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LOCATION AND NUMBER OF PERIODIC ELEMENTS IN $\mathbb{Q}(\sqrt{2})$

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Abstract. In this paper we deal with a quadratic extension of the field of rational numbers. We prove some assertions in connection with the location and quantity of the periodic elements of coefficient systems given in $\mathbb{Q}\left(\sqrt{2}\right)$.

1. Introduction

1.1. Number systems in quadratic extension fields

It is a well-known fact in number theory that every quadratic algebraic extension field of the rational numbers is one of $\mathbb{Q}\left(\sqrt{D}\right)$, where D is a square-free integer. Let I be the set of algebraic integers in $\mathbb{Q}\left(\sqrt{D}\right)$. For some $\beta = c + d\sqrt{D} \in \mathbb{Q}(\sqrt{D})$ let $\overline{\beta} = c - d\sqrt{D}$ be the algebraic conjugate of β, and $r(\beta) = c$ the rational, $i(\beta) = d$ the irrational part of β, where $c, d \in \mathbb{Q}$.

Definition. Let $\alpha \in I$ and E_α $(\subseteq I)$ be a complete residue system mod α containing 0, i.e. such a collection of $f_0 = 0, f_1, ..., f_{t-1} \in I$ for which for every $\gamma \in I$ there exists a unique $f \in E_\alpha$ such that

$$\gamma = \alpha\gamma_1 + f \tag{1}$$

with a suitable $\gamma_1 \in I$. Then we can say that (α, E_α) is a *coefficient system.* From now on we call the elements of E_α digits or coefficients, and α is the base number of (α, E_α).

Definition. We say that (α, E_α) is a *number system* in I if each $\gamma \in I$ can be written as a finite sum

$$\gamma = e_0 + e_1\alpha + ... + e_k\alpha^k, \tag{2}$$

where $e_i \in E_\alpha$, $i = 0, 1, ..., k$.

Naturally, in this case α is the base of the number system (α, E_α). The uniqueness of representation (2) follows from the fact that E_α is a complete residue system mod α. Before examining an arbitrary (α, E_α), we have to introduce certain concepts.

Definition. Let the $J : I \longrightarrow I$ function be defined in the following way: if the equation (1) holds, then $J(\gamma) = \gamma_1$. Then we can speak about *transition* $\gamma \xrightarrow{f} \gamma_1$. J^k denotes the $k - fold$ iterate of J.

Definition. An arbitrary $\pi \in I$ is a *periodic element* if there exists a positive integer k such that $J^k(\pi) = \pi$. Let P be the set of periodic elements, and $G(P)$ the directed graph to be obtained by directing an edge from π to $J(\pi)$ for every $\pi \in P$.

Our long-term objective is to give for an arbitrary $\alpha \in I$, if any, such an E_α digit set that (α, E_α) will constitute a number system. The investigation of graph $G(P)$ and set P is indispensible in finding the appropriate digit set, because, as we will see later, (α, E_α) is a NS if and only if $P = \{0\}$. In this paper we focus our attention on $\mathbb{Q}\left(\sqrt{2}\right)$, and we prove some statements connected with the position, quantity and the modulus of elements of P to an arbitrary $\alpha \in I$.

1.2. Previous achievements in this field

The research in this field has two main directions:

(*) For a given α find such a digit system E_α, if any, for which (α, E_α) is a number system.

(**) For a given α and digit set $E_c = \{0, 1, ..., |N(\alpha)| - 1\}$ decide whether E_c is a complete residue system mod α, and whether (α, E_c) is a number system or not.

$(**)$ was solved for quadratic extension fields in I. Kátai-J. Szabó [1] and in I. Kátai-B. Kovács [2-3], see also W. Gilbert [4]. It was shown that only very special numbers α can serve as bases with such special digit set. In [5], [6] the size, location and stuctural properties of periodic elements were fully described in imaginary quadratic fields.

With respect to problem $(*)$, G. Steidl observed in [7] that in the case of Gaussian integers a good strategy for the choice of an appropriate digit set is to take one for which $\max_{f\in E_\alpha} |f|$ is close to the minimum. Later I. Kátai [8] proved that if I is the set of integers in some imaginary quadratic extension field, then $\alpha \in I$ is a base of a number system with an appropriate digit set if and only if $\alpha \neq 0, |\alpha| \neq 1, |1-\alpha| \neq 1$.

