

INVESTIGATION OF A CONTINUOUS CYCLIC-WAITING PROBLEM BY SIMULATION

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1. Introduction

In practice one often meets so-called queueing systems in which customers arrive, after some waiting they get the necessary service and then leave the system. Because of their importance these problems constitute a special field of probability theory, depending on the inter-arrival and service time distributions, the number of servers and service discipline lead to different mathematical problems and form an important area of applied mathematics, the theory of queues.

For the investigation of queueing systems one has two possibilities. If the system under consideration is simple enough, then it allows a mathematical description, and one can construct a model which may be examined by exact analytical methods. If the system is too complex or its features are too specific, there remains the method of simulation. In the investigation of real systems by simulation verification and validation play an essential role. One way is to use a - possibly simpler - analytical model for which we can obtain exact results, and to compare its characteristics with the simulation one. The parallel use of analytical and simulation methods usually gives enough information about the behaviour of such systems.

In conventional queueing systems the service process runs continuously, after having completed the service of a customer, we immediately take the next one. In systems with vacation after the service e.g. a repair is required, it is a random variable whose distribution does not depend on the service. In this paper we consider a model describing the landing of airplanes. Our systems are different from the above ones, the starting moment of service is determined by the moment of the completion of previous service and the moment of the arrival of the actual customer. Such systems were analytically investigated in the case of Poisson arrivals and exponentially distributed service time in [3],

uniform service time in [4], and for discrete time case in [5,6]. Here we compare the analytical results with data obtained by means of simulation.

2. Formulation of our problem

The problem which is in the focus of our investigation can be described in the following way. There is an airport where the entering airplanes put a landing request to the control tower upon arrival in the airside. Provided there is free system, i.e. the entering entity can be serviced at the moment of the request, the airplane can start landing. However, if the server is busy, i.e. a formerly arrived plane has not accomplished landing yet or other planes are already queueing for being serviced, then the incoming plane starts to circular manoeuvre. The radius of the circle is fixed in a way that it takes the airplane T cycle time to be above the runway again, i.e. the airplane can only put further landing request to the control tower at every nT moment after arrival, where $n \in \mathbb{N}$. Naturally, the request can only be serviced if there is no airplane queueing before it. The reception and service of the incoming planes follow the FIFO rule, according to which the earlier arriving planes are given landing permission earlier. Obviously, this system only operates properly if there are not many planes cyclic queueing.

3. Theoretical results

The above described problem has been investigated by L. Lakatos in several papers. Here we shortly formulate his results to which we can compare our data obtained by means of simulation.

Let us consider a queueing system in which the arriving requests form a Poisson process with parameter λ , the service time distribution is exponential with parameter μ (uniformly distributed on the interval $[c, d]$, where c and d are the multiples of T), and the service of a request can be started only at the moment of its arrival or at moments differing from it by the multiples of cycle time T according to the FIFO rule. The described system will be investigated by means of the embedded Markov chain technique (see e.g. [1]). Let us define an embedded Markov chain whose states correspond to the number of requests in the system at moments just before starting the service of a request $t_k - 0$

(where t_k is the moment of beginning of service of the k -th one). The matrix of transition probabilities for this chain has the form

$$\begin{bmatrix} a_0 & a_1 & a_2 & a_3 & \cdots \\ a_0 & a_1 & a_2 & a_3 & \cdots \\ 0 & b_0 & b_1 & b_2 & \cdots \\ 0 & 0 & b_0 & b_1 & \cdots \\ \vdots & & \vdots & \vdots & \ddots \end{bmatrix},$$

whose elements are determined by the generating functions: in the case of exponential service time distribution

$$(1) \quad A(z) = \sum_{i=0}^{\infty} a_i z^i = \frac{\mu}{\lambda + \mu} + \frac{\lambda}{\lambda + \mu} z \frac{(1 - e^{-\mu T}) e^{-\lambda(1-z)T}}{1 - e^{-(\lambda(1-z) + \mu)T}},$$

$$(2) \quad B(z) = \sum_{i=0}^{\infty} b_i z^i =$$

$$\frac{1}{(1 - e^{-\lambda T})(1 - e^{-(\lambda(1-z) + \mu)T})} \left\{ \frac{1}{2 - z} (1 - e^{-\lambda(2-z)T}) (1 - e^{-(\lambda(1-z) + \mu)T}) - \frac{\lambda}{\lambda(2-z) + \mu} (1 - e^{-(\lambda(2-z) + \mu)T}) (1 - e^{-(1-z)T}) \right\},$$

