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ON THE MONOTONE CONVERGENCE OF A CHEBYSHEFF–HALLEY TYPE METHOD IN PARTIALLY ORDERED TOPOLOGICAL SPACES

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Abstract. We provide sufficient conditions for the monotone convergence of a Chebysheff-Halley-type method in a partially ordered topological space setting.

1. Introduction

In this study we are concerned with the problem of approximating a solution x^* of the nonlinear operator

$$F(x) = 0 \tag{1}$$

in a linear space E_1, where F is defined on a convex subset D of E_1 with values in a linear space E_2.

We showed recently that if E_1 and E_2 are Banach spaces, then under standard Newton-Kantorovich hypotheses the Chebysheff-Halley-type method of the form

$$y_n = x_n - F'(x_n)^{-1}F(x_n), \tag{2}$$

$$C_n = -F'(x_n)^{-1}([x_n, y_n] - [x_n, x_n]), \tag{3}$$

(4)

$$x_{n+1} = y_n - F'(x_n)^{-1}(I - C_n)^{-1}([x_n, y_n] - [x_n, x_n])\,(y_n - x_n), \quad x_0 \in D, \quad n \geq 0$$

converges with order almost three to a locally unique solution $x^* \in D$ of equation (1), [1]. Here $[x, y]$ denotes a divided difference of order one, which is a linear operator.

We introduce and study the monotone convergence of the iterations $\{v_n\}$ and $\{x_n\}$ $(n \geq 0)$ given by

$$F(v_n) + [x_n, x_n](w_n - v_n) = 0, \tag{5}$$

$$F(x_n) + [x_n, x_n](y_n - x_n) = 0, \tag{6}$$

$$[x_n, x_n]([x_n, y_n] - [x_n, x_n])(w_n - v_n) + [x_n, y_n][x_n, x_n](v_{n+1} - w_n) = 0 \tag{7}$$

and

$$[x_n, x_n]([x_n, y_n] - [x_n, x_n])(y_n - x_n) + [x_n, y_n][x_n, x_n](x_{n+1} - y_n) = 0 \tag{8}$$

to approximate a solution x^* of equation (1). We note that in order to compute the iterates in (2)-(4), we usually result to solving (5) and (7) or (6) and (8).

The Chebysheff-Halley method (or the method of tangent hyperbolas) converges with order three [5]-[9]. However, with the exception of some special cases, this method has no practical value in a Banach space setting because it requires an evaluation of the second Fréchet-derivative at each step (which means a number of function evaluations proportional with the cube of the dimension of the space). Discretized versions of the Euler-Chebysheff (which is similar to ours) method were considered by Ul'm [8] and Potra [7]. Ul'm used divided differences of order one and two, whereas Potra used divided differences of order one only. However, Potra used hypotheses on divided differences of order two in his convergence theorem [7, p.91]. The order of convergence of their iterations is 1.839... . The order of convergence of our iterations is almost three. Moreover, we use hypotheses on divided differences of order one only.

II. Monotone convergence

We will assume that the reader is familiar with the meaning of divided difference of order one and the notion of partially ordered topological space (POTL-space) [1], [2], [7], [9]. Moreover, from now on we will assume that E_1 and E_2 are POTL-spaces.

We can now prove the main result.

Theorem 1. *Let F be a nonlinear operator defined on a convex subset D of a regular POTL-space E_1 with values in a POTL-space E_2. Let v_0 and x_0 be two points of D such that*

$$v_0 \leq x_0 \tag{9}$$

and

$$F(v_0) \leq 0 \leq F(x_0). \tag{10}$$

Suppose that F has a divided difference of order one on $D_0 = \langle v_0, x_0 \rangle = \{x \in \in E_1 \mid v_0 \leq x \leq x_0\} \subseteq D$ satisfying

$$A_0 = [x_0, x_0], \quad \bar{A}_0 = [x_0, y_0][x_0, x_0], \quad \textit{for some} \quad y_0 \in \langle v_0, x_0 \rangle \tag{11}$$

have continuous nonnegative left sub-inverses B_0 and $\bar{B}_0$ respectively,

$$[x_0, y] \geq 0 \quad \textit{for all } v_0 \leq y \leq x_0, \tag{12}$$

$$[x, v] \leq [x, y] \quad \textit{if} \quad v \leq y, \tag{13}$$

$$[x, x]([x, x] - [x, y]) \leq [x, y][x, x] \quad \textit{if } y \leq x, \tag{14}$$