It has been shown in [9] that if $|\alpha| \geq 2, |\overline{\alpha}| \geq 2$, then there always exists an E_α for which (α, E_α) is a number system. The digit set was explicitly computed. It has been proved in [10] that we can find an E_α coefficient system for which $G(P)$ has a simple structure. In detail: for an arbitrary integer $\alpha \in \mathbb{Q}(\sqrt{D})$, where $\alpha, \alpha-1$ is not a unit, $1 < \min(|\alpha|, |\overline{\alpha}|) < 2$ and $D > 1$ is a square-free integer, there exists such an E_α coefficient system for which $G(P)$ is the disjoint union of loops, if either $|\alpha| > 2 > |\overline{\alpha}|, \overline{\alpha} > 0$ or $|\overline{\alpha}| > 2 > |\alpha|, \alpha > 0$, and beside the loop $0 \longrightarrow 0$ it contains only circles of order two of type $\pi \longrightarrow (-\pi) \longrightarrow \pi$, if either $|\alpha| > 2 > |\overline{\alpha}|, \overline{\alpha} < 0$ or $|\overline{\alpha}| > 2 > |\alpha|, \alpha < 0$ holds.

2. The presentation of our assertions

2.1. Observations

One can easily observe that if $\alpha \in I$ is a base of a number system in $\mathbb{Q}(\sqrt{D})$ with a suitable digit set then the following assertions are valid:

1. $\alpha \neq 0$,
2. $\alpha \neq$ unit,
3. $1-\alpha \neq$ unit,
4. $|\alpha| > 1, |\overline{\alpha}| > 1$.
5. if $|\alpha|, |\overline{\alpha}| > 1$, then for each $\gamma \in I$ the path $\gamma, J(\gamma), J^2(\gamma), \ldots$ is ultimately periodic,
6. $G(P)$ is a disjoint union of directed circles,
7. (α, E_α) is a number system if and only if $P = \{0\}$.

The assertions 1. and 2. are obvious. Assume that $1-\alpha = \varepsilon = \mathit{unit}$ and (α, E_α) is a number system. Let $f \in E_\alpha, f \neq 0, \gamma = f\overline{\varepsilon}\delta$, where $\delta = \varepsilon\overline{\varepsilon}$. Then $\gamma = f + \alpha\gamma$ and $\gamma \neq 0$, consequently γ cannot be expanded as (2). Assume that $|\alpha| < 1$ and (α, E_α) is a number system. Then the set of γ having the finite representation (2) is bounded, while the whole set I is not bounded, which is a contradiction. Let us observe finally that (α, E_α) is a number system if and only if $(\overline{\alpha}, \overline{E}_\alpha)$ is a number system, where $\overline{E}_\alpha$ consists of the set of the algebraic conjugates of the elements of E_α. This implies that $|\overline{\alpha}| > 1$ is also necessary.

2.2. K-type digit sets

Let us consider now $\mathbb{Q}(\sqrt{2})$. It is known that $\{1, \sqrt{2}\}$ is an integral basis in I. Let $\varepsilon = \pm 1$, $\delta = \pm 1$ and for some $\alpha \in \mathbb{Q}(\sqrt{2})$ let $d = N(\alpha) = \alpha\overline{\alpha}$.

Let $\alpha = a + b\sqrt{2}$ and $E_\alpha^{(\varepsilon,\delta)}$ be the sets of those $f = k + l\sqrt{2}$, $k, l \in \mathbb{Z}$ for which $f\overline{\alpha} = (k + l\sqrt{2})(a - b\sqrt{2}) = (ka - bl2) + (la - kb)\sqrt{2} = r + s\sqrt{2}$ satisfy the following conditions:

- if $(\varepsilon, \delta) = (1, 1)$, then $r, s \in \left(\frac{-|d|}{2}, \frac{|d|}{2}\right]$,
- if $(\varepsilon, \delta) = (-1, -1)$, then $r, s \in \left[\frac{-|d|}{2}, \frac{|d|}{2}\right)$,
- if $(\varepsilon, \delta) = (-1, 1)$, then $r \in \left[\frac{-|d|}{2}, \frac{|d|}{2}\right)$, $s \in \left[\frac{-|d|}{2}, \frac{|d|}{2}\right)$,
- if $(\varepsilon, \delta) = (1, -1)$, then $r \in \left(\frac{-|d|}{2}, \frac{|d|}{2}\right]$, $s \in \left[\frac{-|d|}{2}, \frac{|d|}{2}\right)$.