in the case of uniform service time distribution

$$(3) \quad A(z) = \sum_{i=0}^{\infty} a_i z^i = \frac{e^{-\lambda c} - e^{-\lambda d}}{\lambda(d - c)} + \frac{z}{\lambda(d - c)} (e^{-\lambda c} - e^{-\lambda d}) (1 - e^{-\lambda T}) e^{\lambda T z} \frac{1 - e^{\lambda c z}}{1 - e^{\lambda T z}} + \frac{z}{\lambda(d - c)} \lambda T e^{-\lambda(1-z)T} \frac{e^{-\lambda(1-z)c} - e^{-\lambda(1-z)d}}{1 - e^{-\lambda(1-z)T}} + \frac{z}{\lambda(d - c)} e^{-\lambda d} (1 - e^{\lambda T}) e^{-\lambda(1-z)T} \frac{e^{\lambda c z} - e^{\lambda d z}}{1 - e^{\lambda T z}},$$

$$(4) \quad B(z) = \sum_{i=0}^{\infty} b_i z^i = \frac{e^{-\lambda(1-z)c} - e^{-\lambda(1-z)d}}{[1 - e^{-\lambda(1-z)T}](d - c)(1 - e^{-\lambda T})} \times \left\{ \frac{1}{\lambda(2-z)^2} [1 - e^{-\lambda(2-z)T} + e^{-\lambda(3-2z)T} - e^{-\lambda(1-z)T}] + \frac{T}{2-z} [e^{-\lambda(1-z)T} - e^{-\lambda(2-z)T}] \right\}.$$

The generating function of ergodic distribution $P(z) = \sum_{i=0}^{\infty} p_i z^i$ for this chain has the form

$$(5) \quad P(z) = p_0 \frac{B(z)(\lambda z + \mu) - zA(z)(\lambda + \mu)}{\mu(B(z) - z)},$$

where

$$(6) \quad p_0 = 1 - \frac{\lambda}{\lambda + \mu} \frac{1 - e^{-(\lambda + \mu)T}}{e^{-\lambda T}(1 - e^{-\mu T})}$$

in exponential case and

$$(7) \quad P(z) = p_0 \frac{zA(z) + (a_0 z - a_0 - z)B(z)}{a_0(z - B(z))},$$

where

$$(8) \quad p_0 = \frac{e^{\lambda T} - 1}{\lambda T} \left(1 - \frac{\lambda(c + d + T)}{2} \right)$$

if uniform.

The condition of the existence of ergodic distribution is the fulfilment of inequalities

$$(9) \quad \frac{\lambda}{\mu} < \frac{e^{-\lambda T}(1 - e^{-\mu T})}{1 - e^{-\lambda T}},$$

and

$$(10) \quad \frac{\lambda(c + d + T)}{2} < 1.$$

In order to support the analytical results on the basis of numerical computation, we have produced a computer program modelling the above mentioned cyclic-waiting system. With this program we have generated random data in accordance with the conditions of the examined system. In this paper we present how the values, calculated by the program from the random data, approximate the exact values acquired from the theoretical formulas. The computations have been completed for both exponential and uniform distribution of service times.

4. Computed results

We carried out the experiments with different T and λ values. In the case of exponential distribution service times we used μ , while in the case of uniform distribution service times c and d parameters. For every fixed T , λ , μ , c , d we did 500 independent experiments with different computer generated arrival and service times. On the basis of the above, we examined the probabilities in free systems of having $0, 1, 2 \dots$ airplanes (marked $p_0, p_1, p_2 \dots$ respectively) in the queue at the starting moments of services. In every case we recorded the results in a table where $p_0, p_1, p_2 \dots$ are given in columns, and rows show the number of the incoming airplanes.

For lack of place, we only include two tables here: Figure 1 indicating exponential distribution service times, where $\lambda = 3, \mu = 6, T = 0.1$ and Figure 4 indicating uniform distribution service times, where $\lambda = 4, c = 0.05, d = 0.15, T = 0.1$.

The diagrams in Figure 2, 3, 5 show the calculated results where the horizontal lines from top to bottom express the probabilities p_0, p_1, p_2, p_3 , whose exact values can be seen in the grey box in the upper line. Below the grey box we also give p_4, p_5, p_6, p_7 values which are not included in the diagrams. In the lower lines one can see the average values obtained from numerical results shown on diagrams. The average value for p_0 is a bit increased, for the other probabilities a bit decreased since at their computation the initial values were included, too.

