} = (L_n + X_n - S)^+$ for the queue length for $n = 0, 1, \ldots$, where $x^+ \stackrel{def}{=} max\{0, x\}$.

The sequence of the random variable-pairs $\{(L_n, X_n),\ n = 0, 1, \ldots\}$ forms a homogeneous, aperiodic and irreducible Markov chain with a unique positive steady-state distribution if $E(X) < S$, where $E(X)$ is the expected value of the steady-state size of the arrival-batches – see [11] for the details or see some matrix-type positive recurrence criteria from [9] or [10]. It is easy to verify that this Markov chain is a $GI/G/1$ type $(S, N - S)$-banded process with the queue length being the level and the arriving batch size being the sub-level of the Markov chain.

4.2. The arrival process

For the numerical demonstration, the arrival process is supposed to originate from N number of independent and identical discrete-time on-off sources (a source is generating one customer and no customers if it is in its "on" state and "off" state, respectively, see Fig. 1 for the transition diagram). Then X_n (Markov chain) denotes the number of customers arriving at time t_n, that is the number of sources being in "on" state at time t_n. It can be easily checked that the π_X steady-state distribution of the arrival process can be expressed as $(\pi_X)_i = \binom{N}{i} q^i (1-q)^{N-i}$, for $i = 0, \ldots, N$, where

$$q = \frac{1}{1 + \frac{p_0}{p_1}}$$

(it has a binomial distribution with parameters N and q). The expected value of the steady-state arrival process is hence $E(X) = Nq$ and the positive recurrence criterion for the (L_n, X_n) Markov chain is therefore $E(X) = Nq < S$.

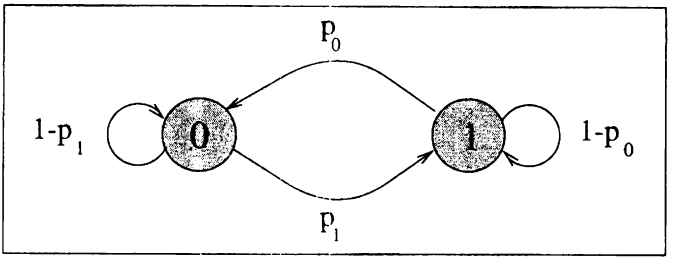

Fig. 1. An on-off source

4.3. Some numerical results

Let us consider the simple queueing model of Section 4.1 with the arrival process of Section 4.2 and suppose $N = 16$ number of on-off sources with the parameters $p_0 = 0.5$ and $p_1 = 0.2$ (the expected number of arrivals in the steady state is $E(X) = \frac{32}{7}$, thus positive queue length distribution can be derived for the server capacity $S = 5, 6, \ldots, 15$).

Three type of algorithms were tested in *MATLAB* for the given configuration: the CSR-method, the CSR-method with block size enlargement (to get a known iterative algorithm for QBD processes, see Ch. 6 of [5], here referred to as the *QBD-method*) and the FS-method.

It turned out for the CSR-method and the QBD-method that the numerical algorithms reach the limiting values in finite steps, see k_{QBD} and k_{CSR} in Table 1 (the minimal nonnegative solutions were "exactly" attained for equations (4) and (5)). This however cannot be expected from the FS-method (remember the probabilistic interpretation), therefore the same number of steps were made for the FS-method as for the CSR-method (because of the computational complexity). Then the differences of these $Q_i'[k_{CSR}]$ approximations from the "exact" Q_i^* solution have been measured in maximum matrix norm, $|A|_\infty \stackrel{def}{=} \max\{|(A)_{ij}|\}$ (see $|Q_i^* - Q_i'[k_{CSR}]|_\infty$ in Table 1) and finally the number of iterations CSR-method needed for providing the same

MATLAB Version 5.3.1.29215*a* (*R*11.1)

precision were determined (these step numbers, $k_{prec.}$, are in the brackets in Table 1).

	k_{QBD}	k_{CSR} ($k_{prec.}$)	$\|Q_i^* - Q_i'[k_{CSR}]\|_\infty$
$S = 5$	29	212 (38)	$6.9583\ 10^{-4}$
$S = 6$	13	74 (28)	$5.7776\ 10^{-7}$
$S = 7$	11	45 (24)	$6.6887\ 10^{-10}$
$S = 8$	9	31 (20)	$3.0170\ 10^{-12}$
$S = 9$	7	24 (18)	$8.6112\ 10^{-15}$
$S = 10$	5	18 (14)	$6.1756\ 10^{-16}$
$S = 11$	5	15 (12)	$3.4694\ 10^{-17}$
$S = 12$	4	12 (10)	$1.3878\ 10^{-17}$
$S = 13$	3	11 (8)	$2.0817\ 10^{-17}$
$S = 14$	3	9 (7)	$2.0817\ 10^{-17}$
$S = 15$	3	7 (5)	$6.9389\ 10^{-18}$

Table 1. Step numbers and precision

Meeting the expectations, a monotone decreasing sequence (in the S server capacity) is got for the number of needed iterations in both "exact" cases (also some monotone decreasing sequence is attained for the precision of the FS-method). The explanation of course is that the smallest server capacity means the highest load of the sources for a given source number (larger queue lengths can develop with higher loads).