It is known from number theory that $E_\alpha^{(\varepsilon,\delta)}$ is a complete residue system mod α. From now on we call the above constructed coefficient sets *K-type digit sets.*

2.3. The formulation of our Theorem

Theorem. *Let $\alpha = a + b\sqrt{2}$ be an arbitrary algebraic integer in $\mathbb{Q}(\sqrt{2})$, for which*

$$\alpha \neq 0,$$
$$\alpha, \alpha \pm 1 \textit{ is not a unit},$$
$$1 < \min(|\alpha|, |\overline{\alpha}|) < 2 \textit{ are valid.}$$

Let, in addition, E_α be an arbitrary K-type digit set. Then the following assertions hold

I. $|a|, |b|\sqrt{2} < \frac{|d|}{2}$.

II. *For each $\pi \in P$ we have that $|\pi| < 1$ if $|\overline{\alpha}| \geq \frac{\sqrt{2}+1}{2}$* and $sgn(a) = sgn(b)$, *and $|\overline{\pi}| < 1$ if $|\alpha| \geq \frac{\sqrt{2}+1}{2}$* and $sgn(a) \neq sgn(b)$.

III. *If there exists such an $f \in E_\alpha$ for which either $0 \neq \pi = \pi\alpha + f$, or $0 \neq \pi = (-\pi)\alpha + f$ holds, i.e. $\pi = p + q\sqrt{D} \in P \setminus \{0\}$ and $|\pi| > 1$ if $sgn(a) = sgn(b)$ or $|\overline{\pi}| > 1$ if $sgn(a) \neq sgn(b)$, then there must exist such an $f' \in E_\alpha$ for which we can find such a π' which satisfies either $\pi' = \pi'\alpha + f'$, or $\pi' = (-\pi')\alpha + f'$, and $i(\pi) = i(\pi')$.*

Note. In fact the Assertion III states that "beside" the periodic elements with large modulus there should be another one on the integer lattice.

2.4. Preparing the proof

Remark 1. Since during the proof we never specify the value of (ε, δ), therefore our proof will hold true for each $E_\alpha^{(\varepsilon,\delta)}$. Observe that if d is an odd number, then we get the same digit set for arbitrary value of (ε, δ). If d is an even number addition we can say that

$E_{-\alpha}^{(\varepsilon,\delta)} = E_\alpha^{(-\varepsilon,-\delta)}$, because $f(-\bar{\alpha}) = -r - s\sqrt{2}$, and

$\overline{E_\alpha^{(\varepsilon,\delta)}} = E_{\overline{\alpha}}^{(\varepsilon,-\delta)}$, because $\overline{f}\alpha = r - s\sqrt{2}$.

Thus, if we have proved our assertions in case $|\alpha| > |\overline{\alpha}|$, we get the theorem in case $|\alpha| < |\overline{\alpha}|$ simply by reversing the roles of $|\alpha|, |\overline{\alpha}|$ and those of $E_\alpha^{(\varepsilon,\delta)}, E_{\overline{\alpha}}^{(\varepsilon,-\delta)}$. Further we assume that $|\alpha| > 2$ and $1 < |\overline{\alpha}| < 2$ and an arbitrary element of the set $\{E_\alpha^{(1,1)}, E_\alpha^{(1,-1)}, E_\alpha^{(-1,1)}, E_\alpha^{(-1,-1)}\}$ will be denoted by E_α.

Remark 2. In [10] we delt with the case $|\alpha|, |\overline{\alpha}| \geq 2$ and now we observe that if $|\alpha|, |\overline{\alpha}| < 2$, then $\alpha = -\overline{\alpha} = \sqrt{2}$ and $\sqrt{2} - 1$ is a unit, therefore we do not need to investigate these two cases. Thus we further assume that $\min(|\alpha|, |\overline{\alpha}|) < 2$ and $\max(|\alpha|, |\overline{\alpha}|) > 2$.