and

$$\bar{B}_n A_n L_n B_n [v_n, x_n] + B_n [v_n, x_n] \leq I \quad \textit{for all } n \geq 0, \tag{15}$$

where

$$L_n = [x_n, x_n] - [x_n, y_n], \tag{16}$$

$$A_n = [x_n, x_n], \quad \bar{A}_n = [x_n, y_n][x_n, x_n] \tag{17}$$

and $B_n, \bar{B}_n$ are continuous left sub-inverses of A_n and $\bar{A}_n$ respectively for all $n \geq 0$.

Then there exist two sequences $\{v_n\}, \{x_n\}$ $(n \geq 0)$ satisfying the approximations (5)-(8),

$$v_0 \leq w_0 \leq v_1 \leq \ldots \leq w_n \leq v_{n+1} \leq x_{n+1} \leq y_n \leq \ldots \leq x_1 \leq y_0 \leq x_0, \tag{18}$$

(19) $\lim_{n\to\infty} v_n = v^*, \quad \lim_{n\to\infty} x_n = x^* \quad and \quad v^*, x^* \in D \quad with \quad v^* \le x^*.$

Moreover, if the operators A_n are inverse nonnegative, then any solution u of the equation $F(x) = 0$ in $\langle v_0, x_0\rangle$ belongs to $\langle v^*, x^*\rangle$.

Proof. Let us define the operator

$$P_1 : \langle 0, x_0 - v_0\rangle \to E_1, \quad P_1(x) = x - B_0(F(v_0) + A_0(x)).$$

This operator is clearly isotone and continuous. We can have in turn

$$\begin{aligned} P_1(0) &= -B_0F(v_0) \ge 0, \\ P_1(x_0 - v_0) &= x_0 - v_0 - B_0F(x_0) + B_0(F(x_0) - F(v_0) - A_0(x_0 - v_0)) \le \\ &\le x_0 - v_0 + B_0([x_0, v_0] - [x_0, x_0])(x_0 - v_0) \le \qquad \text{(by (10))} \\ &\le x_0 - v_0, \end{aligned}$$

since $[x_0, v_0] \le [x_0, x_0]$ by (13).

By Kantorovich's theorem [4], the operator P_1 has a fixed point $z_1 \in \in \langle 0, x_0 - v_0\rangle : P_1(z_1) = z_1$. Set $w_0 = v_0 + z_1$, then we have the estimates

$$F(v_0) + A_0(w_0 - v_0) = 0,$$

$$F(w_0) = F(w_0) - F(v_0) - A_0(w_0 - v_0) \le 0$$

and

$$v_0 \le w_0 \le x_0.$$

We also define the operator

$$P_2 : \langle 0, x_0 - w_0\rangle \to E_1, \quad P_2(x) = x + B_0(F(x_0) - A_0(x)).$$

This operator is isotone and continuous. We can have in turn

$$\begin{aligned} P_2(0) &= B_0F(x_0) \ge 0, \\ P_2(x_0 - w_0) &= x_0 - x_0 + B_0F(w_0) + B_0(F(x_0) - F(w_0) - A_0(x_0 - w_0)) \le \\ &\le x_0 - w_0 + B_0([x_0, w_0] - [x_0, x_0])(x_0 - w_0) \le \qquad \text{(by (10))} \\ &\le x_0 - w_0, \end{aligned}$$

since $[x_0, w_0] \le [x_0, x_0]$ by (13).

By Kantorovich's theorem, there exists $z_2 \in \langle 0, x_0 - w_0 \rangle$ such that $P_2(z_2) = z_2$. Set $y_0 = x_0 - z_1$, then we have the estimates

$$F(x_0) + A_0(y_0 - x_0) = 0,$$

$$F(y_0) = F(y_0) - F(x_0) - A_0(y_0 - x_0) \geq 0$$

and

$$v_0 \leq w_0 \leq y_0 \leq x_0.$$

We now define the operator

$$P_3 \quad \langle 0, x_0 - v_0 \rangle \to E_1, \quad P_3(x) = x - \bar{B}_0(A_0 L_0 B_0 F(v_0) + \bar{A}_0(x)).$$

