Annales Univ. Sci. Budapest., Sect. Comp. 19 (2000) 133-141

A NOTE ON THE PRODUCT OF CONSECUTIVE ELEMENTS OF AN ARITHMETIC PROGRESSION

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1. Introduction

For an integer $x > 1$ we denote by $P(x)$ the greatest prime factor of x and by $\pi(x)$ the number of primes $\leq x$. We consider the equation

$$(n+d)(n+2d)\dots(n+kd) = y^\ell \tag{1.1}$$

in positive integers $d,\ k,\ \ell,\ n,\ y$ subject to $gcd(n,d) = 1$, $k > 2$, $\ell \geq 2$.

P.Erdős and J.L.Selfridge confirm in [1] an old conjecture that equation (1.1) has no solution if $d = 1$. Furthermore, Erdős conjectured that equation (1.1) implies that k is bounded by an absolute constant.

R.Marszalek [2] considered equation (1.1) with $d \geq 2$. He showed that k is bounded if d is fixed. More precisely, he proved that for any solution of (1.1) with $d \geq 2$ we have

$$k < 2\exp[d(d+1)^{1/2}] \qquad \text{if } \ell = 2,$$
$$k < \max\{30000,\ (3/2)\exp[1/2d(d+2)(d+1)^{1/3}]\} \qquad \text{if } \ell = 3,$$
$$k < \max[30000,\ (1/4)d(d+2)(d+1)^{1/2}] \qquad \text{if } \ell = 4,$$
$$k < \max[30000,\ (3/2)(d+1)] \qquad \text{if } \ell \geq 5.$$

The results in this paper considerably improve the results of Marszalek. We will prove the following result

Theorem. *For every integer $d \geq 2$ and $\ell \geq 2$ there exists a constant $k_0(d,\ell)$ such that for $k \geq k_0(d,\ell)$ the equation (1.1) has no solution. For $k_0(d,\ell)$ we can take the following values:*

$$k_0(d,2) = \max[64,\ 2\exp(d)],$$
$$k_0(d,3) = \max[30000,\ (3/2)\exp(d^{4/3})],$$
$$k_0(d,\ell) = \max[30000,\ d] \qquad \text{for } \ell \geq 4.$$

2. Lemmas

For the proof we need the following results.

Lemma 1. (T.N.Shorey and R.Tijdeman [3]) *If* $d > 1$ *and* $(n+d,\ d,\ k) \neq$ $\neq (2,\ 7,\ 3)$, *then* $P(\Delta) > k$, *where* $\Delta = (n+d)(n+2d)\dots(n+kd)$.

Lemma 2. (R.Marszalek [2]) *Let d be a positive integer and let f be a real function for which there exists a positive integer* k_0, *such that f is positive and nondecreasing on the interval* $[k_0, \infty)$. *If the positive integers n and k satisfy*

$$gcd(n,\ d) = 1, \qquad n + d > kf(k),$$
$$k > \max\{k_0,\ 2\pi[1 + d/f(k_0)]\},$$

then

$$\pi[P(\Delta)] > k\{\log[f(k) + d]/[\log(f(k) + d) + \log k]\}.$$

Lemma 3. *The equation* (1.1) *with* $d \geq 2$ *has no solution if* $k \geq$ $\geq \max(d,\ n)$.

Proof. If the equation (1.1) has solution, by Lemma 1 there exists a prime $P > k$ dividing exactly one factor of Δ. Thus

$$n + kd \geq (k+1)^\ell \geq (k+1)^2. \tag{2.1}$$

On the other hand, if $k \geq \max(n,\ d)$ we have

$$n + kd \leq k + k^2 < (k+1)^2. \tag{2.2}$$

However (2.1) contradicts to (2.2). This completes the proof of Lemma 3.

Lemma 3 implies that we may confine ourselves to the case

$$d \leq k < n \tag{2.3}$$

to complete the proof of our theorem.

We assume that $d,\ k,\ n,\ \ell$ and y are positive integers satisfying the equation (1.1). Thus, for $1 \leq i \leq k$ we can write

$$n + id = a_i x_i^\ell, \tag{2.4}$$

where a_i is ℓ-th power-free and its prime factors are less than k.