Remark 3. In [10] we proved that for each $\pi = \left(p + q\sqrt{2}\right) \in P$ $|\pi| < \sqrt{2}$ is true. This implies that $sgn(p) \neq sgn(q)$, otherwise $|\pi| \geq \sqrt{2} + 1$ would be valid. Since the proofs of cases $p > 0$ and $p < 0$ are the same, we can avoid the proof of case $p > 0$ and in what follows we assume that for arbitrary $\pi = \left(p + q\sqrt{2}\right) \in P \backslash \{0\}$ $p < 0$ holds. Then, naturally, the inequalities $q > 0$ and $\overline{\pi} < 0$ are always maintained.

Remark 4. We would point out that in [10] we gave the number systems in the case of $|\alpha|, |\overline{\alpha}| \leq 6 + 3\sqrt{2}$ for each possible $\alpha \in I$. It can be easily checked that our theorem obviously holds for the values $\alpha, \overline{\alpha}$, thus in the following we can assume, without losing generality that $|\alpha| > 6 + 3\sqrt{2}$.

Definition. $x = \dfrac{|d|}{2} - \max\left(|a|, |b|\sqrt{2}\right)$.

The number x plays important role in the estimating of the modulus of the periodic elements. Let us now examine the connection between x and $\overline{\alpha}$. $|\alpha| =$

$= |a| + |b| \sqrt{2} = \frac{|d|}{2} + \frac{|d|}{2} - 2x - |\overline{\alpha}|$, from which $|d| = |\overline{\alpha}| |\alpha| = |\overline{\alpha}| (|d| - 2x - |\overline{\alpha}|) = |\overline{\alpha}| |d| - 2x |\overline{\alpha}| - |\overline{\alpha}|^2$, thus

$$x = \frac{(|\overline{\alpha}| - 1) |d| - |\overline{\alpha}|^2}{2 |\overline{\alpha}|}. \tag{3}$$

3. The proof of our Theorem

3.1. The proof of Assertion I

In order to prove the first part of our theorem we have to investigate two cases.

Case 1. Let $sgn(\alpha) = sgn(\overline{\alpha})$, i. e. $sgn(a + b\sqrt{2}) = sgn(a - b\sqrt{2})$. This implies that $d > 0$ and $|a| > |b| \sqrt{2}$, thus $|\overline{\alpha}| = |a| - |b| \sqrt{2}$. We have got two cases to examine:

Case 1/a. We assume that $\alpha, \overline{\alpha} > 0$. In this case $a, b > 0$ and we can put forward the following relation

$$0 < (\alpha - 1) (\overline{\alpha} - 1) = d - 2a + 1,$$

because $\alpha - 1, \overline{\alpha} - 1$ are two positive numbers. Let us realise that

$$(\alpha - 1) (\overline{\alpha} - 1) \geq 2,$$

because $(\alpha - 1) (\overline{\alpha} - 1) = 1$ would imply that $\alpha - 1$ is a unit. We get that $d - 2a + 1 \geq 2$, from which $\frac{d}{2} \geq a + \frac{1}{2} > b\sqrt{2}$ follows.

Case 1/b. We assume now that $\alpha, \overline{\alpha} < 0$. In this case $a, b < 0$ and we can observe the following relation

$$0 < (\alpha + 1) (\overline{\alpha} + 1) = d + 2a + 1,$$

because $\alpha + 1, \overline{\alpha} + 1$ are two negative numbers. Observe that $(\alpha + 1) (\overline{\alpha} + 1) \geq 2$, because $(\alpha + 1) (\overline{\alpha} + 1) = 1$ would imply that $\alpha + 1$ is a unit. We get that $d + 2a + 1 \geq 2$, from which $\frac{d}{2} \geq -a + \frac{1}{2} > -b\sqrt{2}$ follows. Thus we have proved that $\frac{|d|}{2} > |a|, |b| \sqrt{2}$ in case $sgn(\alpha) = sgn(\overline{\alpha})$.

Case 2. Let $sgn(\alpha) \neq sgn(\overline{\alpha})$ hold true, i. e. $sgn(a + b\sqrt{2}) \neq sgn(a - b\sqrt{2})$. This implies that $d < 0$, $|a| < |b| \sqrt{2}$, thus $|\overline{\alpha}| = |b| \sqrt{2} - |a|$.