The number of steps needed for the QBD-method are much lower than for the CSR-method for example, still one must not forget that higher computational complexity and memory requirement is involved by the block size enlargement.

By equations (9), (10) and (11) plus equations (12), (13) and (14), the stationary distribution of the (L_n, X_n) Markov chain was calculated: π_{CSR} for the "exact" Q_i^*, π_{FS} for the $Q_i'[k_{CSR}]$ approximation and $\pi_{CSR(FS)}$ for the $Q_i[k_{prec.}]$ approximation. The performance measure of main interest, the (first) marginal distribution for the π^L queue length – $(\pi^L_{CSR})_l = \sum_{i=0}^{N} (\pi_{CSR})_{(l,i)}$, – was better approximated by $(\pi^L_{FS})_l = \sum_{i=0}^{N} (\pi_{FS})_{(l,i)}$ than by $(\pi^L_{CSR(FS)})_l =$

$= \sum_{i=0}^{N} (\pi_{CSR(FS)})_{(l,i)}$ since it turned out that $|(\pi^L_{CSR})_l - (\pi^L_{FS})_l| < |(\pi^L_{CSR})_l - (\pi^L_{CSR(FS)})_l|$ for all the calculated queue lengths $l \geq 0$ (the largest difference was of course for the 0 queue length for example $4.4147\ 10^{-5}$ for $S = 5$). It was also checked that the (second) marginal distribution for the arrival process $(\pi^X_{CSR})_i = \sum_{l=0}^{\infty} (\pi_{CSR})_{(l,i)}$ was really a very close approximation of the π_X equilibrium distribution of the arrival process calculated before.

In Fig. 2 the stationary queue length distribution is depicted for the highest loads $S = 5, 6, 7$ (the probabilities of having non-zero queue length decreases in the server capacity S, while the probability of the empty queue increases as expected).

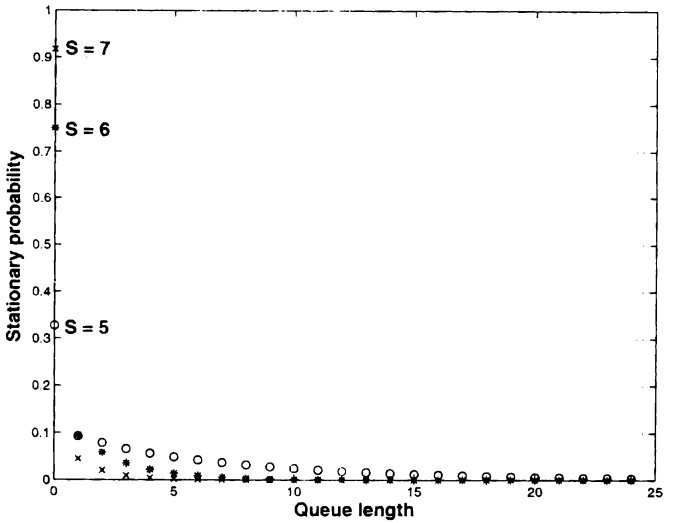

Fig.2. The steady-state queue length distribution

5. Conclusions and future work

A new general iterative algorithm has been derived in the paper for the solution of the general $GI/G/1$ type (g, h)-banded processes (infinite block-structured Markov chains with repetitive structure) based on a probabilistic

interpretation. From theoretical point of view, it gives an exact solution method for the most general $GI/G/1$ type processes.

The main advantage of the proposed numerical procedure compared to an already known general solution method from [4] is that it does not use matrix inversions in the iterational steps (it can be thought of approximating an inverse step by step and not in each step). Its large drawback is that the computational complexity and the storage requirement grows linearly in the number of iterational steps. (The author would like to emphasise the theoretical importance of the novel method in its native form and not the numerical behaviour.)

Also a simple queueing model is numerically evaluated in the paper. The performance of the new method is compared with the other general method and with another more special solution technique (for QBD processes, see for example Ch. 6 of [5]).

A more thorough analysis of the proposed method may result in an enhanced iterative algorithm (maybe with better computational properties). The exact computational requirement and some other aspects like stability of the new numerical procedure might be analysed in the future. The performance of the method for other models could also be checked.

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