This operator is isotone and continuous. We have in turn
(20)

$$\begin{aligned} P_3(0) &= -\bar{B}_0 A_0 L_0 B_0 F(v_0) \geq 0, \\ P_3(x_0 - v_0) &= x_0 - v_0 - \bar{B}_0 A_0 L_0 B_0 F(x_0) + \bar{B}_0[A_0 L_0 B_0(F(x_0) - F(v_0)) - \\ &\quad - [x_0, y_0][x_0, x_0](x_0 - v_0)]. \end{aligned}$$

By (10) we get

$$-\bar{B}_0 A_0 L_0 B_0 F(x_0) \leq 0 \tag{21}$$

and by (14)

$$A_0 L_0 B_0[x_0, v_0](x_0 - v_0) \leq A_0 L_0 B_0 A_0(x_0 - v_0) \leq A_0 L_0(x_0 - v_0) \leq \bar{A}_0(x_0 - v_0),$$

which together with (20) and (21) gives

$$P_3(x_0 - v_0) \leq x_0 - v_0.$$

By Kantorovich's theorem there exists $z_3 \in \langle 0, x_0 - v_0 \rangle$ such that $P_3(z_3) = z_3$. Set $v_1 = w_0 + z_3$, then we have the estimate

$$[x_0, x_0]L_0(w_0 - v_0) + \bar{A}_0(v_1 - w_0) = 0.$$

Furthermore, we can define the operator

$$P_4 : \langle 0, x_0 - v_0 \rangle \to E_1, \quad P_4(x) = x + \bar{B}_0(A_0 L_0 B_0 F(x_0) - \bar{A}_0(x)).$$

This operator is isotone and continuous. We have in turn

(22)
$$\begin{aligned} P_4(0) =& \bar{B}_0 A_0 L_0 B_0 F(x_0) \geq 0, \\ P_4(x_0 - v_0) =& x_0 - v_0 + \bar{B}_0 A_0 L_0 B_0 F(v_0) + \bar{B}_0[A_0 L_0 B_0 (F(x_0) - F(v_0)) - \\ & - \bar{A}_0 (x_0 - v_0)]. \end{aligned}$$

By (10) we get

$$\bar{B}_0 A_0 L_0 B_0 F(v_0) \leq 0 \tag{23}$$

and by (14)

$$A_0 L_0 B_0 [x_0, v_0] \leq A_0 L_0 B_0 A_0 \leq A_0 L_0 \leq \bar{A}_0, \tag{24}$$

which together with (22) and (23) gives

$$P_4(x_0 - v_0) \leq x_0 - v_0.$$

By Kantorovich's theorem there exists $z_4 \in \langle 0, x_0 - v_0 \rangle$ such that $P_4(z_4) = z_4$. Set $x_1 = y_0 - z_4$, then we have the estimate

$$A_0 L_0 (y_0 - x_0) + \bar{A}_0 (x_1 - y_0) = 0.$$

We also have by (10) and (15) (for $n = 0$)

$$\begin{aligned} v_1 - w_0 &= w_0 + \bar{B}_0 A_0 L_0 (w_0 - v_0) - w_0 = \bar{B}_0 A_0 L_0 (w_0 - v_0) \geq 0, \\ x_1 - y_0 &= y_0 + \bar{B}_0 A_0 L_0 (y_0 - x_0) - y_0 = \bar{B}_0 A_0 L_0 (y_0 - x_0) \leq 0, \\ v_1 - x_1 &= w_0 + \bar{B}_0 A_0 L_0 (w_0 - v_0) - (y_0 + \bar{B}_0 A_0 L_0 (y_0 - x_0)) = \\ &= w_0 - y_0 - \bar{B}_0 A_0 L_0 (v_0) + \bar{B}_0 A_0 L_0 (v_0 - B_0 F(v_0)) - \\ &\quad - \bar{B}_0 A_0 L_0 (x_0 - B_0 F(x_0)) + \bar{B}_0 A_0 L_0 (x_0) = \\ &= (I - B_0[v_0, x_0] - \bar{B}_0 A_0 L_0 B_0 [v_0, x_0])(v_0 - x_0) \leq 0. \end{aligned}$$

Hence, we get $v_0 \leq w_0 \leq v_1 \leq x_1 \leq y_0 \leq x_0$.