Considering not more than 500 independent experiments and the cases 60 arriving airplanes shows that the computed results clearly approximate the exact values.

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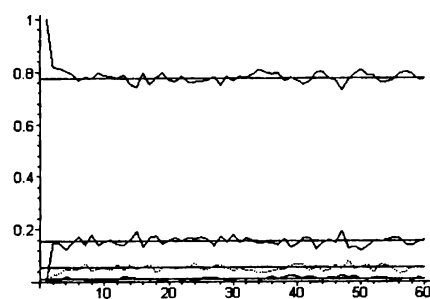
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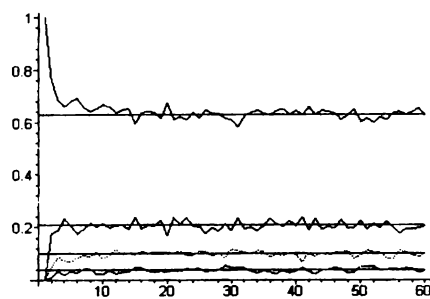
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	p0	p1	p2	p3	p4	p5	p6	p7	p8	p9	p10	p11	p12	p13	p14	p15	p16	p17
1	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
2	0.638	0.200	0.112	0.028	0.018	0.000	0.004	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
3	0.544	0.236	0.136	0.044	0.022	0.006	0.000	0.002	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
4	0.476	0.228	0.178	0.064	0.030	0.012	0.006	0.004	0.002	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
5	0.482	0.258	0.140	0.088	0.050	0.016	0.008	0.004	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
6	0.502	0.198	0.152	0.076	0.040	0.018	0.008	0.004	0.000	0.002	0.002	0.000	0.000	0.000	0.000	0.000	0.000	0.000
7	0.470	0.192	0.174	0.094	0.036	0.020	0.004	0.002	0.000	0.002	0.002	0.000	0.000	0.000	0.000	0.000	0.000	0.000
8	0.410	0.250	0.162	0.086	0.048	0.034	0.006	0.002	0.000	0.002	0.002	0.000	0.000	0.000	0.000	0.000	0.000	0.000
9	0.428	0.218	0.170	0.078	0.042	0.026	0.006	0.004	0.000	0.002	0.002	0.000	0.000	0.000	0.000	0.000	0.000	0.000
10	0.450	0.214	0.136	0.110	0.040	0.034	0.006	0.006	0.000	0.002	0.002	0.000	0.000	0.000	0.000	0.000	0.000	0.000
11	0.442	0.222	0.162	0.082	0.042	0.018	0.002	0.002	0.000	0.002	0.002	0.000	0.000	0.000	0.000	0.000	0.000	0.000
12	0.422	0.244	0.160	0.086	0.040	0.026	0.016	0.002	0.000	0.002	0.002	0.000	0.000	0.000	0.000	0.000	0.000	0.000
13	0.420	0.258	0.140	0.074	0.044	0.032	0.012	0.004	0.002	0.004	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
14	0.438	0.198	0.164	0.084	0.058	0.024	0.012	0.014	0.000	0.006	0.002	0.000	0.000	0.000	0.000	0.000	0.000	0.000
15	0.450	0.202	0.164	0.092	0.058	0.026	0.014	0.014	0.000	0.006	0.002	0.000	0.000	0.000	0.000	0.000	0.000	0.000
16	0.448	0.186	0.164	0.072	0.066	0.024	0.014	0.014	0.006	0.006	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
17	0.446	0.184	0.158	0.068	0.058	0.028	0.010	0.016	0.004	0.004	0.002	0.002	0.000	0.000	0.000	0.000	0.000	0.000
18	0.432	0.216	0.148	0.080	0.044	0.034	0.020	0.010	0.006	0.002	0.002	0.006	0.000	0.000	0.000	0.000	0.000	0.000
19	0.482	0.248	0.152	0.088	0.042	0.020	0.006	0.004	0.002	0.002	0.002	0.000	0.000	0.000	0.000	0.000	0.000	0.000
20	0.406	0.210	0.194	0.076	0.044	0.032	0.010	0.008	0.008	0.008	0.000	0.004	0.000	0.000	0.000	0.000	0.000	0.000
21	0.398	0.238	0.158	0.078	0.058	0.030	0.016	0.018	0.004	0.004	0.002	0.004	0.000	0.000	0.000	0.000	0.000	0.000
22	0.406	0.222	0.140	0.102	0.060	0.016	0.026	0.008	0.008	0.006	0.004	0.000	0.000	0.002	0.000	0.000	0.000	0.000
23	0.426	0.184	0.158	0.108	0.032	0.038	0.016	0.020	0.008	0.004	0.002	0.000	0.002	0.000	0.000	0.000	0.000	0.000
24	0.384	0.204	0.180	0.082	0.060	0.038	0.022	0.018	0.006	0.002	0.002	0.002	0.000	0.000	0.000	0.000	0.000	0.000
25	0.378	0.214	0.158	0.088	0.034	0.018	0.018	0.018	0.006	0.002	0.002	0.002	0.000	0.000	0.000	0.000	0.000	0.000
26	0.400	0.210	0.158	0.092	0.050	0.042	0.018	0.018	0.006	0.002	0.002	0.002	0.000	0.000	0.000	0.000	0.000	0.000
27	0.404	0.210	0.140	0.100	0.052	0.038	0.028	0.014	0.010	0.004	0.002	0.002	0.000	0.000	0.000	0.000	0.000	0.000
28	0.404	0.190	0.156	0.100	0.054	0.038	0.028	0.014	0.010	0.002	0.002	0.002	0.000	0.000	0.000	0.000	0.000	0.000