Lemma 4. *The products $a_i a_j$ are all distinct provided*

$$k \geq d \qquad \text{for} \quad \ell > 3, \tag{1}$$

$$k \geq (3/2)\exp(d^{4/3}) \qquad \text{for} \quad \ell = 3. \tag{2}$$

Proof. By Lemma 1 and (2.3) we have

$$(k+1)^{\ell} \leq n + kd \leq n + k^2.$$

Therefore

$$k^{\ell} < n \qquad \text{if} \quad \ell \geq 3. \tag{2.5}$$

For $1 \leq i, j, r, s \leq k$ and $\langle i, j\rangle \neq \langle r, s\rangle$ we have $gcd(n+id,\ n+rd) < k$, $gcd(n+id,\ n+sd) < k$ and by (2.5) $n+id > k^2$. If $n+id$ divides $(n+rd)(n+sd)$, then $gcd[n+id,\ (n+rd)(n+sd)] = n+id > k^2$. However, this is not possible. So, it follows that $n+id$ cannot divide $(n+rd)(n+sd)$. Hence the products $(n+id)(n+jd)$ and $(n+rd)(n+sd)$ are distinct.

Suppose that for some $1 \leq i, j, r, s \leq k$ and $\langle i, j\rangle \neq \langle r, s\rangle$ one has $a_i a_j = a_r a_s$. Putting $T = (n+id)(n+jd) - (n+rd)(n+sd)$ (which we may assume to be positive) and $A = a_i a_j$, we get

$$(n+id)(n+jd) = a_i a_j x^{\ell} = Ax^{\ell},$$

$$(n+rd)(n+sd) = a_r a_s y^{\ell} = Ay^{\ell}.$$

Hence $Ax^{\ell} > Ay^{\ell}$, and therefore $x \geq y+1$. Thus $T \geq A[(y+1)^{\ell} - y^{\ell}] > A\ell y^{\ell-1}$. Since $Ay^{\ell} \geq (n+d)^2$ and A is an integer, so we obtain

$$T > \ell(n+d)^{2(\ell-1)/\ell}. \tag{2.6}$$

On the other hand

$$T \leq (n+kd)^2 - (n+d)^2 = 2kdn + k^2d^2 - 2nd - d^2.$$

Using (2.5) we get

$$2nd > 2k^{\ell}d \geq 2k^3 d > k^2 d^2.$$

So

$$T < 2kdn. \tag{2.7}$$

By (2.6) and (2.7) it follows

$$\ell(n+d)^{2(\ell-1)/\ell} < 2kdn < 2kd(n+d).$$

Then

$$\ell^{\ell}(n+d)^{\ell-2} < 2^{\ell}k^{\ell}d^{\ell}. \tag{2.8}$$

Now we have to consider separately the cases $\ell > 3$ and $\ell = 3$. If $\ell > 3$ and $k \geq d$, then

$$3^{\ell}(n+d)^2 \leq \ell^{\ell}(n+d)^{\ell-2} < 2^{\ell}k^{\ell}d^{\ell} \leq 2^{\ell}k^{2\ell}.$$

However, this contradicts to (2.5).

In the case $\ell = 3$ by (2.5) we see that $n+d > k(k^2-d)$. This enables us to utilize Lemma 2 for $f(k) = k^2 - d$. Therefore there exists a prime P dividing Δ such that $\pi(P) > 2/3k$. By $x > \pi(x)\log\pi(x)$, this gives

$$P > (2/3)k\log(2k/3) \geq (2/3)kd^{4/3} \qquad \text{for } k \text{ satisfying (2)}.$$

From (2.4) and the fact that P divides only one factor of Δ, we get

$$n + kd > [(2/3)kd^{4/3}]^3. \tag{2.9}$$

Since

$$n + kd = n + d + (k-1)d,$$

then from (2.8) and (2.9)

$$[(2/3)kd^{4/3}]^3 < n + d + k^2 < [(2/3)kd]^3 + k^2.$$

This implies $d < 2$, and Lemma 4 is proved.