By hypotheses (13) it follows that the operators $A_n, \bar{A}_n$ have continuous nonnegative left sub-inverses B_n and $\bar{B}_n$ for all $n \geq 0$. Proceeding by induction we can show that there exist two sequences $\{v_n\}, \{x_n\}$ $(n \geq 0)$ satisfying (5)-(8) and (18) in a regular space E_1, and as such, they converge to some $v^*, x^* \in D$. That is, we have $\lim_{n\to\infty} v_n = v^*$, $\lim_{n\to\infty} x_n = x^*$ and $v^* \leq x^*$.

If $v_0 \leq u \leq x_0$ and $F(u) = 0$, then we can immediately have in turn

$$A_0(y_0 - u) = A_0(x_0 - B_0 F(x_0)) - A_0(u) = \\ = A_0(I - B_0[x_0, u])(x_0 - u) \geq 0,$$

since $B_0[x_0, u] \leq B_0 A_0 \leq I$.

Similarly, we show $A_0(w_0 - u) \leq 0$. Since the operator A_0 is inverse nonnegative, then it follows from the above that $w_0 \leq u \leq y_0$. Proceeding by induction we deduce that $w_n \leq u \leq y_n$, from which it follows that $w_n \leq v_n \leq w_{n+1} \leq u \leq \leq y_{n+1} \leq x_n \leq y_n$ for all $n \geq 0$. That is, we have $v_n \leq u \leq x_n$ for all $n \geq 0$. Hence, we get $v^* \leq u \leq x^*$, which completes the proof of the theorem.

Remark. Let us assume:

(i) There exists $c \in [0, 1]$ such that

$$[x, x]([x, x] - [x, y]) \leq c[x, y][x, x] \quad \text{if } y \leq x. \tag{25}$$

(ii) B_0 is a continuous nonnegative sub-inverse of A_0.

Under hypotheses (i) and (ii), condition (15) reduces to showing that

$$\bar{B}_0 A_0 L_0 B_0[v_0, x_0] + B_0[v_0, x_0] \leq I. \tag{26}$$

But by (14) and (25) we get

$$\bar{B}_0 A_0 L_0 B_0[v_0, x_0] + B_0[v_0, x_0] \leq c\overline{B_0 A_0} B_0[v_0, x_0] + B_0[v_0, x_0] \leq (c+1) B_0[v_0, x_0]. \tag{27}$$

Then by (ii), (26) will be true if

$$(c + 1)[z, x] \leq [x, x] \quad \text{for } z \leq x. \tag{28}$$

Therefore, conditions (14) and (15) can be replaced by the stronger (but easier to verify) (25), (ii) and (28).

Moreover, we can do even better.

Theorem 2. *Let F be a nonlinear operator defined on a convex subset D of a regular POTL-space, E_1, with values in a POTL-space, E_2. Let v_0 and z_0 be two points of D such that*

$$v_0 \leq x_0 \quad \textit{and} \quad F(v_0) \leq 0 \leq F(x_0).$$

Suppose that F has a divided difference of order one on $D_0 = \langle v_0, x_0 \rangle \subseteq D$ satisfying $A_0 = [x_0, y_0]$ has a continuous nonnegative left sub-inverse B_0, the operators $A_n, M_n = [x_n, y_n]$ are inverse nonnegative,

$$[x_0, y] \geq 0 \quad \text{for all } v_0 \leq y \leq x_0,$$

$$[x, v] - [x, y] \leq 0 \quad \text{if } v \leq y, \tag{29}$$

$$[z, w] + [w, q] - [z, z] - [v, z] \geq 0 \quad \text{if } v \leq w \leq z \text{ for some } q \in \langle v, z \rangle.$$

Then

(i) the points v_{n+1} and $x_{n+1} \geq 0$ are solutions of (7) and (8) respectively if and only if they are solutions of

$$([x_n, y_n] - [x_n, x_n])(w_n - v_n) + [x_n, x_n](v_{n+1} - w_n) = 0 \tag{30}$$

and

$$([x_n, y_n] - [x_n, x_n])(y_n - x_n) + [x_n, x_n](x_{n+1} - y_n) = 0 \tag{31}$$

for all $n \geq 0$;

(ii) there exist two sequences $\{v_n\}, \{x_n\}$ $(n \geq 0)$ satisfying the approximations (5)-(8), $v_0 \leq w_0 \leq v_1 \leq \ldots \leq w_n \leq v_{n+1} \leq x_{n+1} \leq y_n \leq \ldots \leq x_1 \leq \leq y_0 \leq x_0$, $\lim_{n\to\infty} v_n = v^*$, $\lim_{n\to\infty} x_n = x^*$ and $v^*, x^* \in D_0$ with $v^* \leq x^*$.