29	0.428	0.202	0.130	0.088	0.064	0.044	0.024	0.010	0.006	0.004	0.002	0.000	0.000	0.000	0.000	0.000	0.000	0.000
30	0.460	0.164	0.142	0.098	0.050	0.040	0.024	0.006	0.010	0.000	0.004	0.000	0.002	0.000	0.000	0.000	0.000	0.000
31	0.432	0.188	0.162	0.082	0.058	0.038	0.014	0.012	0.008	0.008	0.008	0.002	0.000	0.000	0.000	0.000	0.000	0.000
32	0.426	0.198	0.136	0.098	0.058	0.024	0.018	0.018	0.010	0.008	0.002	0.002	0.002	0.000	0.000	0.000	0.000	0.000
33	0.406	0.194	0.158	0.102	0.044	0.034	0.026	0.010	0.012	0.004	0.002	0.000	0.000	0.000	0.000	0.000	0.000	0.000
34	0.410	0.186	0.182	0.084	0.048	0.040	0.018	0.012	0.010	0.002	0.002	0.002	0.002	0.000	0.002	0.000	0.000	0.000
35	0.398	0.232	0.142	0.100	0.058	0.038	0.020	0.010	0.006	0.004	0.000	0.002	0.000	0.000	0.000	0.000	0.000	0.000
36	0.408	0.208	0.132	0.094	0.060	0.040	0.018	0.014	0.010	0.006	0.000	0.006	0.004	0.000	0.000	0.000	0.000	0.000
37	0.410	0.190	0.148	0.110	0.054	0.024	0.024	0.018	0.010	0.002	0.004	0.008	0.000	0.000	0.000	0.000	0.000	0.000
38	0.390	0.184	0.160	0.108	0.056	0.036	0.016	0.028	0.002	0.008	0.010	0.002	0.000	0.000	0.000	0.000	0.000	0.000
39	0.380	0.198	0.154	0.120	0.058	0.036	0.010	0.010	0.008	0.006	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
40	0.384	0.164	0.190	0.110	0.062	0.030	0.026	0.014	0.010	0.004	0.006	0.000	0.000	0.000	0.000	0.000	0.000	0.000
41	0.350	0.228	0.188	0.088	0.068	0.038	0.014	0.010	0.004	0.006	0.002	0.004	0.000	0.000	0.000	0.000	0.000	0.000
42	0.380	0.240	0.114	0.114	0.060	0.050	0.022	0.002	0.008	0.004	0.006	0.000	0.000	0.000	0.000	0.000	0.000	0.000
43	0.398	0.200	0.178	0.088	0.068	0.024	0.014	0.010	0.008	0.002	0.008	0.000	0.000	0.000	0.000	0.000	0.000	0.000
44	0.418	0.222	0.128	0.092	0.058	0.036	0.022	0.012	0.000	0.006	0.004	0.002	0.000	0.000	0.000	0.000	0.000	0.000
45	0.434	0.184	0.180	0.078	0.068	0.020	0.018	0.006	0.006	0.006	0.006	0.002	0.000	0.000	0.000	0.000	0.000	0.000
46	0.402	0.208	0.166	0.094	0.050	0.032	0.020	0.016	0.008	0.002	0.000	0.000	0.000	0.000	0.002	0.000	0.000	0.000
47	0.410	0.186	0.172	0.098	0.062	0.038	0.022	0.014	0.010	0.000	0.000	0.000	0.000	0.002	0.000	0.000	0.000	0.000
48	0.432	0.168	0.160	0.100	0.060	0.038	0.014	0.022	0.004	0.000	0.000	0.000	0.000	0.002	0.000	0.000	0.000	0.000
49	0.412	0.224	0.142	0.100	0.058	0.038	0.022	0.012	0.004	0.000	0.000	0.000	0.002	0.000	0.000	0.000	0.000	0.000
50	0.396	0.212	0.150	0.106	0.058	0.028	0.028	0.014	0.004	0.002	0.002	0.000	0.000	0.000	0.000	0.000	0.000	0.000
51	0.420	0.188	0.160	0.088	0.068	0.038	0.020	0.014	0.008	0.002	0.002	0.000	0.000	0.000	0.000	0.000	0.000	0.000
52	0.410	0.220	0.140	0.092	0.050	0.042	0.028	0.012	0.002	0.004	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
53	0.428	0.214	0.138	0.098	0.058	0.038	0.028	0.008	0.002	0.002	0.004	0.002	0.000	0.000	0.000	0.000	0.000	0.000
54	0.426	0.160	0.150	0.130	0.062	0.038	0.016	0.004	0.004	0.000	0.008	0.000	0.002	0.000	0.000	0.000	0.000	0.000
55	0.380	0.184	0.182	0.110	0.058	0.038	0.014	0.014	0.008	0.006	0.002	0.002	0.000	0.000	0.000	0.000	0.000	0.000
56	0.384	0.212	0.184	0.096	0.048	0.032	0.016	0.008	0.008	0.004	0.004	0.000	0.004	0.000	0.000	0.000	0.000	0.000
57	0.378	0.242	0.152	0.098	0.068	0.038	0.014	0.014	0.008	0.008	0.002	0.002	0.000	0.000	0.000	0.000	0.000	0.000
58	0.418	0.194	0.158	0.084	0.044	0.040	0.024	0.016	0.008	0.006	0.002	0.004	0.002	0.000	0.000	0.000	0.000	0.000
59	0.400	0.204	0.158	0.078	0.058	0.038	0.014	0.016	0.008	0.000	0.000	0.000	0.000	0.002	0.000	0.000	0.000	0.000
60	0.400	0.166	0.166	0.114	0.070	0.032	0.020	0.014	0.008	0.002	0.002	0.004	0.000	0.002	0.000	0.000	0.000	0.000