Then $sgn(\alpha) = sgn(a) = sgn(b) \neq sgn(\overline{\alpha})$ holds, which implies that $|\alpha| = \left|a + b\sqrt{2}\right| = |2a - \overline{\alpha}| = 2|a| + |\overline{\alpha}|$. From this we get $|d| = |\alpha||\overline{\alpha}| > |\alpha| > 2|a| + 1$, thus $\frac{|d|}{2} > |a|$. On the other hand, $|\overline{\alpha}| = |b|\sqrt{2} - |a| < 2$, i.e. $|b|\sqrt{2} < |a| + 2$. This implies that if $|d| \geq 2|a| + 4$ then $\frac{|d|}{2} \geq |a| + 2 > |b|\sqrt{2}$. We have to investigate just two cases, namely, either $|d| = 2|a| + 2$ or $|d| = 2|a| + 3$.

Case 2/a. We assume first that $|d| = 2|a| + 2$. If $a > 0$, that is $d = -2a - 2$, then $(\alpha + 1)(\overline{\alpha} + 1) = d + 2a + 1 = -1$. If $a < 0$, that is $d = 2a - 2$, then $(\alpha - 1)(\overline{\alpha} - 1) = d - 2a + 1 = -1$. We have arrived at the conclusion that in this case $\alpha - 1$, or $\alpha + 1$ is a unit, thus $|d| = 2|a| + 2$ never holds.

Case 2/b. Let us assume now that $|d| = 2|a| + 3$. Let us observe that if $|\overline{\alpha}| = |b|\sqrt{2} - |a| < \frac{3}{2}$, then $\frac{|d|}{2} = \frac{2|a|+3}{2} = |a| + \frac{3}{2} > |b|\sqrt{2}$, thus we have completed the proof. In the opposite case, if $|\overline{\alpha}| = |b|\sqrt{2} - |a| \geq \frac{3}{2}$, then we can write that $(|\alpha| + 1)(|\overline{\alpha}| - 1) = |d| + |\overline{\alpha}| - |\alpha| - 1 = |d| + |b|\sqrt{2} - |a| - |b|\sqrt{2} - |a| - 1 = |d| - 2|a| - 1 = 2|a| + 3 - 2|a| - 1 = 2$. From this we get that $|\alpha| + 1 = \frac{2}{|\overline{\alpha}|-1}$. Naturally, this is impossible because $|\alpha| > 10$. We have finished the Assertion I.

Corollary 1. $\pm 1 \in E_\alpha$. Let us assume that $f = 1$. From this we get that $f\overline{\alpha} = \overline{\alpha} = a - b\sqrt{2} = r + s\sqrt{2}$. Thus $|r| = |a| < \frac{|d|}{2}$, and $|s| = |b| < \frac{|d|}{2}$. We can obtain the same result for $f = -1$.

3.2. The proof of Assertion II

We assume now that $|\overline{\alpha}| \geq \frac{\sqrt{2}+1}{2}$ and let $\pi = p + q\sqrt{2}$ be an arbitrary periodic number. Then in according to [10]

$$\text{either} \quad \pi = \pi\alpha + f, \quad \text{or} \quad \pi = -\pi\alpha + f \tag{4}$$

holds for an $f \in E_\alpha$. From (4) we get that

$$\text{either} \quad f = \pi(1 - \alpha), \quad \text{or} \quad f = \pi(1 + \alpha).$$

Thus

$$|f| \geq |\pi|(|\alpha| - 1),$$

from which it follows that

$$|\pi| \leq \frac{|f|}{|\alpha| - 1} = \frac{|f\overline{\alpha}|}{|d| - |\overline{\alpha}|}.$$

In a similar way, we get for $\overline{\pi}$:

$$|\overline{\pi}| \leq \frac{|\overline{f}\alpha|}{|d| - |\alpha|}.$$

Now we can estimate the value of q:

$$|\pi - \overline{\pi}| = 2\,|q|\,\sqrt{2} \leq \frac{|f\overline{\alpha}|}{|d| - |\overline{\alpha}|} + \frac{|\overline{f}\alpha|}{|d| - |\alpha|}. \tag{5}$$

We have to examine two cases.