Moreover, any solution u of the equation $F(x) = 0$ in $\langle v_0, x_0 \rangle$ belongs to $\langle v^*, x^* \rangle$.

Proof.

(i) Let v_{n+1} be a solution of (7), then by (13)

$$[x_n, y_n]\,[([x_n, y_n] - [x_n, x_n])(w_n - v_n) + [x_n, x_n](v_{n+1} - w_n)] \leq 0$$

from which it follows that since $M_n = [x_n, y_n]$ are inverse nonnegative

$$([x_n, y_n] - [x_n, x_n])(w_n - v_n) + [x_n, x_n](v_{n+1} - w_n) \leq 0. \tag{32}$$

Also, by (13) we get

$$[x_n, x_n]\,[([x_n, y_n] - [x_n, x_n])(w_n - v_n) + [x_n, x_n](v_{n+1} - w_n)] \leq 0$$

from which it follows that since $A_n = [x_n, x_n]$ are inverse nonnegative

$$([x_n, y_n] - [x_n, x_n])(w_n - v_n) + [x_n, x_n](v_{n+1} - w_n) \geq 0. \tag{33}$$

From (32) and (33) we deduce that v_{n+1} be a solution of (30).

Conversely, let v_{n+1} be a solution of (30). Then we have in turn

$$\begin{aligned} 0 = M_n\, [([x_n, y_n] - [x_n, x_n])(w_n - v_n) + A_n(v_{n+1} - w_n)] \leq \\ \leq A_n([x_n, y_n] - [x_n, x_n])(w_n - v_n) + M_n A_n(v_{n+1} - w_n), \end{aligned} \tag{34}$$

$$\begin{aligned} 0 = A_n\, [([x_n, y_n] - [x_n, x_n])(w_n - v_n) + A_n(v_{n+1} - w_n)] \geq \\ \geq A_n([x_n, y_n] - [x_n, x_n])(w_n - v_n) + M_n A_n(v_{n+1} - w_n). \end{aligned} \tag{35}$$

From (34) and (35) we now deduce that v_{n+1} is a solution of (7). The proof for approximations (8) and (31) follows similarly.

(ii) The proof of this part, as almost identical to the proof of Theorem 1, is omitted.

The proof of the theorem is now complete.

In what follows we shall give some natural conditions under which the points v^* and x^* are solutions of the equation $F(x) = 0$.

Theorem 3. *Under the hypotheses of Theorem 1 (or 2), suppose that F is continuous at v^* and x^*. If one of the following conditions is satisfied*

(a) $x^* = y^*$;

(b) E_1 is normal and there exists an operator $Q : E_1 \rightarrow E_2$ ($Q(0) = 0$) which has an isotone inverse continuous at the origin and such that $A_n \leq T$ for sufficiently large n;

(c) E_2 is normal and there exists an operator $R : E_1 \rightarrow E_2$ ($R(0) = 0$) continuous at the origin and such that $A_n \leq R$ for sufficiently large n;

(d) the operators A_n are equicontinuous for all $n \geq 0$;

(e) E_2 is normal and $[u, v] \leq [x, y]$ if $u \leq x$ and $v \leq y$.

Then we have

$$F(v^*) = F(x^*) = 0.$$

Proof.

(a) Using the continuity of F and $F(v_n) \leq 0 \leq F(x_n)$ we get $F(v^*) \leq v^* \leq$ $\leq F(v^*)$. That is, we obtain $F(x^*) = F(v^*) = 0$.