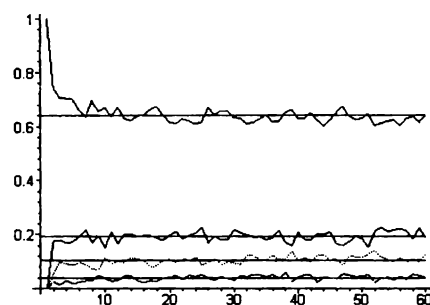
Figure 1.

T=0.1 $\lambda=2, \mu=10$

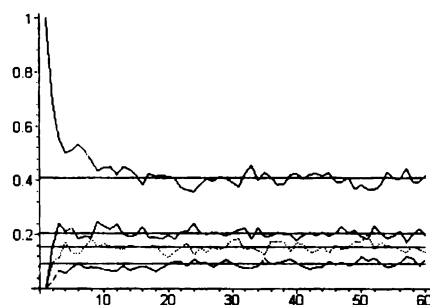
p_0	p_1	p_2	p_3
0.77495767	0.15499153	0.05322665	0.01296313
0.78266667	0.15076667	0.05063333	0.01276667
p_4	p_5	p_6	p_7
0.00297725	0.00068136	0.00015604	0.00003574
0.00220000	0.00063333	0.00026667	0.00000000

 $\lambda=2, \mu=6$

p_0	p_1	p_2	p_3
0.62732244	0.20910748	0.10125836	0.03891468
0.64343333	0.20146667	0.09533333	0.03573333
p_4	p_5	p_6	p_7
0.01461428	0.00548564	0.00205935	0.00077310
0.01390000	0.00670000	0.00250000	0.00076667