Let G be the set of primes p dividing Δ with $p \leq k-1$. For every $p \in G$ we choose a $u(p) \in \{1, 2, \ldots, k\}$ such that

$$\operatorname{ord}_p[n + u(p)d] = \max\{\operatorname{ord}_p(n+jd)\}, \tag{2.10}$$

where $1 \leq j \leq k$. We denote by H the set of all elements from $\{1, 2, \ldots, k\}$ which do not appear in the range of u. Then we have

Lemma 5.

$$\prod_{j\in H} a_j \mid (k-1)! \tag{2.11}$$

Proof. For each prime $p \in G$, if $1 \le j \le k$ and $j \ne u(p)$, we have

$$\mathrm{ord}_p(n+jd) \le \mathrm{ord}_p[u(p)-j], \tag{2.12}$$

since if $p^m \mid n+jd$, then (2.10) and $gcd(n,\ d) = 1$ imply $p^m \mid u(p)-j$. Hence

$$\mathrm{ord}_p\left[\prod_{1\le j\le k,\ j\ne u(p)} (n+jd)\right] \le \mathrm{ord}_p\left[\prod_{1\le j\le k,\ j\ne u(p)} (u(p)-j)\right] =$$

$$= \mathrm{ord}_p[(u(p)-1)!(k-u(p)!] \le \mathrm{ord}_p[(k-1)!].$$

Thus, (2.11) follows from

$$\mathrm{ord}_p\left(\prod_{j\in H} a_j\right) \le \mathrm{ord}_p\left[\prod_{j\in H}(n+jd)\right] \le \mathrm{ord}_p\left[\prod_{1\le j\le k,\ j\ne u(p)} (n+jd)\right].$$

Note that

$$|H| \ge k - \pi(k-1), \tag{2.13}$$

where $|A|$ denotes the cardinality of set A.

Lemma 6. (P.Erdős and J.L.Selfridge [1]) *Let $b_1 < b_2 < \ldots < b_k$ be positive integers such that the products $b_i b_j$ are all distinct. Then for $k \ge 30000$*

$$\prod_{i\in D} b_i > k!, \tag{2.14}$$

where D is any subset of $\{1,2,\ldots,k\}$ satisfying $|D| \ge k - \pi(k)$.

Lemma 7. *If $k \ge 2\exp(q)$ and $q \ge 5$, then*

$$3^{(k-6)/4} q^{(k-1)/(q-1)} > 2^{(k+6)/3} k^4, \tag{2.15}$$

where k and q are positive integers.

Proof. First we prove that

$$3^{(k-6)/4} > 2^{(k+6)/3}, \qquad \text{if } k \ge 2\exp(5). \tag{2.16}$$

If (2.16) is false, then

$$3^{(k-6)/4} \le 2^{(k+6)/3}.$$

So

$$4(k+6)\log 2 \geq 3(k-6)\log 3.$$

This implies

$$k(3\log 3 - 4\log 2) \leq 24\log 2 + 18\log 3. \tag{2.17}$$

However it is impossible for $k \geq 2\exp(5)$. Thus we have (2.16).

Next we prove that if $k \geq 2\exp(q)$ and $q \geq 5$, then

$$q^{(k-1)/(q-1)} > k^4. \tag{2.18}$$

If $k \geq 2\exp(q)$ and $q \geq 5$, then

$$q^{(k-1)^{1/2}} > k.$$

Thus

$$(k-1)^{1/2} > (\log k)/(\log q).$$

Since

$$(k-1)^{1/2} \leq (k-1)/4(q-1),$$

we have

$$(\log k)/(\log q) < (k-1)/4(q-1).$$

Consequently, (2.18) is true.

3. Proof of the Theorem

a) The case $\ell \geq 3$. Lemma 4 enables us to apply Lemma 6 to the set H given by Lemma 5. Thus in the case $\ell \geq 3$, since (2.11) and (2.14) are in contradiction for k satisfying (1), (2) and $k \geq 30000$, we have proved: if

$$k \geq \max\{30000,\ 3/2\exp(d^{4/3})\} \qquad \text{for } \ell = 3,$$
$$k \geq \max\{30000,\ d\} \qquad \text{for } \ell > 3,$$

then the equation (1.1) has no solution.

b) The case $\ell = 2$. Now suppose that the theorem is false for $\ell = 2$. We shall first prove that if $k \geq 2\exp(d)$ and $i \neq j$, then $a_i \neq a_j$. Suppose that $a_i = a_j$ for some $i \neq j$. Assuming that $x_i \geq x_j + 1$, we have

$$d(k-1) = (n+kd) - (n+d) \geq (n+id) - (n+jd) = a_j(x_i^2 - x_j^2) > 2x_j a_j \geq$$

$$\geq 2(n+d)^{1/2}.$$

Hence

$$(n+d) < [d^2(k-1)^2]/4. \tag{3.1}$$

On the other hand, by Lemma 1, we have $n+d > k(k-d)$. Thus we may utilize Lemma 2 for $f(k) = k-d$. Therefore there exists a prime P dividing Δ, such that $\pi(P) > 1/2k$, which by $x > \pi(x)\log\pi(x)$ gives $P > (kd)/2$ for $k \geq 2\exp(d)$.