Case 1. Let $|f\overline{\alpha}| = |r| + |s|\,\sqrt{2}$ hold true. Then

$$|\overline{f}\alpha| = \Big||r| - |s|\,\sqrt{2}\Big| \leq \frac{|d|}{2}\sqrt{2}$$

follows. From this we get that

$$\frac{|f\overline{\alpha}|}{|d| - |\overline{\alpha}|} \leq \frac{\sqrt{2}+1}{2}\,\frac{|d|}{|d| - |\overline{\alpha}|} = \frac{\sqrt{2}+1}{2}\,\frac{|\alpha|}{|\alpha| - 1} < \sqrt{2},$$

and

$$\frac{|\overline{f}\alpha|}{|d| - |\alpha|} \leq \frac{\sqrt{2}}{2}\,\frac{|d|}{|d| - |\alpha|} = \frac{\sqrt{2}}{2}\,\frac{|\overline{\alpha}|}{|\overline{\alpha}| - 1},$$

because of the condition of Assertion II, we get that

$$\frac{\sqrt{2}}{2}\,\frac{|\overline{\alpha}|}{|\overline{\alpha}| - 1} \leq \frac{\sqrt{2}}{2}\,\frac{\frac{\sqrt{2}+1}{2}}{\frac{\sqrt{2}+1}{2} - 1} = \frac{3}{2}\sqrt{2} + 2.$$

If we substitute this value to (5), we can see that

$$2\,|q|\,\sqrt{2} \leq \sqrt{2} + \frac{3}{2}\sqrt{2} + 2 = \frac{5}{2}\sqrt{2} + 2,$$

and this implies

$$|q| < 2.$$

Let us observe that if $q = 0$ were valid, we would get that either $|\pi| > \sqrt{2}$ or $\pi \in E_\alpha$, because ± 1 is a digit, which follows from Corollary 1. Thus exclusively the $|q| = 1$ statement can be true. Then $|p| < 3$ must hold, since $|\pi| < \sqrt{2}$. We can check easily that in this case $|\pi| < 1$.

Case 2. Now we assume that $|f\overline{\alpha}| = \left||r| - |s|\sqrt{2}\right|$. Since $|\alpha| > 10$, we get that

$$|\pi| = \frac{|f\overline{\alpha}|}{|d| - |\overline{\alpha}|} = \frac{\left||r| - |s|\sqrt{2}\right|}{|d| - |\overline{\alpha}|} \leq \frac{\sqrt{2}}{2}\frac{|d|}{|d| - |\overline{\alpha}|} = \frac{\sqrt{2}}{2}\frac{|\alpha|}{|\alpha| - 1} < 1.$$

We have completed the proof of Assertion II.

3.3. The proof of Assertion III

Lemma 1. *In P there are at most two periodic numbers with identical irrational parts. If there exist $\pi = p + q\sqrt{D} \in P$, $\pi' = p' + q\sqrt{D} \in P$ and $\pi \neq \pi'$, then for $n = \pi' - \pi$:*
(i) $|n| = |p - p'| = 1$,
(ii) $sgn(r') \neq sgn(r)$.

Proof. Case 1. We assume that $G(P)$ is the disjoint union of loops. Then we know from [10] that for an arbitrary $\pi \in \mathbb{Q}\left(\sqrt{2}\right)$, $\pi \in P$ means that $\pi = \pi\alpha + f$, where $f \in E_\alpha$. In addition we assume that there exists $\pi' = \pi'\alpha + f'$ such that $q = q'$ and $p' \neq p$. Then $p' = p + n$ for some $n \in \mathbb{Z}$. We can write:

$$f' = \pi'(1 - \alpha) = (\pi + n)(1 - \alpha) = f + n - n\alpha.$$

From this we get

$$f'\overline{\alpha} = r' + s'\sqrt{2} = f\overline{\alpha} + n\overline{\alpha} - nd,$$

which implies that

$$r' = r + n(a - d),$$
$$s' = s + nb.$$

Let us observe that $\frac{|d|}{2} > |a|$ implies that $|a - d| > \frac{|d|}{2}$. Now we assume that $sgn(r) = sgn\left(r'\right)$. Then $|r - r'| = ||r| - |r'|| > |n|\frac{|d|}{2}$ which implies that $n = 0$. This is a contradiction, thus we reached that $sgn(r) \neq sgn\left(r'\right)$. Then $|d| \geq |r - r'| = |r| + |r'| > |n|\frac{|d|}{2}$. From this we can see that $|n| = 1$. We have finished the proof of Lemma 1 in the Case 1.