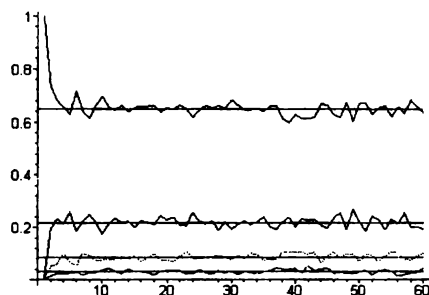
 $\lambda=3, \mu=10$

p_0	p_1	p_2	p_3
0.64150727	0.19245218	0.10315385	0.03985753
0.64666667	0.18900000	0.10306667	0.03880000
p_4	p_5	p_6	p_7
0.01460987	0.00534001	0.00195299	0.00071434
0.01513333	0.00500000	0.00163333	0.00043333

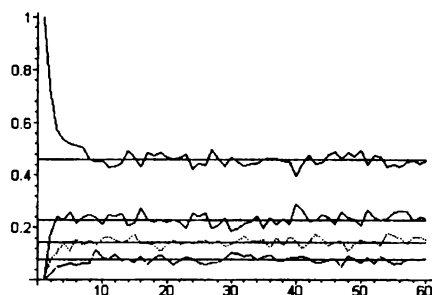
 $\lambda=3, \mu=6$

p_0	p_1	p_2	p_3
0.40819457	0.20409729	0.15475224	0.09416051
0.43336667	0.20166667	0.14966667	0.08683333
p_4	p_5	p_6	p_7
0.05611184	0.03342485	0.01991275	0.01186306
0.05120000	0.03066667	0.01926667	0.01136667

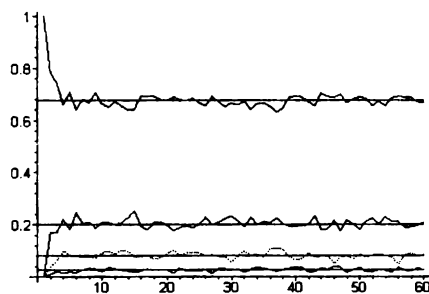
Figure 2.

T=0.05 $\lambda=2, \mu=6$

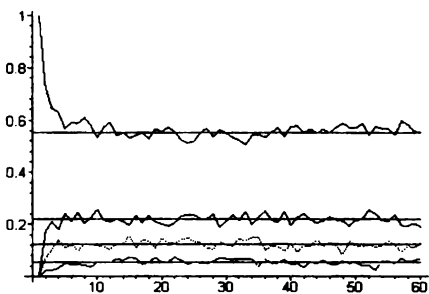
P_0	P_1	P_2	P_3
0.64855487	0.21618496	0.08743206	0.03098908
0.65573333	0.21486667	0.08470000	0.02983333
P_4	P_5	P_6	P_7
0.01091058	0.00384123	0.00135238	0.00047613
0.01053333	0.00300000	0.00096667	0.00036667

 $\lambda=3, \mu=6$

P_0	P_1	P_2	P_3
0.45853187	0.22926593	0.14235242	0.07771303
0.47593333	0.22843333	0.14136667	0.07463333
P_4	P_5	P_6	P_7
0.04215684	0.02286801	0.01240489	0.00672911
0.03870000	0.02183333	0.01066667	0.00476667

 $\lambda=3, \mu=10$

P_0	P_1	P_2	P_3
0.67431521	0.20229456	0.08253285	0.02748416
0.68040000	0.20143333	0.08050000	0.02583333
P_4	P_5	P_6	P_7
0.00899670	0.00294420	0.00096356	0.00031535
0.00796667	0.00266667	0.00093333	0.00016667

 $\lambda=4, \mu=10$

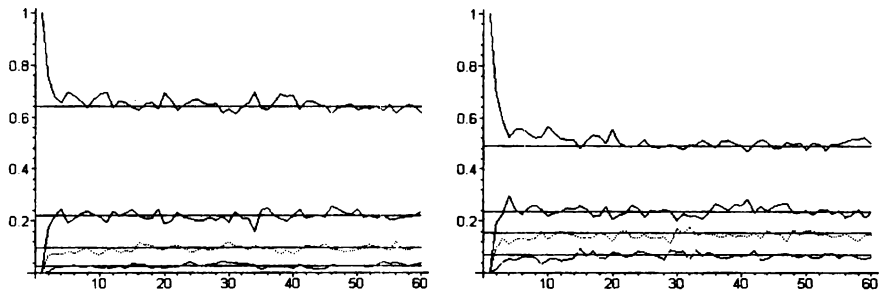
P_0	P_1	P_2	P_3
0.55351606	0.22140642	0.12314327	0.05618341
0.57040000	0.21440000	0.11923333	0.05330000
P_4	P_5	P_6	P_7
0.02521897	0.01131734	0.00507912	0.00227947
0.02486667	0.01006667	0.00436667	0.00213333