(b) By (4) and (6)

$$0 \geq F(v_n) = A_n(v_n - w_n) \geq Q(v_n - w_n),$$

$$0 \leq F(x_n) = A_n(x_n - y_n) \leq Q(x_n - y_n).$$

Hence, we get

$$0 \geq Q^{-1}F(v_n) \geq v_n - w_n, \quad 0 \leq Q^{-1}F(x_n) \leq x_n - y_n.$$

Since E_1 is normal and $\lim_{n\to\infty}(v_n - w_n) = \lim_{n\to\infty}(x_n - y_n) = 0$, we have $\lim_{n\to\infty} Q^{-1}F(v_n) = \lim_{n\to\infty} Q^{-1}F(x_n) = 0$. Hence, by continuity, we get $F(v^*) = = F(x^*) = 0$.

(c) As above we get

$$0 \geq F(v_n) \geq R(v_n - w_n), \quad 0 \leq F(x_n) \leq R(x_n - y_n).$$

Using the normality of E_2 and the continuity of F and R we get $F(v^*) = = F(x^*) = 0$.

(d) From the equicontinuity of the operator A_n we have $\lim_{n\to\infty} A_n(v_n - -w_n) = \lim_{n\to\infty} A_n(x_n - y_n) = 0$. Hence, by (4) and (6), $F(v^*) = F(x^*) = 0$.

(e) Using hypotheses (11)-(15) we get in turn

$$\begin{aligned}0 \leq F(y_n) &= F(y_n) - F(x_n) - A_n(y_n - x_n) = \\ &= (A_n - [y_n, x_n])(x_n - y_n) \leq ([x_0, x_0] - [x^*, x^*])(x_n - y_n).\end{aligned}$$

Since E_2 is normal and $\lim_{n\to\infty}(x_n - y_n) = 0$, we get $\lim_{n\to\infty} F(x_n) = 0$.

Moreover, from hypotheses (13)

$$[x^*, x^*](x_n - x^*) \leq [x^*, x_n](x_n - x^*) = F(x_n) - F(x^*) \leq [x_0, x_0](x_n, x^*)$$

and by the normality of E_2, $F(x^*) = \lim_{n\to\infty} F(x_n)$. Hence, we get $F(x^*) = 0$. The result $F(v^*) = 0$ can be obtained similarly.

The proof of the theorem is now complete.

As in Theorems 1 and 2, we can prove the following result (see also [7, Th.6.2]):

Theorem 4. *Assume that the hypotheses of Theorem 2 are true. Then the approximations*

$$y_n = x_n - B_n F(x_n),$$

$$x_{n+1} = y_n + B_n L_n(y_n - x_n), \quad L_n = [x_n, x_n] - [x_n - y_n],$$
$$w_n = v_n - B_n F(v_n)$$

and

$$v_{n+1} = w_n + B_n + L_n(w_n - v_n),$$

where the operators B_n, which are nonnegative subinverses of A_n, generate two sequences $\{v_n\}$ and $\{x_n\}$ satisfying approximations (5)-(8) and (18). Moreover, for any solution $u \in \langle v_0, x_0 \rangle$ of the equation $F(x) = 0$ we have

$$u \in \langle v_n, x_n \rangle \quad n \geq 0.$$

Furthermore, assume that the following are true:

(a) E_2 is a POTL-space and E_1 is a normal POTL-space;

(b) $\lim_{n\to\infty} x_n = x^*$ and $\lim_{n\to\infty} v_n = v^*$;

(c) F is continuous at v^* and x^*

and

(d) there exists a continuous nonsingular nonnegative operator T such that $B_n \geq T$ for sufficiently large n.

Then

$$F(v^*) = F(x^*) = 0.$$

Remarks.

(a) Similar results can immediately follow if the divided difference $[x_0, x_0]$ is replaced by $[x_0, z_0]$, $v_0 \leq z_0 \leq x_0$ in (11), $[x_n, x_n]$ is replaced by $[x_n, y_{n-1}]$ ($n \geq$ ≥ 1) in (5)-(8).

(b) Our conditions coincide with (49) and (50) in [7, p.98]. In case $E_1 =$ $= E_2 = IR$, our conditions are satisfied if and only if F is differentiable on D_0 and F, F' are convex on D_0.

(c) It follows from all the above that our method uses similar or simpler conditions than the ones in all previous results [4]-[9] and the order of convergence is faster [3]. Note that the results in [6] have been obtained for Euler-Chebysheff method only.

References

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