Case 2. In this case $P \backslash \{0\}$ contains exclusively such elements π for which

$$\pi = (-\pi)\alpha + f.$$

Here $\overline{\alpha} < 0$ holds, therefore $sgn(a) \neq sgn(d)$, thus we get that

$$r' = r + n\,(a + d)\,,$$
$$s' = s - nb.$$

The proof is the same as the above one, therefore we will not go into further details. We have finished the proof of Lemma 1.

In the rest of this paper we limit our discussion to the case $\alpha, \overline{\alpha} > 0$. The proofs of the other cases can be completed in a similar way, therefore, we do not present them here, and from now on we assume that $\alpha, \overline{\alpha} > 0$. In addition, let us assume that there exists such a $\pi = p + q\sqrt{D} \in P \setminus \{0\}$ for which $|\pi| > 1$. We have to examine two cases.

Case 1. Let $\pi < 0$ hold true. We can write the following relations

$$f = \pi\,(1 - \alpha) > 0,$$
$$\overline{f} = \overline{\pi}\,(1 - \overline{\alpha}) > 0,$$

from which

$$f\overline{\alpha} = r + s\sqrt{2} > 0,$$
$$\overline{f}\alpha = r - s\sqrt{2} > 0$$

follows. Thus we can see that $r > 0$ és $|r| > |s|\,\sqrt{2}$. We have two cases to examine:

If $s > 0$, then

$$|f\overline{\alpha}| = |\pi|\,(|d| - |\overline{\alpha}|) = |r| - |s|\sqrt{2} < \frac{|d|}{2}$$

holds and from this we get that

$$|\pi| < \frac{|d|}{2\,(|d| - |\overline{\alpha}|)} < 1.$$

If $s < 0$, then

$$\left|\overline{f}\alpha\right| = |\overline{\pi}|\,(|d| - |\alpha|) = |\overline{\pi}|\,(2x + |\overline{\alpha}|) = |r| - |s|\sqrt{2},$$

thus

$$|s|\sqrt{2} = |r| - |\overline{\pi}|\,(2x + |\overline{\alpha}|)\,.$$

We know that

$$|f\overline{\alpha}| = |\pi| \left(|d| - |\overline{\alpha}|\right) = |r| + |s|\sqrt{2} = 2|r| - |\overline{\pi}| \left(2x + |\overline{\alpha}|\right),$$

therefore we get for π that

$$|\pi| \leq \frac{|d| - |\overline{\pi}| \left(2x + |\overline{\alpha}|\right)}{|d| - |\overline{\alpha}|} < 1,$$

because in this case $|\overline{\pi}| \geq \sqrt{2} + 1$. It is obvious that $\pi < -1$ never holds, thus we proved Assertion III in case 1.

Case 2. Now let us assume that $\pi > 0$. Then the following inequalities are valid:

$$f = \pi \left(1 - \alpha\right) < 0,$$
$$\overline{f} = \overline{\pi} \left(1 - \overline{\alpha}\right) > 0,$$

from which

$$f\overline{\alpha} = r + s\sqrt{2} < 0,$$
$$\overline{f}\alpha = r - s\sqrt{2} > 0$$

follows. Thus we can observe easily that

(6) $$r, s < 0 \quad \text{and} \quad |r| < |s|\sqrt{2}.$$

Let us write f in the form $f = k + l\sqrt{2}$. In [9] we proved that $sgn(k) = sgn(l)$, from which

(7) $$|l|\sqrt{2} > |k| \quad \text{and} \quad k, l < 0$$

follows.

What can we say about the modulus of k and l? Since $|\pi| > 1$ holds, therefore

(8) $$|f| = |k| + |l|\sqrt{2} > |1 - \alpha| = |\alpha| - 1$$

is also true. The question is whether the following inequalities hold true:

(9) $$|k| > |a| - 1 \quad \text{and} \quad |l| > |b|.$$

The answer is yes, since if we assume indirectly that $|l| \leq |b|$, we get the relation

$$|k| > |a| - 1$$

because of (8). But then

$$|k| > |a| - 1 > |b|\sqrt{2} \geq |l|\sqrt{2}.$$

This contradicts (7), thus $|l| > |b|$ must be true.