Figure 3.

	p0	p1	p2	p3	p4	p5	p6	p7	p8	p9	p10	p11	p12	p13	p14	p15	p16	p17
1	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
2	0.690	0.194	0.096	0.016	0.004	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
3	0.688	0.228	0.134	0.042	0.008	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
4	0.524	0.298	0.110	0.054	0.012	0.002	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
5	0.554	0.242	0.130	0.050	0.022	0.002	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
6	0.556	0.226	0.134	0.062	0.020	0.002	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
7	0.638	0.252	0.128	0.084	0.014	0.004	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
8	0.522	0.262	0.132	0.060	0.020	0.002	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
9	0.528	0.244	0.160	0.034	0.028	0.008	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
10	0.566	0.220	0.130	0.052	0.024	0.006	0.002	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
11	0.540	0.230	0.148	0.058	0.012	0.006	0.004	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
12	0.518	0.236	0.162	0.048	0.016	0.016	0.004	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
13	0.514	0.250	0.152	0.052	0.012	0.016	0.004	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
14	0.514	0.246	0.146	0.058	0.026	0.010	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
15	0.492	0.258	0.136	0.062	0.018	0.012	0.002	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
16	0.510	0.246	0.134	0.066	0.030	0.010	0.004	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
17	0.532	0.218	0.138	0.062	0.014	0.012	0.002	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
18	0.528	0.220	0.156	0.060	0.022	0.008	0.004	0.002	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
19	0.500	0.256	0.144	0.062	0.030	0.006	0.004	0.004	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
20	0.554	0.206	0.122	0.082	0.022	0.006	0.008	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
21	0.502	0.222	0.164	0.078	0.022	0.014	0.002	0.004	0.002	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
22	0.490	0.230	0.166	0.062	0.032	0.008	0.006	0.006	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
23	0.498	0.260	0.136	0.068	0.028	0.010	0.004	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
24	0.492	0.248	0.134	0.084	0.028	0.004	0.004	0.002	0.002	0.000	0.002	0.000	0.000	0.000	0.000	0.000	0.000	0.000
25	0.514	0.214	0.148	0.080	0.014	0.014	0.010	0.000	0.002	0.002	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
26	0.488	0.242	0.134	0.070	0.034	0.014	0.012	0.000	0.004	0.002	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
27	0.484	0.236	0.138	0.066	0.040	0.020	0.012	0.004	0.002	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
28	0.486	0.244	0.138	0.052	0.046	0.018	0.008	0.002	0.006	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
29	0.498	0.238	0.112	0.088	0.036	0.018	0.004	0.006	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
30	0.488	0.202	0.172	0.082	0.026	0.018	0.006	0.004	0.000	0.002	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
31	0.476	0.226	0.152	0.084	0.030	0.012	0.008	0.006	0.002	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
32	0.494	0.218	0.178	0.052	0.032	0.012	0.010	0.002	0.002	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
33	0.498	0.220	0.148	0.060	0.030	0.010	0.008	0.002	0.006	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
34	0.514	0.208	0.156	0.068	0.034	0.012	0.006	0.002	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
35	0.498	0.234	0.160	0.062	0.022	0.004	0.004	0.002	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
36	0.486	0.266	0.146	0.066	0.018	0.016	0.000	0.002	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
37	0.510	0.258	0.138	0.062	0.010	0.010	0.002	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
38	0.512	0.240	0.150	0.058	0.030	0.006	0.002	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
39	0.500	0.262	0.140	0.068	0.020	0.008	0.002	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
40	0.492	0.260	0.144	0.064	0.036	0.002	0.002	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
41	0.470	0.284	0.140	0.068	0.030	0.006	0.002	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
42	0.498	0.232	0.156	0.086	0.020	0.006	0.002	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
43	0.518	0.258	0.134	0.060	0.030	0.006	0.002	0.002	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
44	0.506	0.238	0.164	0.056	0.018	0.012	0.002	0.002	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
45	0.484	0.266	0.162	0.070	0.028	0.008	0.000	0.002	0.002	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
46	0.484	0.264	0.150	0.072	0.022	0.004	0.002	0.000	0.000	0.002	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
47	0.508	0.268	0.118	0.072	0.022	0.006	0.002	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
48	0.494	0.232	0.170	0.066	0.018	0.010	0.006	0.002	0.000	0.000	0.002	0.000	0.000	0.000	0.000	0.000	0.000	0.000
49	0.504	0.240	0.146	0.062	0.030	0.008	0.004	0.002	0.000	0.000	0.002	0.000	0.000	0.000	0.000	0.000	0.000	0.000
50	0.476	0.240	0.164	0.070	0.028	0.010	0.008	0.002	0.000	0.000	0.002	0.000	0.000	0.000	0.000	0.000	0.000	0.000
51	0.492	0.224	0.160	0.068	0.022	0.012	0.002	0.002	0.000	0.000	0.002	0.000	0.000	0.000	0.000	0.000	0.000	0.000
52	0.498	0.242	0.146	0.058	0.036	0.010	0.002	0.002	0.006	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
53	0.474	0.246	0.160	0.072	0.032	0.010	0.006	0.004	0.002	0.000	0.002	0.000	0.000	0.000	0.000	0.000	0.000	0.000
54	0.500	0.228	0.140	0.076	0.022	0.018	0.012	0.000	0.000	0.002	0.000	0.002	0.000	0.000	0.000	0.000	0.000	0.000
55	0.498	0.240	0.142	0.058	0.024	0.010	0.010	0.000	0.000	0.002	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
56	0.508	0.214	0.152	0.072	0.022	0.010	0.016	0.002	0.002	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
57	0.518	0.224	0.136	0.070	0.020	0.020	0.016	0.002	0.002	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
58	0.518	0.244	0.122	0.058	0.032	0.012	0.008	0.000	0.004	0.000	0.000	0.002	0.000	0.000	0.000	0.000	0.000	0.000
59	0.524	0.214	0.148	0.064	0.032	0.014	0.006	0.002	0.000	0.002	0.000	0.002	0.000	0.000	0.000	0.000	0.000	0.000
60	0.498	0.232	0.146	0.062	0.036	0.012	0.008	0.002	0.000	0.002	0.000	0.000	0.000	0.002	0.000	0.000	0.000	0.000