Since P divides only one factor of Δ which is a square, we get $n+kd >$ $> P^2 > (k^2d^2)/4$. Thus

$$(n+d) > [(k^2d^2)/4] - (k-1)d. \tag{3.2}$$

By (3.1) and (3.2) we have

$$[k^2d^2/4] - (k-1)d < [d^2(k-1)^2]/4. \tag{3.3}$$

Thus (3.3) gives

$$2k(d-2) < d-4.$$

However, this is not possible for $d \geq 2$. Thus for $k \geq 2\exp(d)$ the a's are distinct and square-free. So by Lemma 5

$$\prod_{1\leq j\leq k} a_j \mid (k-1)! \prod_{p<k} p. \tag{3.4}$$

Let us for a prime q put $g_q = \operatorname{ord}_q\left(\prod_{1\leq j\leq k} a_j\right)$ and $h_q = \operatorname{ord}_q[(k-1)!]$. Then by (3.4) if $g_2 \geq h_2$, then we have

$$\prod_{1\leq j\leq k} a_j \mid (k-1)! 2^{g_2-h_2} \prod_{p<k} p,$$

and if $g_2 < h_2$, then there exists an integer w which satisfies $\operatorname{ord}_2(w) > h_2 - g_2$ and

$$w \prod_{1\leq j\leq k} a_j = (k-1)! \prod_{p<k} p.$$

So we get

$$\prod_{1\leq j\leq k} a_j \mid (k-1)! 2^{g_2-h_2} \prod_{p<k} p. \tag{3.5}$$

Similarly, we have

$$\prod_{1\le j\le k} a_j \mid (k-1)!2^{g_2-h_2}3^{g_3-h_3}\prod_{p<k} p. \tag{3.6}$$

If 2 cannot divide d and 3 also cannot divide d, then there is a prime $q \ge 5$ such that $q|d$. Therefore q cannot divide a_i. Thus

$$\prod_{1\le j\le k} a_j \mid (k-1)!2^{g_2-h_2}3^{g_3-h_3}q^{-h_q}\prod_{p<k} p. \tag{3.7}$$

On the other hand, for a prime q we have

$$g_q \le [k/(q+1)] + \log_q k + 1 \qquad \text{(cf. [2] p.221)}$$

and also

$$h_q \ge [(k-1)/(q-1)] - \log_q k \qquad \text{(cf. [2] p.221)}.$$

Therefore

$$g_2 - h_2 \le -(2/3)k + 2\log_2 k + 2, \qquad g_3 - h_3 \le -(1/4)k + 2\log_3 k + (3/2). \tag{3.8}$$

Further, using the above inequality,

$$\prod_{p<k} p < 3^k, \qquad \text{for } k = 1, 2, \ldots \tag{3.9}$$

(see for example [4]) and the fact that the product of k consecutive square-free integers is greater than $k!(3/2)^k$ for $k \ge 64$ (see [1]), we obtain

$$3^{(k-6)/4}q^{(k-1)/(q-1)} < 2^{(k+6)/3}k^4, \tag{3.10}$$

which is in contradiction with Lemma 7. If $2|d$ or $3|d$, then by (3.6) and (3.8) we get

$$3^{(k-6)/4} < 2k^2 \qquad \text{or} \qquad 3^{(k-1)/2} < 2^{(k+6)/3}k^2$$

which also give a contradiction for $k \ge \max[64,\ 2\exp(d)]$. If $2|d$ and $3|d$, then we get

$$3^{(k-1)/2} < 2k.$$

This leads also to a contradiction. So we complete the proof of the theorem in the case $\ell = 2$.

References

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(Received November 29, 1994)

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