On the other hand l, b are rational integers, therefore

$$|l|\sqrt{2} \geq |b|\sqrt{2} + \sqrt{2}. \tag{10}$$

We can write from (6) that

$$\left|\overline{f}\alpha\right| = |\overline{\pi}|\left(|d| - |\alpha|\right) = |s|\sqrt{2} - |r|,$$

from which

$$\left|\overline{f}\right| = \frac{|\overline{\pi}|\left(|d| - |\alpha|\right)}{|\alpha|} = |\overline{\pi}|\left(|\overline{\alpha}| - 1\right) = \frac{|s|\sqrt{2} - |r|}{|\alpha|} \leq \frac{\frac{|d|}{2}\sqrt{2}}{|\alpha|} \leq \frac{\sqrt{2}}{2}|\overline{\alpha}|$$

is obtained, and because of Assertion II $|\overline{\alpha}| < \frac{\sqrt{2}+1}{2}$ holds true. Thus

$$\left|\overline{f}\right| = |l|\sqrt{2} - |k| \leq \frac{\sqrt{2}}{2}|\overline{\alpha}| < \frac{\sqrt{2}}{2}\frac{\sqrt{2}+1}{2} = \frac{2+\sqrt{2}}{4}. \tag*{[11]}$$

(10) and (11) impliy the next relation for k:

$$|k| = |l|\sqrt{2} - \left|\overline{f}\right| > |b|\sqrt{2} + \sqrt{2} - \frac{2+\sqrt{2}}{4} = |a| - |\overline{\alpha}| + \sqrt{2} - \frac{2+\sqrt{2}}{4}.$$

We get with substitution that

$$|k| > |a| - \frac{\sqrt{2}+1}{2} + \sqrt{2} - \frac{2+\sqrt{2}}{4},$$

from which we can compute easily that

$$|k| > |a| - 1.$$

Why do the inequalities (9) play very important role in the proof? It will be clear if we consider the $f' = k' + l'\sqrt{2}$ integer for which $f' = f + \alpha - 1$ is valid. It is sure that f' exists and because of (9)

$$|k'| = |k| - |a| + 1,$$
$$|l'| = |l| - |b|.$$

Let us consider the following product

$$f'\overline{\alpha} = \left(k + l\sqrt{2} + a + b\sqrt{2} - 1\right)\left(a - b\sqrt{2}\right) =$$
$$= ak + al\sqrt{2} + a^2 + ab\sqrt{2} - a - bk\sqrt{2} - 2bl - ab\sqrt{2} - 2b^2 + b\sqrt{2}.$$

We should recall that $d = \alpha\overline{\alpha} = a^2 - 2b^2$, and $r = ak - 2bl, s = al - bk$, thus

$$\begin{aligned} r' &= ak - 2bl + a^2 - 2b^2 - a = r + d - a, \\ s' &= al - bk + b = s + b. \end{aligned} \tag{12}$$

It is clear that $r' > 0$, and we know from (6) that $s, r < 0$. It is true in addition that $|r| > x$, because $|f'\overline{\alpha}| = |\pi|\,(|d| - |\overline{\alpha}|) = |r| + |s|\,\sqrt{2}$ and from this

$$\min(|r|) \geq 2x + |\alpha| - \frac{|d|}{2}\sqrt{2} > 2x + \frac{|d|}{\frac{\sqrt{2}+1}{2}} - \frac{|d|}{2}\sqrt{2} > 2x$$

follows. We have got that

$$|r'| < \frac{|d|}{2},$$
$$|s'| < \frac{|d|}{2},$$

because of (12). What does it mean? This means that $f' \in E_\alpha$, therefore the existence of f implies the existence of f'. It means that if f exists, then there exists an f' digit in E_α, for which

$$f' = f + \alpha - 1.$$

But then

$$f' = \pi\,(1 - \alpha) + \alpha - 1 = (\pi - 1)\,(1 - \alpha) = \pi'\,(1 - \alpha)\,,$$

i. e. $\pi' \in P$. Thus we have finished the proof of Assertion III and the Theorem.

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