Figure 4.

T=0.1, c=0.05, d=0.15



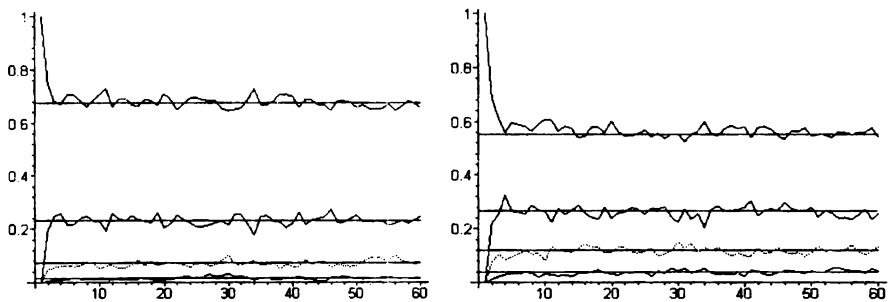
$\lambda=3$

P_0	P_1	P_2	P_3
0.64140782	0.22116389	0.09839155	0.02882717
0.65720000	0.21380000	0.09103333	0.02716667
P_4	P_5	P_6	P_7
0.00757934	0.00195262	0.00050280	0.00012968
0.00780000	0.00183333	0.00083333	0.00033333

$\lambda=4$

P_0	P_1	P_2	P_3
0.49182470	0.23702282	0.15395949	0.06896404
0.55230000	0.23453333	0.14096667	0.06366667
P_4	P_5	P_6	P_7
0.02850998	0.01165353	0.00476471	0.00194972
0.02476667	0.00960000	0.00450000	0.00146667

T=0.05, c=0.05, d=0.15



$\lambda=3$

P_0	P_1	P_2	P_3
0.67430935	0.23250867	0.07316093	0.01605422
0.68390000	0.22740000	0.06926667	0.01580000
P_4	P_5	P_6	P_7
0.00319686	0.00062098	0.00012014	0.00002325
0.00276667	0.00066667	0.00020000	0.00000000

$\lambda=4$

P_0	P_1	P_2	P_3
0.55350690	0.26674904	0.12168297	0.04027163
0.57383333	0.25743333	0.11406667	0.03626667
P_4	P_5	P_6	P_7
0.01239609	0.00376003	0.00113857	0.00034487
0.01226667	0.00410000	0.00123333	0.00060000

Figure 5.

